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TRANSFERRING ALGEBRA STRUCTURES UP TO HOMOLOGY EQUIVALENCE

LEIF JOHANSSON and LARRY LAMBE

(This article is dedicated to Jim Stasheff on his 60th birthday)

Abstract

Given a strong deformation retract M of an algebra A, there are several apparently distinct ways ([9], [19], [13], [24], [15], [18], [17]) of constructing a coderivation on the tensor coalgebra of M in such a way that the resulting complex is quasi isomorphic to the classical (differential tor) [7] bar construction of A. We show that these methods are equivalent and are determined combinatorially by an inductive formula first given in a very special setting in [16]. Applications to de Rham theory and Massey products are given.

1. Preliminaries and Notation

Throughout this paper, R will denote a commutative ring with unit. The term *(co)module* is used to mean a differential graded (co)module over R and *maps* between modules are graded maps. When we write $\otimes$ we mean $\otimes_R$. The usual (Koszul) sign conventions are assumed. The degree of a homogeneous element m of some module is denoted by $|m|$. Algebras are assumed to be connected and coalgebras simply connected. (Co)algebras are assumed to have (co)units. (Co)algebras are, unless otherwise stated, assumed to be (co)augmented. The differential in an (co)algebra is a graded (co)derivation.

The R-module of maps from M to N (for R-modules M and N) is denoted by $\hom(M, N)$ (if the context requires it, we will use a subscript to denote the ground ring). The differential in this module is given by $D(f) = df - (-1)^{|f|} fd$. Note that D is a derivation with respect to the composition operation whenever it is defined. In particular, $End(M) = \hom(M, M)$ is an algebra.

If A is an algebra and C is a coalgebra, the module $\hom(C, A)$ is an algebra with respect to the operation defined by the following diagram

$$
\begin{tikzcd}
C \arrow[r, "f \cup g"] \arrow[d, "\Delta"'] & A \\
C \otimes C \arrow[r, "f \otimes g"] & A \otimes A \arrow[u, "m"']
\end{tikzcd}
\tag{1}
$$

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This product is called the *cup* or *convolution* product. The unit of this algebra is given by $\sigma\epsilon$, the composition of the counits of C and A.

Maps $\tau \in \hom(C, A)$ which satisfy $D(\tau) = \tau \cup \tau$, $\epsilon\tau = 0$ and $\tau\eta = 0$, where ϵ is the augmentation and η is the coaugmentation, are called *twisting cochains*. Any twisting cochain factors through the *universal twisting cochain* defined as the map $\pi \in \hom(\bar{B}(A), A)$ sending homogeneous elements of tensor degree 1 to their desuspension and other elements to 0. This map is a twisting cochain, and given another twisting cochain $\tau \in \hom(C, A)$, the following diagram commutes:

(2)
$$\begin{tikzcd} \bar{B}(A) \arrow[r, "\pi"] & A \\ C \arrow[u, "c(\tau)"] \arrow[ur, "\tau"'] & \end{tikzcd}$$

Here $c(\tau)$ is the unique coalgebra map induced by τ.

Given a twisting cochain $\tau : C \longrightarrow A$, the *cap* product $\tau \cap \cdot$ is defined by the following diagram:

(3)
$$\begin{tikzcd} C \otimes A \arrow[r, "\tau \cap \cdot"] \arrow[d, "\Delta \otimes 1"'] & C \otimes A \\ C \otimes C \otimes A \arrow[r, "1 \otimes \tau \otimes 1"] & C \otimes A \otimes A \arrow[u, "1 \otimes m"'] \end{tikzcd}$$

By a classical result of E. Brown, [5] τ is a twisting cochain if and only if $d_C \otimes 1 + 1 \otimes d_A + \tau \cap \cdot$ is a differential on $C \otimes A$, which together with this differential is denoted $C \otimes_\tau A$ and is referred to as the *twisted tensor product* (along τ).

Given two modules M and A, M is called a strong deformation retract (SDR) of A if and only if there are maps as in the following diagram:

(4)
$$M \underset{f}{\overset{\nabla}{\rightleftarrows}} (A, \phi).$$

Here $\phi : A \longrightarrow A$ is a homotopy: $1 - \nabla f = d\phi - \phi d$ between ∇f and the identity on A. Finally $f\nabla = 1$. Without loss of generality (as demonstrated in [20]) one may assume in addition that the following relations (called the *side conditions*) hold:

(5)
$$\phi\phi = 0, \quad \phi\nabla = 0, \quad f\phi = 0.$$

For a given module M we let $T(M)$ denote $\sum \otimes^n (s\bar{M})$, where $\bar{M}$ denotes the submodule of elements in positive degrees, and s denotes suspension. We let $T^c(M)$ denote $T(M)$ with the standard coproduct given by

$$\Delta([m_1, \dots, m_n]) = \sum [m_1, \dots, m_i] \otimes [m_{i+1}, \dots, m_n].$$

We will deonte the submodule of $T^c(M)$ tensor-degree n by $T^c_n(M)$.

