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ON WEIGHTED MULTIDIMENSIONAL EMBEDDINGS FOR MONOTONE FUNCTIONS

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Abstract

We characterize the inequality

$$\left(\int_{\mathsf{R}_+^N} f^q u\right)^{1/q} \le C\left(\int_{\mathsf{R}_+^N} f^p v\right)^{1/p}, \qquad 0 < q,\ p < \infty,$$

for monotone functions $f \ge 0$ and nonnegative weights u and v. The case $q < p$ is new and the case $0 < p \le q < \infty$ is extended to a modular inequality with N-functions. A remarkable fact concerning the calculation of C is pointed out.

1. Introduction

Let $\mathsf{R}_+^N := \{(x_1, \dots, x_N); x_i \ge 0, i = 1, 2, \dots, N\}$ and $\mathsf{R}_+ := \mathsf{R}_+^1$. Assume that $f : \mathsf{R}_+^N \to \mathsf{R}_+$ is monotone which means that it is monotone with respect to each variable. We denote $f \downarrow$, when f is decreasing (=nonincreasing) and $f \uparrow$ when f is increasing (=nondecreasing).

Given $0 < p, q < \infty$ and the weights $u \ge 0$ and $v \ge 0$ we consider the inequality

$$\left(\int_{\mathsf{R}_+^N} f^q u\right)^{1/q} \le C\left(\int_{\mathsf{R}_+^N} f^p v\right)^{1/p} \tag{1}$$

for all $f \downarrow$ or $f \uparrow$.

In the one dimensional case the inequality (1) was characterized in ([4], Proposition 1) for both alternative cases $0 < p \le q < \infty$ and $0 < q < p < \infty$ as follows:

(a) If $N = 1$, $0 < p \le q < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$A_0 := \sup_{t>0} \left(\int_0^t u\right)^{1/q} \left(\int_0^t v\right)^{-1/p} < \infty \tag{2}$$

and the constant $C = A_0$ is sharp.

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(b) If $N = 1$, $0 < q < p < \infty$, $1/r = 1/q - 1/p$, then (1) is true for all $f \downarrow$ if and only if

$$B_0 := \left(\int_0^\infty \left(\int_0^t u \right)^{r/p} \left(\int_0^t v \right)^{-r/p} u(t)\, dt \right)^{1/r} < \infty. \tag{3}$$

Moreover,

$$\left(\frac{q^2}{pr} \right)^{1/p} B_0 \le C \le \left(\frac{r}{q} \right)^{1/r} B_0$$

and

$$B_0^r = \frac{q}{r} \frac{\left(\int_0^\infty u \right)^{r/q}}{\left(\int_0^\infty v \right)^{r/p}} + \frac{q}{p} \int_0^\infty \left(\int_0^t u \right)^{r/q} \left(\int_0^t v \right)^{-r/q} v(t)\, dt. \tag{4}$$

(c) The same characterizations are valid, when $f \uparrow$, with the only replacement of the integrals over $[0, t]$ by the integrals over $[t, \infty]$.

Since the one dimensional inequality (1) expresses the embedding of classical Lorentz spaces, the further generalizations and references in this directions can be found in [2]. The multidimensional case was treated in ([1], Theorem 2.2), where, in particular the inequality (1) was characterized in the case $0 < p \le q < \infty$ and the sharp value of the constant C was given as

$$C = A_N := \sup_{D \in \mathscr{D}_d} \frac{\left(\int_D u \right)^{1/q}}{\left(\int_D v \right)^{1/p}} \tag{5}$$

and supremum is taken over the set $\mathscr{D}_d$ of all "decreasing" domains. Moreover it was shown ([1], Theorem 2.5) that if $u(x)$ and $v(x)$ are product weights, i.e., if

$$u(x) = u_1(x_1) \dots u_N(x_N), \qquad v(x) = v_1(x_1) \dots v_N(x_N), \tag{6}$$

then the constant C can be calculated in the following way:

$$C = A_N^{(1)} := \sup_{a_i > 0} \frac{\left(\int_0^{a_1} \dots \int_0^{a_N} u \right)^{1/q}}{\left(\int_0^{a_1} \dots \int_0^{a_N} v \right)^{1/p}}. \tag{7}$$

It was also pointed out in [1], Example 3.1, that if $u(x)$ and $v(x)$ are not product weights, then the equality $A_N = A_N^{(1)}$ is not true in general. In fact, in this paper we even prove the remarkable fact that theconstants A_N and $A_N^{(1)}$ are not comparable in general (for $N \ge 2$).