2. A Review of Past Results

Given a quasi isomorphism $M \longrightarrow A$, where A is an algebra it is natural to try to associate to it some construction on M which is quasi isomorphic to $\bar{B}(A)$. Specifically one seeks a coalgebra differential ∂ on $T^c(M)$ and a twisting cochain

$$T^c(M) \xrightarrow{\tau} A, \tag{6}$$

such that $c(\tau)$ is a quasi isomorphism. It is common practice to call the tensor coalgebra $T^c(M)$ along with a coderivation ∂ an A_∞-structure (on M) [17], [15].

When one has an SDR, as in (4) with A an algebra, this problem has a long history. It was first solved in a special case, viz. when the differential in M is zero (so that $M \cong H(A)$) and the characteristic of the ground ring is zero by K. T. Chen in [8] and [9] using "iterated integrals". The special case of zero differential in M was also done by T. V. Kadeishvili in [19] and independently of this by V. Smirnov in [24].

Again in the special case where the differential of M is trivial, Gugenheim [13] gave an inductive construction of a twisting cochain τ and a coderivation ∂ also solving the problem by abstracting the algebraic content of Chen's work. This gave a purely algebraic method of achieving Chen's original result which was not restricted to characteristic zero. We call this the *obstruction method*.

The general problem (no restriction on the differential of M) was solved in [15] using what is called the *tensor trick* and [17] using a generalization of the obstruction method.

The method of [15], first occurred in the literature in [14] for a special class. It was independently discovered and used in [18] to obtain A_∞-structures.

Seemingly unrelated at first, Gugenheim and Munkholm [16] gave an inductive formula for lifting cochains when both objects in the general problem are (co)algebras and *one* of the maps is a (co)algebra map. In this case, we have an SDR (4) with A an algebra, M an algebra, f an algebra map, and $C \xrightarrow{\eta} M$ a twisting cochain. The Gugenheim-Munkholm formula gives a way to "lift" η to a twisting cochain $C \xrightarrow{\widehat{\eta}} A$. All of this will be explained below.

In particular, one can lift the universal twisting cochain (6) in the special case using the Gugenheim-Munkholm formula, but the tensor trick and the obstruction method *also* give ways to lift the universal twisting cochain in this case.

Thus one has three a priori distinct methods for defining the lift $\hat{\pi}$. The purpose of the paper of Gugenheim and Lambe [14] was to show that these three methods are essentially the same in this special setting (i.e. f multiplicative) – they each give the *same* $\widehat{\tau}$. The purpose of this paper is to generalize this result to the case where the differential on M is *not necessairly* zero. This is interesting since the proof the validity of the method in [16] *required* multiplicativity of f.

In the course of our work, we will show that the methods used in [15] and [17] to construct solutions to the problem above, are equivalent, yielding the same A_∞-structure and twisting cochain. Not only that, it will be shown that the same inductive formula given in [16] to lift twisting cochains works in the more general setting (of *homotopy twisting cochains*). Thus we have found a complete generalization of [14] to the setting of [15]. We will begin by reviewing the methods used in [15] and [17] and then review and generalize the formula of [16].

2.1. The Obstruction Method

Gugenheim and Stasheff in [17] extended the results of Chen, Kadeishvili, Smirnov and Gugenheim to the case where the differential in M is possibly non-zero by using a non-standard filtration of $T^c(M)$ to prove convergence of inductive formulas for τ and ∂. The details of this construction are summarized conviently in [14] and we refer the reader to this paper and the original for detrails. We give equivalent formulas in section 4.2 for the construction of τ and ∂. The differential constructed this way is the one denoted by ∂_{di} in theorem 4.1. Finally, we note that these ideas are also relavent to the work in [1].

2.2. The Tensor Trick

We will begin by introducing a notation which will be used throughout this paper. For a map $f : T^c(X)- > X$ let $\omega(f) : T^c(X)- > T^c(X)$ be the unique coextension of f as a coderivation, such that $\pi_1\omega(f) = sf$ where π_1 is the projection to X and s as usual is the suspension.

This method for constructing τ and ∂ uses the *perturbation lemma* [6], [11], [2] in the following way. First one applies the free tensor coalgebra functor $T^c(\cdot)$ to (4). $T^c(\nabla)$ and $T^c(f)$ are the obvious maps and $T^c(\phi)$ on n tensors is given by

$$T^c_n(\phi) = (\phi \otimes 1 \otimes \cdots \otimes 1) + \cdots$$
$$+ (\nabla f \otimes \cdots \otimes \nabla f \otimes \phi \otimes 1 \otimes \cdots \otimes 1) + \cdots + (\nabla f \otimes \cdots \otimes \nabla f \otimes \phi).$$

We will often write $\phi^{\otimes} = T^c(\phi)$ and similarly $\nabla^{\otimes} = T^c(\nabla)$, etc. Note that ϕ is uniquely determined by the fact that it is a *skew-derivation*, i.e.

$$\Delta\phi^{\otimes} = (\phi^{\otimes} \otimes 1 + \nabla f \otimes \phi^{\otimes})\Delta. \tag{7}$$

This follows from the commutativity of the diagram

$$\begin{tikzcd} T^c(A) \arrow[r, "\Delta"] \arrow[d, "p_{a+b}"'] & T^c(A) \otimes T^c(A) \arrow[d, "p_a \otimes p_b"] \\ T^c_a(A) \otimes T^c_b(A) \arrow[r, "\lambda_{a,b}"] & T^c_{a+b}(A) \end{tikzcd} \tag{8}$$

using an argument similar to that in §2 of [14]. Here p_i is the obvious projection and $\lambda_{a,b}$ the obvious natural isomorphism.

Now on the resulting SDR

$$\begin{tikzcd} T^c(M) \arrow[r, shift left, "\nabla^{\otimes}"] & (T^c(A), \phi^{\otimes}) \arrow[l, shift left, "f^{\otimes}"] \end{tikzcd} \tag{9}$$

one considers the transference problem given by changing the differential on $T^c(A)$ by the unique coderivation ∂^{alg} determined by $s(\pi \cup \pi)$ (which is exactly the “algebra” part of the bar construction differential). The perturbation lemma yields the desired result, viz., a limit SDR

$$\begin{tikzcd} (T^c(M), \partial_\infty) \arrow[r, shift left, "\nabla^{\otimes}_\infty"] & ((\bar{B}(A), \partial), \phi^{\otimes}_\infty) \arrow[l, shift left, "f^{\otimes}_\infty"] \end{tikzcd} \tag{10}$$

where ∂ is the ordinary “differential-Tor” bar construction differential,

$$\partial_\infty = \omega(-d) + f^{\otimes}(t_1 + \cdots + t_n + \cdots)\nabla^{\otimes}, \tag{11}$$

$$\nabla^{\otimes}_\infty = \nabla^{\otimes} + \phi^{\otimes}(t_1 + \cdots + t_n + \cdots)\nabla^{\otimes}, \tag{12}$$

$$f^{\otimes}_\infty = f^{\otimes} + f^{\otimes}(t_1 + \cdots + t_n + \cdots)\phi^{\otimes}, \tag{13}$$

$$\phi^{\otimes}_\infty = \phi^{\otimes} + \phi^{\otimes}(t_1 + \cdots + t_n + \cdots)\phi^{\otimes}, \tag{14}$$

where $t = \partial^{alg}$, and $t_n = (t\phi^{\otimes})^{n-1}t$, for $n \geq 1$. Note the use of $\omega(-d)$ above, which is the same thing as the "tensor-differential" part of the bar-construction. In the notation of theorem 4.1 $\partial_\infty = \partial_{tt}$ and $\pi_A(\nabla_\infty^{\otimes})$ is the twisting cochain in (6). While this is straightforward enough to do, it is not obvious that this method actually produces a coderivation. That this is indeed the case was shown in [15] and independently in [18].

REMARK 2.1. As an aside note that if ϕ is an algebra homotopy, i.e. if $\phi(ab) = a\phi b + \phi a\pi b$ then $\phi(\nabla a \nabla b) = 0$. In this case $\hat{\tau} = \pi\nabla$ is a twisting cochain and the A_∞-structure collapses to an algebra-structure on M.

2.3. *The Gugenheim-Munkholm Formula*

There is a formula for *lifting* (see the diagram below) twisting cochains across an SDR, in the special case when both A and M are algebras and f is an algebra map. Consider the universal twisting cochain $\pi : \bar{B}(M) \longrightarrow M$ (here π is defined as before by desuspending tensor degree 1 elements and sending elements of higher degree to 0). Gugenheim and Munkholm gave an inductive formula [16] for *lifting* π to a twisting cochain $\hat{\pi}$.

$$\begin{tikzcd} A \arrow[d, shift right, "f"'] & \\ M \arrow[u, shift right, "\nabla"'] & \bar{B}(M) \arrow[l, "\pi"] \arrow[ul, "\hat{\pi}"'] \end{tikzcd} \tag{15}$$

One should think of the operator $\hat{\cdot}$ as a function going from $\hom(\bar{B}(M), M)$ to $\hom(\bar{B}(M), A)$. By the universal property (2) of π, this gives a lift of any twisting cochain $C \longrightarrow M$ to A (for a given coalgebra C).