Section 2 of the present paper is devoted to the modular inequality of the form

$$(8)\quad \Phi_2^{-1}\left(\int_{\mathbb{R}_+^N} \Phi_2\left(\omega(x) f(x)\right) u(x)\, dx\right) \le \Phi_1^{-1}\left(\int_{\mathbb{R}_+^N} \Phi_1\left(C f(x)\right) v(x)\, dx\right),$$

where Φ_1 and Φ_2 are N-functions [3] such that

$$(9)\qquad \sum_n \Phi_2 \circ \Phi_1^{-1}(a_n) \le K \Phi_2 \circ \Phi_1^{-1}\left(\sum_n a_n\right)$$

for all $a_n \ge 0$ with a constant $K \ge 1$ independent on $\{a_n\}$.

In Section 3 we consider the particular case of (1), when $N = 2$, $0 < p \le q < \infty$, $u(x, y) = u(xy)$, $v(x, y) = v(xy)$ and find an explicit criterion for this case. One important consequence of this result is that there is no uniform constant $c > 0$ such that $cA_N^{(1)} \ge A_N$, i.e., A_N and $A_N^{(1)}$ are not comparable in general.

The case $0 < q < p < \infty$ of (1) is characterized in Section 4.

Conventions and notations. Products and quotients of the forms $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are taken to be 0. $\mathbb{Z}$ stands for the set of all integers and χ_E denotes the characteristic function of a set E.

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2. A modular integral inequality

Let $0 \le h(x) \downarrow$ and $t > 0$. Denote

$$D_{h,t} := \{x \in \mathbb{R}_+^N;\ h(x) > t\},$$

and

$$\mathscr{D}_d := \bigcup_{0 \le h\downarrow} \bigcup_{t>0} D_{h,t}.$$

The set $\mathscr{D}_d$ consists of all "decreasing" domains $D_{h,t}$. In particular, $\chi_{D_{h,t}}$ is decreasing in each variable.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative, convex function such that

$$\lim_{x\to 0} \frac{\Phi(x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

Following [3] we call Φ an N-function. In particular,

$$\Phi(x) \le \frac{1}{a}\Phi(ax) \qquad \text{for all} \quad a \ge 1, \quad x > 0. \tag{10}$$

THEOREM 2.1. *Let* Φ_1, Φ_2 *be two N-functions satisfying* (9). *Given weight functions* $\omega(x) \ge 0$, $u(x) \ge 0$, $v(x) \ge 0$ *the inequality* (8) *holds for all* $0 \le f \downarrow$ *if and only if there exists a constant* $A = A(\Phi_1, \Phi_2, u, v, \omega)$ *such that, for all* $\varepsilon > 0$ *and* $D_{h,t} \in \mathscr{D}_d$

$$\Phi_2^{-1}\left(\int_{D_{h,t}} \Phi_2\left(\varepsilon\omega(x)\right) u(x)\,dx\right) \le \Phi_1^{-1}\left(\Phi_1\left(A\varepsilon\right)\int_{D_{h,t}} v(x)\,dx\right). \tag{11}$$

PROOF. The necessity follows, if we replace f in (8) by $f = \varepsilon\chi_{D_{h,t}}$.
For sufficiency we define for a fixed $f \downarrow$

$$\Delta_n := \{x \in \mathsf{R}_+^N;\ 2^n < f(x) \le 2^{n+1}\}, \qquad n \in \mathsf{Z},$$
$$D_n := \{x \in \mathsf{R}_+^N;\ f(x) > 2^n\},$$

and note that

$$D_n \supset D_{n+1},\ D_n = \bigcup_{k\ge n}\Delta_k, \qquad \mathsf{R}_+^N = \bigcup_n \Delta_n.$$

Obviously, $\Delta_n \bigcap \Delta_k = \emptyset$ for $n \ne k$. We have, using (10)

$$\begin{aligned}
\int_{\mathsf{R}_+^N} \Phi_2\left(\omega(x)f(x)\right)u(x)\,dx &\le \frac{1}{K}\int_{\mathsf{R}_+^N} \Phi_2\big(K\omega(x)f(x)\big)u(x)\,dx \\
&= \frac{1}{K}\sum_n \int_{\Delta_n} \Phi_2\big(K\omega(x)f(x)\big)u(x)\,dx \\
&\le \frac{1}{K}\sum_n \int_{\Delta_n} \Phi_2\left(2^{n+1}K\omega(x)\right)u(x)\,dx \\
&\le \frac{1}{K}\sum_n \int_{D_n} \Phi_2\left(2^{n+1}K\omega(x)\right)u(x)\,dx
\end{aligned}$$

[applying (11) with $\varepsilon = 2^{n+1}K$]

$$\le \frac{1}{K}\sum_n \Phi_2 \circ \Phi_1^{-1}\left(\Phi_1\left(AK2^{n+1}\right)\int_{D_n} v\right)$$