PROPOSITION 2.2 (Gugenheim-Munkholm Construction). Given an SDR as in (4) with both A and M algebras and f an algebra map. Consider the universal twisting cochain $\bar{B}(M) \xrightarrow{\pi} M$, i.e. the map which is zero on all terms of tensor degree not equal to one and such that $\pi[sm] = m$. Define $\hat{\pi}_n$ by the following inductive formula:

$$\begin{aligned} \hat{\pi}_0 &= 0 \\ \hat{\pi}_1 &= \nabla\pi \\ \hat{\pi}_n &= \sum_{i+j=n} \phi(\hat{\pi}_i \cup \hat{\pi}_j). \end{aligned} \tag{16}$$

Define $\hat{\pi}_n : \bar{B}(M) \to A$ on all of $\bar{B}(M)$ by taking it to be 0 outside of tensor degree n. Let $\hat{\pi} = \sum \hat{\pi}_n$.

Throughout this paper, the Gugenheim-Munkholm construction above for the map $\hat{\pi}$ will be referred to as either the *GM construction* or the *GM formula*.

It is shown in [16] that $\hat{\pi}$ is a twisting cochain (they actually prove the dual theorem). The proof *requires* the multipliciativity of f.

3. The Generalized GM Construction

Now consider again the general case where we are given an SDR (4) with A an algebra, but neither f nor ∇ are assumed to be multiplicative. Consider the diagram

$$\begin{tikzcd} A \arrow[d, shift right, "f"'] \\ M \arrow[u, shift right, "\nabla"'] & T^c(M) \arrow[l, "\pi"'] \end{tikzcd} \tag{17}$$

Even though there is *no multiplicative assumption* on M or on f, we can construct (at least formally) a map $\hat{\pi} : T^c(M) \longrightarrow A$ by using (16). It is natural to wonder what significance this has. To understand the answer, some notation will be introduced.

3.1. A Nonassociative Operation

We will use the notation $\bar{a} = (-1)^{|a|}a$ throughout this section. Note that the generalized GM-formula can be written as the inductive formula

$$\begin{aligned} \hat{\pi} &= \phi(\hat{\pi} \cup \hat{\pi}) + \nabla\pi \\ &= \phi m(\hat{\pi} \otimes \hat{\pi})\Delta + \nabla\pi. \end{aligned} \tag{18}$$

In general the composite

$$A \otimes A \xrightarrow{m} A \xrightarrow{\phi} A$$

is a non-associative product in A which we write as $a * b = \phi(ab)$.

In this way, the higher terms of $\hat{\pi}$ can be seen to arise as the obstructions to the associativity of $*$ on the image of ∇. For this, note that clearly

$$\begin{aligned}\hat{\pi}_n[m_1|\ldots|m_n] &= \sum_{i=1}^{n-1}(-1)^{|[m_1|\ldots|m_n]|}\,\phi(\hat{\pi}_i[m_1|\ldots|m_i]\cdot\hat{\pi}_{n-i}[m_{i+1}|\ldots|m_n])\\ &= \sum_{i=1}^{n-1}(-1)^{|[m_1|\ldots|m_n]|}\,\hat{\pi}_i[m_1|\ldots|m_i]*\hat{\pi}_{n-i}[m_{i+1}|\ldots|m_n]\\ &= \sum_{i=1}^{n-1}\hat{\pi}_i\,\overline{[m_1|\ldots|m_i]}*\hat{\pi}_{n-i}[m_{i+1}|\ldots|m_n],\end{aligned}$$

so that

$$\begin{aligned}\hat{\pi}_2[m_1|m_2] &= -\nabla(\overline{m_1})*\nabla(m_2)\\ \hat{\pi}_3[m_1|m_2|m_3] &= \nabla(\overline{m_1})*\big(\nabla(\overline{m_2})*\nabla(m_3)\big)-\big(\nabla(\overline{m_1})*\nabla(m_2)\big)*\nabla(m_3)\end{aligned}$$

and so on. The main result of this paper is to show that these higher obstructions are exactly reproduced in the tensor trick and in the obstruction method. In essence, these higher obstructions are what create the associated A_∞ structure.

4. The Main Result

In this section, we prove the following

THEOREM 4.1. *Given an SDR as in (4), where $N = A$ is an algebra, let ∂_{tt} denote the A_∞-structures on M given by the tensor trick, ∂_{om} denote the one given by the obstruction method (cf section 2.1), and let ∂_{ggm} denote the map given by the generalized Gugenheim Munkholm construction (cf section 3). we have that*

$$\partial_{tt} = \partial_{ggm} = \partial_{om}.$$

4.1. The Tensor Trick Revisited

Consider the SDR (4) and suppose that A is an algebra and let t be the initiator (in the language of [2]) for the SDR (9).

We begin with an algebraic observation that depends on the side conditions (5).

LEMMA 4.2. *On the image of $\nabla^{\otimes}$, we have that*

$$\Delta(\phi^{\otimes}t)^k = \Big((\phi^{\otimes}t)^k\otimes 1+(\phi^{\otimes}t)^{k-1}\nabla f\otimes(\phi^{\otimes}t)+(\phi^{\otimes}t)^{k-2}\nabla f\otimes(\phi^{\otimes}t)^2 + \cdots + (\phi^{\otimes}t)\nabla f\otimes(\phi^{\otimes}t)^{k-1}+\nabla f\otimes(\phi^{\otimes}t)^k\Big)\Delta.$$

PROOF. The result easily follows by induction on k using fact that $\phi^{\otimes}$ is a skew-derivation (7) and that t is a derivation and the side conditions. Note that since $(\phi^{\otimes}t)$ is of total degree 0 no signs are introduced.

By essentially just rearranging the terms in Lemma 4.2, we obtain

COROLLARY 4.3. *Let* $\nabla^{\otimes^m}$ *be the restriction of* $\nabla^{\otimes}$ *to terms of tensor degree* m*, then*

$$\begin{aligned} t(\phi^{\otimes}t)^k\nabla^{\otimes^{k+2}} = t\Big(&\phi(t\phi^{\otimes})^{k-1}t\nabla^{\otimes^{k+1}} \otimes \nabla \\ &+ \phi(t\phi^{\otimes})^{k-2}t\nabla^{\otimes^{k}} \otimes \phi t\nabla^{\otimes^{2}} + \phi(t\phi^{\otimes})^{k-3}t\nabla^{\otimes^{k-1}} \otimes \phi(t\phi^{\otimes})t\nabla^{\otimes^{3}} \\ &+ \cdots + \phi t\nabla^{\otimes^{2}} \otimes \phi(t\phi^{\otimes})^{k-2}t\nabla^{\otimes^{k}} + \nabla \otimes \phi(t\phi^{\otimes})^{k-1}t\nabla^{\otimes^{k+1}}\Big). \end{aligned}$$

PROOF. Using the commutative diagram (8), we see that $p_2\Delta(\phi^{\otimes}t)^k = \lambda_{1,1}^{-1}(p_1 \otimes p_1)\Delta(\phi^{\otimes}t)^k$. But $(\phi^{\otimes}t)^k$ restricted to the image of $\nabla^{\otimes^{k+2}}$ lands in $T_2^c(A)$, so the result follows by rearranging the terms of Lemma 4.2 along with the fact that $\nabla f \nabla f = \nabla f$ and that t applied to a term of tensor degree 2 is just multiplication in A.

Using an easy inductive argument, we now have

PROPOSITION 4.4. *For all* $k \geq 0$,

$$\pi_A(t\phi^{\otimes})^k t\nabla^{\otimes^{k+2}} = \sum_{i+j=k+2} \hat{\pi}_i \cup \hat{\pi}_j.$$

and

COROLLARY 4.5. *For all* $k \geq 0$,

$$\pi_A(\phi^{\otimes}t)^{k+1}\nabla^{\otimes^{k+2}} = \hat{\pi}_{k+2}. \tag{19}$$

PROOF. The proof of 4.4 and 4.5 will proceed by induction on k. We obtain 4.4 in degree k using 4.3 and 4.5 in degree $< k$ to identify the terms of the right-hand side of 4.3. Using 4.4 in degree k we have that

$$\pi_A(\phi^{\otimes}t)^{k+1}\nabla^{\otimes^{k+1}} = \phi\pi_A\big((t\phi^{\otimes})^k t\nabla^{\otimes^{k+2}}\big) = \phi \sum_{i+j=k+2} \hat{\pi}_i \cup \hat{\pi}_j = \hat{\pi}_{k+2}$$

thus proving 4.5 in degree $k+1$.

In a similar manner, we have

COROLLARY 4.6. *For all $s \geq 1$,*

$$f\pi_A\big((t\phi^{\otimes})^s t\nabla^{\otimes^{s+1}}\big) = f\sum_{i+j=s+1}\hat{\pi}_i \cup \hat{\pi}_j. \tag{20}$$

By noting tensor degree, we immediately have

COROLLARY 4.7. *For all $s \geq 1$,*

$$\pi_A(\nabla_\infty^{\otimes}) \mid_{T^c_{s+1}(M)} = \hat{\pi}_{s+1}. \tag{21}$$

and

COROLLARY 4.8. *For all $s \geq 1$,*

$$\pi_A\big(\partial_\infty \mid_{T^c_{s+1}(M)}\big) = \sum_{i+j=s+1} f(\hat{\pi}_i \cup \hat{\pi}_j). \tag{22}$$

REMARK 4.9. Note that by the above results it is clear that ∂_∞ is the coderivation induced by the maps $\sum_{i+j=n} sf(\hat{\pi}_i \cup \hat{\pi}_j)$ so that we have a proof of this fact independent of [15] and [18].