[applying (9)]

$$\le \Phi_2 \circ \Phi_1^{-1}\Big(\sum_n \Phi_1\left(AK2^{n+1}\right)\sum_{k\ge n}\int_{\Delta_k} v\Big)$$

$$= \Phi_2 \circ \Phi_1^{-1}\Big(\sum_k \Big(\int_{\Delta_k} v\Big)\sum_{n\le k}\Phi_1\left(AK2^{n+1}\right)\Big)$$

[using the convexity of Φ_1]

$$\le \Phi_2 \circ \Phi_1^{-1}\Big(\sum_k \Big(\int_{\Delta_k} v\Big)\Phi_1\big(4AK2^k\big)\Big)$$

$$\le \Phi_2 \circ \Phi_1^{-1}\Big(\sum_k \int_{\Delta_k}\Phi_1\big(4AKf(x)\big)v(x)\,dx\Big)$$

$$= \Phi_2 \circ \Phi_1^{-1}\Big(\int_{\mathsf{R}_+^N}\Phi_1\big(4AKf(x)\big)v(x)\,dx\Big).$$

Thus, the least possible constant C in (8) satisfies

$$A \le C \le 4AK.$$

Theorem 2.1 is proved.

3. Explicit criteria for some cases

As we mentioned in the Introduction in the case of product weights (see (6)) the least possible constant C in (1) satisfies (7). The natural and important question is whether the constants A_N (5) and $A_N^{(1)}$ (7) are comparable in the general case. Clearly, $A_N^{(1)} \le A_N$, but the converse inequality $A_N \le cA_N^{(1)}$ with a constant c independent on weights was so far uncertain. Below we give a negative answer to this question with the help of the following result:

THEOREM 3.1. *Let* $0 < p \le q < \infty$ *and* $u(s) \ge 0$, $v(s) \ge 0$ *be two measurable functions on* R_+ *such that* $U(t) := \int_0^t u < \infty$, $V(t) := \int_0^t v < \infty$ *for all* $t > 0$.

Then the inequality

$$\Big(\int_{\mathsf{R}_+^2} f^q(x,y)u(xy)\,dx\,dy\Big)^{1/q} \le C\Big(\int_{\mathsf{R}_+^2} f^p(x,y)v(xy)\,dx\,dy\Big)^{1/p} \tag{12}$$

holds for all $f(x,y) \ge 0$ *decreasing in* x *and* y *with a finite constant* $C > 0$ *independent on* f *if and only if*

$$\mathscr{A} = \mathscr{A}_{p,q} := \sup_{t>0}\left(\frac{U(t)}{V(t)}\right)^{1/q}\left(\int_0^t V(x)\frac{dx}{x}\right)^{1/q-1/p} < \infty. \tag{13}$$

Moreover,

$$C = \mathscr{A}, \qquad \text{if} \quad p = q \tag{14}$$

and

$$2^{-1/p}\mathscr{A} \le C \le \left(\frac{p}{q}\right)^{1/q} \mathscr{A}, \qquad \text{if} \quad p < q \tag{15}$$

PROOF. We know from (5) that $C = I$, where

$$I = \sup_{t>0, h\downarrow} I_h(t) := \sup_{t>0, h\downarrow} \frac{\left(\int_0^t dx \int_0^{h(x)} u(xy)\, dy\right)^{1/q}}{\left(\int_0^t dx \int_0^{h(x)} v(xy)\, dy\right)^{1/p}} \tag{16}$$

and thus, by changing variables, we find that

$$I_h(t) = \frac{\left(\int_0^t U\big(xh(x)\big)\, \frac{dx}{x}\right)^{1/q}}{\left(\int_0^t V\big(xh(x)\big)\, \frac{dx}{x}\right)^{1/p}}. \tag{17}$$

We begin with the upper bound. By using (13) we obtain

$$\int_0^t U\big(xh(x)\big)\, \frac{dx}{x} \le \mathscr{A}^q \int_0^t \left(\int_0^{xh(x)} V(s)\, \frac{ds}{s}\right)^{q/p-1} V\big(xh(x)\big)\, \frac{dx}{x}$$

[changing the variables: $s = h(x)\xi$]

$$= \mathscr{A}^q \int_0^t \left(\int_0^x V\big(\xi h(x)\big)\, \frac{d\xi}{\xi}\right)^{q/p-1} V\big(xh(x)\big)\, \frac{dx}{x}$$

[$h(x) \le h(\xi)$ if $\xi \in (0, x)$]

$$\le \mathscr{A}^q \int_0^t \left(\int_0^x V\big(\xi h(\xi)\big)\, \frac{d\xi}{\xi}\right)^{q/p-1} V\big(xh(x)\big) \frac{dx}{x}$$

$$= \frac{p}{q}\mathscr{A}^q \left(\int_0^t V\big(\xi h(\xi)\big)\, \frac{d\xi}{\xi}\right)^{q/p}.$$

This implies that

$$I_h(t) \le \left(\frac{p}{q}\right)^{1/q} \mathscr{A}$$

for all $t > 0$ and $h \downarrow$. Thus, (16) brings the upper bound (15) and, in particular, $C \leq \mathscr{A}$ when $p = q$.