4.2. The Obstruction Method Revisited

In [17], Gugenheim and Stasheff constructed a sequence of maps τ_n and ∂_n converging to a twisting cochain and coderivation respectively. We will show that their τ_n and ∂_n converges to the maps given in 3. We will begin by giving an alternative formulation of the obstruction method which the reader can easily see is equivalent with the one given in [17].

A sequence of maps $\tau_n : T^c(M) \longrightarrow A$ and $\partial_n : T^c(M) \longrightarrow T^c(M)$ for $n \geq 2$ is defined by the following inductive scheme:

$$\begin{aligned} \tau_2 &= \nabla\pi \\ \tau_{n+1} &= \tau_n - \phi(D_n(\tau_n) - \tau_n \cup \tau_n) \\ \partial_2 &= \omega(-d_M\pi) \\ \partial_{n+1} &= \partial_n - \omega(f(D_n(\tau_n) - \tau_n \cup \tau_n)), \end{aligned} \tag{23}$$

where $D_n(\gamma) = \gamma\partial_n + d\gamma$. The twisting cochain and derivation are then the limits of the sequences τ_n and ∂_n as $n \to \infty$.

REMARK 4.10. In [17] the expression for τ and ∂ differs from the above but the difference between the two is 0 in the limit. The convergence of the above sequence follows from the results in this section.

We will show that this construction reduces to the GM construction on the universal twisting cochain. We begin by proving some lemmas.

LEMMA 4.11. $\phi D_n(\tau_2) = 0$

PROOF. $\phi D_n(\tau_2) = \phi(d\nabla\pi - \nabla\pi\,\partial_n) = 0$ since ∇ is a chain map and since $\phi\nabla = 0$.

LEMMA 4.12. $\tau_{n+1} = \tau_2 + \phi(\tau_n \cup \tau_n)$ *for* $n \geq 2$.

PROOF. We will prove this by induction on n. For $n = 3$ we have

$$\begin{aligned}\tau_3 &= \tau_2 - \phi(D_2(\tau_2) - \tau_2 \cup \tau_2)\\ &= \tau_2 - \phi D_2(\tau_2) + \phi(\tau_2 \cup \tau_2)\\ &= \tau_2 + \phi(\tau_2 \cup \tau_2),\end{aligned}$$

which follows from lemma 4.12 together with the fact that ∇ is a chain map. For the inductive step (assuming the lemma is true up to and including n) we have

$$\begin{aligned}\tau_{n+1} &= \tau_n - \phi(D_n(\tau_n) - \tau_n \cup \tau_n)\\ &= \tau_n - (\tau_n - \tau_2) + \phi(\tau_n \cup \tau_n)\\ &= \tau_2 + \phi(\tau_n \cup \tau_n)\end{aligned}$$

We use the induction hypothesis when showing that $\phi D_n(\tau_n) = \tau_n - \tau_2$ by the following calculation:

$$\begin{aligned}\phi D_n(\tau_n) &= \phi d\tau_n - \phi\tau_n\partial_n\\ &= -d\phi(\tau_2 - \phi(\tau_{n-1} \cup \tau_{n-1})) + \tau_n\\ &\qquad - \nabla f(\tau_2 + \phi(\tau_{n-1} \cup \tau_{n-1})) - \phi\tau_n\partial_n\\ &= \tau_n - \tau_2\end{aligned}$$

Treating ∂_n in the same way we will start with the following lemma.

LEMMA 4.13. $f D_n \tau_2 = d_M\pi + \pi\,\partial_n$ *for* $n \geq 2$.

PROOF. The proof is simple enough,

$$f D_n\tau_2 = f d\nabla\pi + f\nabla\pi\,\partial_n = d\pi + \pi\,\partial_n$$

Here we have used that ∇ and f are chain maps and that $f\nabla = 1$.

LEMMA 4.14. *For* $n \geq 2$ *we have* $\partial_{n+1} = \partial_2 + \omega f(\tau_n \cup \tau_n)$

PROOF. As before we will proceed by induction on n. For $n = 2$ we have

$$\begin{aligned}\partial_3 &= \partial_2 - \omega(f(D_2(\tau_2)) - \tau_2 \cup \tau_2)\\ &= \partial_2 - \omega(d\pi) - \partial_2 + \omega(f(\tau_2 \cup \tau_2))\\ &= \partial_2 + \omega f(\tau_2 \cup \tau_2)\end{aligned}$$

For the inductive step (assuming that the lemma holds up to and including n) we have

$$\begin{aligned}\partial_{n+1} &= \partial_n - \omega(f(D(\tau_n))) + \omega(f(\tau_n \cup \tau_n))\\ &= \partial_n - \partial_2 - \partial_n + \omega(f(\tau_n \cup \tau_n))\\ &= \partial_2 + \omega(f(\tau_n \cup \tau_n))\end{aligned}$$

In this calculation we used lemma 4.13 and the fact that $\omega f D(\tau_2) = \partial_2 + \omega\pi\partial_n = \partial_2 + \partial_n$ where the last step uses the uniqueness of coextension as a coderivation.