For the lower bound let $0 < \delta < t < \infty$ and $h_\delta(s)$ be defined as follows

$$h_\delta(s) = \begin{cases} 1 & \text{if } 0 \leq s < \delta. \\ \dfrac{\delta}{s} & \text{if } \delta \leq s < t. \\ 0 & \text{if } s \geq t. \end{cases}$$

Then, by using (17), we find in the case $p < q$ that

$$I^q_\delta(t) := I^q_{h_\delta}(t) = \frac{\int_0^\delta U(x)\,\frac{dx}{x} + U(\delta)\log\frac{t}{\delta}}{\left(\int_0^\delta V(x)\,\frac{dx}{x} + V(\delta)\log\frac{t}{\delta}\right)^{q/p}}. \tag{18}$$

Since $\log\frac{t}{\delta}$ takes all the values of $(0, \infty)$, when $t > \delta$, we can choose such a t_δ so that

$$\log\frac{t_\delta}{\delta} = \frac{1}{V(\delta)}\int_0^\delta V(x)\,\frac{dx}{x}.$$

With this t_δ (18) gives

$$I^q_\delta(t_\delta) = \frac{\int_0^\delta U(x)\frac{dx}{x} + \frac{U(\delta)}{V(\delta)}\int_0^\delta V(x)\frac{dx}{x}}{2^{q/p}\left(\int_0^\delta V(x)\frac{dx}{x}\right)^{q/p}} \geq 2^{-q/p}\frac{U(\delta)}{V(\delta)}\left(\int_0^\delta V(x)\frac{dx}{x}\right)^{1-q/p}.$$

Since $\delta > 0$ is arbitrary this implies that

$$C \geq 2^{-1/p}\mathscr{A}, \qquad p < q.$$

In the case $p = q$ we find from (18), that

$$I^p_\delta(t) = \frac{\int_0^\delta U(x)\,\frac{dx}{x} + U(\delta)\log\frac{t}{\delta}}{\int_0^\delta V(x)\,\frac{dx}{x} + V(\delta)\log\frac{t}{\delta}}$$

and observe that the right hand side tends to $U(\delta)/V(\delta)$, when $t \to \infty$, so that

$$C \geq \mathscr{A}, \qquad p = q$$

and the proof is finished.

Now, let $\mathscr{I}$ denote the constant given by (7) when $N = 2$ and $u(x, y) = u(xy)$, $v(x, y) = v(xy)$. Thus,

$$\mathscr{I} := \sup_{0<a,\, b<\infty} \frac{\left(\int_0^a \int_0^b u(xy)\, dx\, dy\right)^{1/q}}{\left(\int_0^a \int_0^b v(xy)\, dx\, dy\right)^{1/p}}.$$

Moreover by using (17) with $h(x) \equiv b$ and changing variable we obtain

$$\mathscr{I} := \sup_{t>0} \frac{\left(\int_0^t U(x)\, \frac{dx}{x}\right)^{1/q}}{\left(\int_0^t V(x)\, \frac{dx}{x}\right)^{1/p}}.$$

Obviously, Theorem 3.1 yields

$$\mathscr{I} \le I \le (p/q)^{1/q} \mathscr{A}$$

and since I and $\mathscr{A}$ are comparable because of (14) and (15) the question is whether there exists a constant $c > 0$ independent on u and v such that

$$\mathscr{A} \le c\, \mathscr{I}. \tag{19}$$

Applying the l'Hôspital test we note, that

$$\lim_{t\to 0} \frac{\int_0^t U(x)\, \frac{dx}{x}}{\left(\int_0^t V(x)\, \frac{dx}{x}\right)^{q/p}} = \frac{p}{q} \lim_{t\to 0} \frac{U(t)}{V(t)} \left(\int_0^t V(x)\, \frac{dx}{x}\right)^{1-q/p}$$

and a similar equality is valid for the limits at infinity. Since the functions involved are continuous, we conclude, that $\mathscr{A}$ and $\mathscr{I}$ are comparable in a sense, that if $\mathscr{I} < \infty$, then $\mathscr{A} < \infty$. However, the estimate (19) is no longer uniform, which can be seen from the following example:

EXAMPLE 3.2. Let $0 < \varepsilon < 1$ and let $V_\varepsilon(t)$ and $U_0(t)$ be defined by

$$U_0(t) = t \qquad \text{if} \quad 0 < t < \infty \tag{20}$$

and

$$V_\varepsilon(t) = \begin{cases} t^{\varepsilon} & \text{if } 0 < t \le 1, \\ t^{1/\varepsilon} & \text{if } t > 1. \end{cases} \tag{21}$$