The astute reader will have noticed the following:

PROPOSITION 4.15. *Let ∂_n and τ_n be as before, then $\tau_n \mid_{T^n} = \hat{\pi}_{n-1}$, and*

$$\tau_n \mid_{T^{\leq n}} = \sum_{i=2}^{n} \hat{\pi}_{i-1}$$

PROOF. The proof of this is a simple induction on $n \geq 3$. Note that for $n = 2$, wehave $\tau_2 = \nabla\pi = \hat{\pi}_1$. For $n = 3$ we have $\tau_3 = \hat{\pi}_1 + T^c(\phi)(\hat{\pi}_1 + \hat{\pi}_1) = \hat{\pi}_1 + \hat{\pi}_2$. Assuming the result is true up to and including n notice that in $\tau_n \cup \tau_n$ we have the homogeneous sum $\sum_{i+j=n} \hat{\pi}_i \cup \hat{\pi}_j$. The extra terms in $\tau_n \cup \tau_n$ are all maps taking more than n arguments (remember the shift in degree from the double induction τ to the GM construction $\hat{\pi}$).

This proposition concludes the proof of theorem 4.1.

5. An Application to de Rham Theory

Let X be a simplicial complex. Sullivan has defined a rational version of the de Rham complex [27] denoted by $\Lambda^*(X)$. On the other hand, we have the (normalized) simplicial cochains with rational coefficients $C^*(X; \mathbb{Q})$.

It is well known (Sullivan–de Rham's theorem) that the transformation

$$\Lambda^*(X) \xrightarrow{\kappa} C^*(X; \mathbb{Q}) \tag{24}$$

given by

$$\kappa(\alpha)(s) = \int_s \alpha \tag{25}$$

where $s \in C_k(X; \mathbb{Q})$, $\alpha \in \Lambda^k(X)$, yields an isomorphism in homology [4], [3], [23].

We will from now on assume that X is a finite complex which is connected as a space. This is strictly for convenience. Everything we say generalizes to an arbitrary connected semi-simplicial complex.

Suppose that X has vertices $\{v_0, \ldots, v_n\}$ and embed X into the standard simplex Δ_n with corresponding vertices. Let $\{t_0, \ldots, t_n\}$ be the barycentric coordinates of Δ_n. Penna [22] has proven the following:

PROPOSITION 5.1.

$$\Lambda^0(\Delta_n) = \mathbb{Q}[t_0, \ldots, t_n]/(t_0 + \ldots + t_n - 1)$$

and if

$$\Lambda^k(\Delta_n) \xrightarrow{r^k} \Lambda^k(X)$$

is the onto map given by restriction, we have

$$\ker(r^0) = (t_{i_1} \ldots t_{i_p} \mid (v_{i_1}, \ldots, v_{i_k}) \notin X)$$

(the ideal of all monomials such that the corresponding simplex is not a simplex of X). Furthermore,

$$\ker(r^k) = (t_{i_1} \ldots t_{i_p} dt_{j_1} \ldots dt_{j_q} \mid (v_{i_1}, \ldots, v_{i_p}, v_{j_1}, \ldots, v_{j_q}) \notin X).$$

REMARK 5.2. Note that the representation of $\Lambda^0(X)$ as $\Lambda^0(\Delta_{N-1})/\ker(r^0)$ is related to the Stanley-Reisner ring of the simplicial complex X *[25]*.

REMARK 5.3. As is well-known (e.g. [10, pp. 158]), if C and D are free chain complexes (as modules over the ground ring) then any onto map inducing an isomorphism in homology is the projection of an SDR.

Using this remark we have the following:

LEMMA 5.4. *Let X and $\Lambda^*(X) \xrightarrow{\kappa} C^*(X; \mathbb{Q})$ be as above. There is an SDR*

$$C^*(X; \mathbb{Q}) \underset{\kappa}{\overset{\nabla}{\rightleftarrows}} (\Lambda^*(X), \phi). \tag{26}$$

Explicit formulae can be given for this SDR, but we will only give the inclusion here. Full details and applications will appear elsewhere. We have

LEMMA 5.5. *Let* e_0 *be the zero vector in* R^n *and* $\{e_i\}$ *be the standard basis vectors so that* Δ_n *is the convex hull of* $\{e_0, e_1, \dots, e_n\}$. *Then an explicit formula for a map* ∇ *as above is determined as follows.*

$$\nabla((e_{i_0}\dots e_{i_k})^*) = k!\sum_{j=0}^{k}(-1)^j t_{i_j} dt_{i_0}\dots \hat{d}t_{i_j}\dots dt_{i_k} \tag{27}$$

where $(e_{i_0}\dots e_{i_k})^*$ *is the cochain dual to the simplex* $(e_{i_0}\dots e_{i_k}) \in X^{[k]}$, *i.e. the cochain which has value* 1 *on* $(e_{i_0}\dots e_{i_k})$ *and* 0 *on all other* k*-simplices of* X.