Then

$$\mathscr{A}_{p,p}^{p} = \sup_{t>0} \frac{U_0(t)}{V_\varepsilon(t)} = 1.$$

We have

$$\int_0^t U_0(x)\,\frac{dx}{x} = t, \qquad t > 0$$

and

$$\int_0^t V_\varepsilon(x)\,\frac{dx}{x} = \begin{cases} \frac{1}{\varepsilon}t^\varepsilon & \text{if } 0 < t \le 1, \\ \frac{1}{\varepsilon} + \varepsilon(t^{1/\varepsilon} - 1) & \text{if } t > 1. \end{cases}$$

Thus,

$$\mathscr{I}_\varepsilon^p(t) := \frac{\int_0^t U_0(x)\,\frac{dx}{x}}{\int_0^t V_\varepsilon(x)\,\frac{dx}{x}} = \begin{cases} \varepsilon t^{1-\varepsilon} & \text{if } 0 < t \le 1 \\ \dfrac{t}{\frac{1}{\varepsilon} + \varepsilon(t^{1/\varepsilon} - 1)} & \text{if } t > 1 \end{cases}$$

and

$$\mathscr{I}_\varepsilon^p := \sup_{t>0} \mathscr{I}_\varepsilon^p(t) = \frac{\varepsilon}{1+\varepsilon}\left(\frac{1+\varepsilon}{\varepsilon}\right)^\varepsilon \to 0, \qquad \text{when} \quad \varepsilon \to 0.$$

Consequently, there exists no constant $c > 0$, independent on u and v such that, in general, the inequality (19) is true, i.e., so that $\mathscr{A} \le c\mathscr{I}$. In particular, this means that the constants $A_N^{(1)}$ and A_N from the introduction are not equivalent in general.

4. The case $0 < q < p < \infty$

Throughout this section we let $h(x) \ge 0$, $h \ne 0$ a.e., denote a decreasing function on R_+^N and $t > 0$ and use the following notations:

$$D_{h,t} := \{x \in \mathsf{R}_+^N;\, h(x) > t\}$$

and for an increasing sequence $\{t_k\} \subset \mathsf{R}_+$ we set

$$D_k = D_{h,k} := \{x \in \mathsf{R}_+^N;\, h(x) > t_k\}, \;\; k \in \mathsf{Z}.$$

Obviously, $D_k \supset D_{k+1}$ and we define

$$\Delta_k = \Delta_{h,k} := D_k \setminus D_{k+1}.$$

Hence, $\Delta_k \bigcap \Delta_n = \emptyset$, $k \ne n$ and $\mathsf{R}_+^N = \bigcup_k \Delta_k$.

Let $0 < q < p < \infty$ and $r \in \mathsf{R}_+$ be determined from the equation $1/r = 1/q - 1/p$.

If $u(x) \geq 0$ and $v(x) \geq 0$ are measurable functions on R_+^N we define the following quantities:

$$B^r := \sup_{0 \leq h \downarrow} \int_0^\infty \Big(\int_{D_{h,t}} v \Big)^{-r/p} d\Big(-\Big(\int_{D_{h,t}} u \Big)^{r/q} \Big), \tag{22}$$

and

$$\mathscr{B}^r := \sup_{0 \leq h \downarrow} \sup_{\{t_k\} \uparrow} \sum_k \Big(\int_{\Delta_k} u \Big)^{r/q} \Big(\int_{D_k} v \Big)^{-r/p}. \tag{23}$$

THEOREM 4.1. *Let* $0 < q < p < \infty$.

(i) *The inequality* (1) *is valid for all decreasing functions with a finite constant* $C > 0$ *independent of* f *if and only if* $\mathscr{B} < \infty$. *Moreover,*

$$\mathscr{B} \leq C \leq 4^{1/q} \mathscr{B}. \tag{24}$$

(ii) *The following inequality is true:*

$$\mathscr{B} \leq B \leq 2^{1/q} (2^{r/q} + 2^{r/p})^{1/r} \mathscr{B}. \tag{25}$$

(iii) *The following representation takes place:*

$$B^r := \frac{\Big(\int_{\mathsf{R}_+^N} u \Big)^{r/q}}{\Big(\int_{\mathsf{R}_+^N} v \Big)^{r/p}} + \sup_{0 \leq h \downarrow} \int_0^\infty \Big(\int_{D_{h,t}} u \Big)^{r/q} d\Big(\Big(\int_{D_{h,t}} v \Big)^{-r/p} \Big). \tag{26}$$

PROOF. For a fixed $0 \leq h \downarrow$ and an increasing sequence $\{t_k\}$ we define the function $f_h(x)$ by

$$f_h(x) = \sum_k \Big(\sum_{n \leq k} \Big(\int_{\Delta_n} u \Big)^{r/q} \Big(\int_{D_n} v \Big)^{-r/q} \Big)^{1/p} \chi_{\Delta_k}(x).$$

Then $f_h(x) \geq 0$ is a decreasing function and

$$\int_{\mathsf{R}_+^N} f_h^p v = \sum_k \Big(\sum_{n \leq k} \Big(\int_{\Delta_n} u \Big)^{r/q} \Big(\int_{D_n} v \Big)^{-r/q} \Big) \int_{\Delta_k} v$$

[changing the order of sums]

$$= \sum_n \Big(\int_{\Delta_n} u \Big)^{r/q} \Big(\int_{D_n} v \Big)^{-r/q} \sum_{k \geq n} \int_{\Delta_k} v$$

[using $\sum_{k\geq n}\int_{\Delta_k} v = \int_{D_n} v$, $-r/q+1 = -r/p$]

$$= \sum_n \left(\int_{\Delta_n} u\right)^{r/q} \left(\int_{D_n} v\right)^{-r/p} := \mathscr{B}^r_{h,\{t_k\}}. \tag{27}$$

Suppose now that (1) is valid with a finite constant $C > 0$, and assume temporarily that $\mathscr{B} \in (0,\infty)$. Then, for any $h \downarrow$ and $\{t_k\}$ such that $\mathscr{B}^r_{h,\{t_k\}} > 0$, we obtain by using the representation formula (27),

$$\begin{aligned}
C^q \left(\mathscr{B}_{h,\{t_k\}}\right)^{qr/p} &= C^q \left(\int_{\mathsf{R}^N_+} f_h^p v\right)^{q/p} \geq \int_{\mathsf{R}^N_+} f_h^q u \\
&= \sum_k \int_{\Delta_k} u \left(\sum_{n\leq k} \left(\int_{\Delta_n} u\right)^{r/q} \left(\int_{D_n} v\right)^{-r/q}\right)^{q/p}
\end{aligned}$$

[reducing the interior sum to one term with $k = n$]

$$\geq \sum_k \left(\int_{\Delta_k} u\right)^{r/q} \left(\int_{D_n} v\right)^{-r/p} = \mathscr{B}^r_{h,\{t_k\}}.$$

Hence,

$$C \geq \mathscr{B}_{h,\{t_k\}}$$

and the lower bound (24) follows. The temporary assumption $\mathscr{B} \in (0,\infty)$ can be removed in the usual way (see [4], p. 178).

Next we consider the upper bound. Given $f \downarrow$ we define

$$U(t) = \int_{D_{f,t}} u; \qquad V(t) = \int_{D_{f,t}} v.$$

Obviously, $U(t)$ and $V(t)$ are decreasing functions.

Now we construct a special increasing sequence $\{\tau_k\} \subset \mathsf{R}_+$ as follows: Put

$$\begin{aligned}
\tau_0 &= 1, \\
\tau_{k+1} &= \inf\left\{t : \min\left(\frac{V(\tau_k)}{V(t)}, \frac{U(\tau_k)}{U(t)}\right) = 2\right\}, \qquad k \geq 0, \\
\tau_{k-1} &= \sup\left\{t : \min\left(\frac{V(t)}{V(\tau_k)}, \frac{U(t)}{U(\tau_k)}\right) = 2\right\}, \qquad k \leq 0,
\end{aligned}$$

and let

$$(28)\qquad \begin{aligned} \mathsf{Z}_1 &= \left\{k \in \mathsf{Z} : V(\tau_{k+1}) = \frac{1}{2} V(\tau_k)\right\}, \\ \mathsf{Z}_2 &= \left\{k \in \mathsf{Z} : U(\tau_{k+1}) = \frac{1}{2} U(\tau_k)\right\}. \end{aligned}$$

We assume without a loss of generality that

$$(29)\qquad \mathsf{Z} = \mathsf{Z}_1 \bigcup \mathsf{Z}_2$$

and note that $\mathsf{Z}_1 \bigcap \mathsf{Z}_2 = \emptyset$. Now, we write

$$I := \int_{\mathsf{R}_+^N} f^q u = \sum_k \int_{\Delta_{k-1}} f^q u$$

where

$$\Delta_k := \Delta_{f,k} = D_{f,k} \setminus D_{f,k+1} := D_k \setminus D_{k+1}.$$

Since

$$\tau_k < f(x) \le \tau_{k+1}, \qquad x \in \Delta_k,$$

we find

$$I \le \sum_k \tau_k^q \int_{\Delta_{k-1}} u = \sum_k \frac{\tau_k^q \left(\int_{\Delta_{k-1}} u\right)^{q/p}}{\left(\sum_{n \le k} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n)\right)^{q/r}} \cdot \left(\int_{\Delta_{k-1}} u\right)^{1-q/p} \left(\sum_{n \le k} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n)\right)^{q/r}$$

[applying Hölder's inequality with $\frac{p}{q}$ and $\frac{r}{q}$]

$$\le \left(\sum_k \frac{\tau_k^p \int_{\Delta_{k-1}} u}{\left(\sum_{n \le k} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n)\right)^{p/r}}\right)^{q/p} \cdot \left(\sum_k \left(\int_{\Delta_{k-1}} u\right) \sum_{n \le k} \left(\int_{\Delta_{k-1}} u\right)^{r/q} V^{-r/p}(\tau_n)\right)^{q/r} := I_1^{q/p} I_2^{q/r}.$$

We have

$$\sum_{n \le k} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n) \ge \left(\int_{\Delta_{k-1}} u\right)^{r/p} V^{-r/p}(\tau_n).$$

Thus,

$$I_1 \le \sum_k \tau_k^p V(\tau_k).$$

We also note that the sequence $\{\tau_k\}$ is constructed in such a way that

$$V(\tau_k) \ge 2V(\tau_{k+1}), \quad U(\tau_k) \ge 2U(\tau_{k+1}) \qquad \text{for all} \quad k \in \mathbb{Z}. \tag{30}$$

Therefore, in particular,

$$V(\tau_k) = \int_{\Delta_k} v + V(\tau_{k+1}) \ge 2V(\tau_{k+1}).$$

Hence,

$$V(\tau_{k+1}) \le \int_{\Delta_k} v$$

and, consequently,

$$V(\tau_k) \le 2\int_{\Delta_k} v. \tag{31}$$

This implies that

$$I_1 \le 2\sum_k \tau_k^p \int_{\Delta_k} v \le 2\sum_k \int_{\Delta_k} f^p v \le 2\int_{\mathbb{R}_+^N} f^p v.$$

Now we return to the estimate of I_2. Write

$$\begin{aligned} I_2 &= \sum_n \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n) \sum_{k \ge n} \int_{\Delta_{k-1}} u \\ &= \sum_n \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n) U(\tau_{n-1}) := I_{2,1} + I_{2,2}, \end{aligned}$$

where, using (29), we put

$$I_{2,1} = \sum_{n: n-1 \in \mathbb{Z}_1} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n) U(\tau_{n-1}),$$

and

$$I_{2,2} = \sum_{n: n-1 \in \mathbb{Z}_2} \left(\int_{\Delta_{n-1}} u\right)^{r/p} V^{-r/p}(\tau_n) U(\tau_{n-1}).$$

Similar to the derivation of (31) we obtain that

$$U(\tau_k) \le 2 \int_{\Delta_k} u. \tag{32}$$

Hence, by using (28) and (32), we find that

$$I_{2,1} \le 2 \sum_{n:n-1\in \mathsf{Z}_1} \left(\int_{\Delta_{n-1}} u \right)^{r/q} \left(\frac{1}{2} V(\tau_{n-1}) \right)^{-r/p}$$
$$\le 2^{1+r/p} \sum_n \left(\int_{\Delta_n} u \right)^{r/q} V^{-r/p}(\tau_n) \le 2^{1+r/p} \mathscr{B}^r.$$

For the second term we use again (28) and (32). We have

$$U(\tau_{n-1}) = 2U(\tau_n), \qquad n-1 \in \mathsf{Z}_2,$$
$$\int_{\Delta_{n-1}} u = U(\tau_{n-1}) - U(\tau_n) = U(\tau_n) \le 2 \int_{\Delta_n} u.$$

Thus,

$$I_{2,2} \le 2^{1+r/p} \sum_{n:n-1\in \mathsf{Z}_2} \left(\int_{\Delta_n} u \right)^{r/q} V^{-r/p}(\tau_n) \le 2^{1+r/p} \mathscr{B}^r.$$

Summarizing the above estimates we obtain the upper bound

$$\left(\int_{\mathsf{R}^N_+} f^q u \right)^{1/q} \le 4^{1/q} \mathscr{B} \left(\int_{\mathsf{R}^N_+} f^p v \right)^{1/p}$$

and the part (i) of the Theorem 4.1 is proved.

For the proof of the lower bound (25) we fix $0 \le h(x) \downarrow$ and define

$$\Delta_{k,t} = \{x : t < h(x) \le t_{k+1}\}.$$

Then

$$\left(\int_{\Delta_k} u \right)^{r/q} = \int_{t_k}^{t_{k+1}} d\left(- \left(\int_{\Delta_{k,t}} u \right)^{r/q} \right) = \frac{r}{q} \int_{t_k}^{t_{k+1}} \left(\int_{\Delta_{k,t}} u \right)^{r/p} d\left(- \int_{\Delta_{k,t}} u \right).$$

Since

$$\int_{\Delta_{k,t}} u \le U(t);\ d\left(- \int_{\Delta_{k,t}} u \right) = d\left(-U(t) + \int_{D_{k+1}} u \right) = d\big(-U(t)\big)$$

we obtain

$$\left(\int_{\Delta_k} u\right)^{r/q} \leq \frac{r}{q}\int_{t_k}^{t_{k+1}} U^{r/p}(t)\, d\big(-U(t)\big) = \int_{t_k}^{t_{k+1}} d\big(-U^{r/q}(t)\big).$$

Applying this estimate and that

$$\left(\int_{D_k} v\right)^{-r/p} = V^{-r/p}(t_k) \leq V^{-r/p}(t), \qquad t \in [t_k, t_{k+1}],$$

we find

$$\begin{aligned}\sum_k \left(\int_{\Delta_k} u\right)^{r/q}\left(\int_{D_k} v\right)^{-r/p} &\leq \sum_k \int_{t_k}^{t_{k+1}} V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big)\\ &\leq \int_0^\infty V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big) \leq B^r.\end{aligned}$$

Thus,

$$\mathscr{B} \leq B.$$

For the proof of the upper bound (25) we observe that for $0 \leq h(x) \downarrow$ and an increasing sequence $\{t_k\} \subset \mathsf{R}_+$ we have

$$\begin{aligned}B_h^r := \int_0^\infty V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big) &= \sum_k \int_{t_k}^{t_{k+1}} V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big)\\ &\leq \sum_k V^{-r/p}(t_{k+1})U^{r/q}(t_k) := \mathscr{I}.\end{aligned}$$

Now suppose that $\{t_k\}$ is taken in the same way as the sequence $\{\tau_k\}$ was taken in the proof of part (i), that is $t_k = \tau_k$, $k \in \mathsf{Z}$. Then

$$\mathscr{I} = \sum_{k\in \mathsf{Z}_1} + \sum_{k\in \mathsf{Z}_2} := \mathscr{I}_1 + \mathscr{I}_2.$$

Therefore, by using (30), (31) and (32), we find that

$$\mathscr{I}_1 \leq 2^{r/q+r/p} \sum_{k\in\mathsf{Z}_1} V^{-r/p}(\tau_k)\left(\int_{\Delta_k} u\right)^{r/q},$$

$$\mathscr{I}_2 \leq 2^{2r/q} \sum_{k\in\mathsf{Z}_2} V^{-r/p}(\tau_{k+1})\left(\int_{\Delta_{k+1}} u\right)^{r/q}.$$

Thus,

$$\mathscr{I} \le 2^{r/q}(2^{r/q+r/p}) \sum_{k\in\mathsf{Z}} V^{-r/p}(\tau_k)\left(\int_{\Delta_k} u\right)^{r/q} \le 2^{r/q}(2^{r/q+r/p})\mathscr{B}^r.$$

This implies that

$$B \le 2^{1/q}(2^{r/q+r/p})^{1/r}\mathscr{B}$$

and, hence, the upper bound (25) is proved.

For the proof of part (iii) we suppose first that $B < \infty$. Then by putting, for a fixed $0 \le h(x) \downarrow$,

$$V(t) = \int_{D_{h,t}} v, \qquad U(t) = \int_{D_{h,t}} u,$$

we see that

$$\infty > B^r \ge \int_\tau^\infty V^{-r/q}(t)\, d\big(-U^{r/q}(t)\big) \to 0, \qquad \tau \to \infty.$$

Hence,

$$\int_\tau^\infty V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big) \ge V^{-r/q}(\tau)U^{r/q}(\tau) \to 0, \qquad \tau \to \infty.$$

This implies, by integration by parts, that

$$\int_0^\infty V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big) = \frac{U^{r/q}(0)}{V^{r/p}(0)} + \int_0^\infty U^{r/q}(t)\, dV^{-r/p}(t)$$

and the inequality

$$\infty > B^r \ge \frac{\left(\int_{\mathsf{R}_+^N} u\right)^{r/q}}{\left(\int_{\mathsf{R}_+^N} v\right)^{r/p}} + \sup_{0\le h} \int_0^\infty \left(\int_{D_{h,t}} u\right)^{r/q} d\left(\int_{D_{h,t}} v\right)^{-r/p} \tag{33}$$

follows.

Now suppose that the right hand side of (26) is finite. Then, for a fixed $h \downarrow$, integration by parts gives

$$\int_0^\infty U^{r/q}(t)\, dV^{-r/p}(t) \ge -\frac{U^{r/q}(0)}{V^{r/p}(0)} + \int_0^\infty V^{-r/p}(t)\, d\big(-U^{r/q}(t)\big)$$

and we obtain the reversed inequality to (33). Thus, also (26) is proved and the proof is complete.

EXAMPLE 4.2. Let $v = u \in L^1(\mathsf{R}_+^N)$. Then $B^r = \frac{r}{q}\int_{\mathsf{R}_+^N} v$.

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