Note the similarity with the proof [23, pp. 148–151] where a partition of unity is used in place of our barycentric coordinate functions. The straightforward proof makes use of the relations given in Penna's theorem and will be omitted.

Now recall that a map $f : A \longrightarrow A'$ between algebras is said to be *strongly homotopy multiplicative* [16] if there exists a twisting cochain τ_f making the following diagram commute:

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ \bar{B}(A) \arrow[u, "\pi"] \arrow[r, "\tau_f"'] & A' \end{tikzcd} \tag{28}$$

where π is the universal twisting cochain.

We generalize this notion by saying that a map $f : A \longrightarrow A'$ between A_∞ algebras is *strongly homotopy* A_∞ if there is a *homotopy* twisting cochain $\tau_f : (T^c(A), \partial) \longrightarrow A'$ analogous to the situation in diagram (28).
Recall that a homotopy twisting cochain is a module map $\tau : (T^c(A), \partial) \longrightarrow A'$ such that the induced coalgebra map $c(\tau) : (T^c(A), \partial) \longrightarrow (T^c(A'), \partial)$ is a chain map. Using the above and our main theorem along with the full thrust of the tensor trick we have the following theorem.

THEOREM 5.6. *Let* X *be a connected simplicial complex.*

1. *The GM formula applied to* ∇ *from above defines a graded commutative* A_∞*-structure on the dual cochain complex* $C^*(X; \mathsf{Q})$.
2. *The map* κ *from de Rham's theorem is strongly homotopy* A_∞.
3. *The map* ∇ *from above is strongly homotopy* A_∞.

REMARK 5.7. This should be compared with the main result of [12] where κ is shown to be strongly homotopy multiplicative using acyclic models.

PROOF. Convergence of the obstruction method is given in [17, pp. 244], but we have proven that this is equivalent to the tensor trick and our main theorem gives the connection of this with the GM formula. Thus we have an SDR

$$(T^c(C^*(X;\mathbb{Q})), \widetilde{\partial}) \underset{\widetilde{\kappa}}{\overset{\widetilde{\nabla}}{\rightleftarrows}} (\bar{B}(\Lambda^*(X)), \widetilde{\phi}) \tag{29}$$

where $\widetilde{\nabla}$ and $\widetilde{\kappa}$ are the coalgebra maps given by the tensor trick.

The first part of the theorem follows from our main theorem. The second part follows by the fact that $\pi \tilde{B}\kappa$ is a twisting cochain extending κ and the last part follows similarly.

6. Relations with Massey products

In this section we will use the formula for an A_∞-structure (derived from an SDR) given in section 3 to study Massey products. Throughout this section we will assume that in (4), A an algebra and $M = H(A)$ the cohomology of A. When the ground ring is a field (see 5.3) such an sdr always exists. In this setting ∇ may be interpreted as a uniform choice of a representative in each homology class.

Consider the term $m_2 = \pi\partial_\infty|_{T^2} = f(\nabla\pi \cup \nabla\pi)$. This map is the ordinary cup product in cohomology. The next term of the A_∞-structure, $m_3 = \pi\partial_\infty|_{T^3}$ is given by

$$f(\nabla\pi \cup \phi(\nabla\pi \cup \nabla\pi) + \phi(\nabla\pi \cup \nabla\pi) \cup \nabla\pi). \tag{30}$$

This expression is reminiscent of $\langle a, b, c\rangle$, the classical Massey product. In fact we have the following

PROPOSITION 6.1. *The elements $\phi(\nabla\pi \cup \nabla\pi)[a, b]$ and $\phi(\nabla\pi \cup \nabla\pi)[b, c]$ constitute a defining system for $\langle a, b, c\rangle$, whenever this Massey product is defined.*

PROOF. Let $a, b, c \in H(A)$ be elements such that $ab = bc = 0$ so that $\langle a, b, c\rangle$ is defined. Since ϕ is a homotopy, $\nabla a \nabla b$ projects to the class of 0 and the product of two cycles is a cycle we have $d\phi(\nabla\bar{a}\nabla b) = \nabla\bar{a}\nabla b$. Since d_A is a derivation and since the multiplication in A is associative,the argument to f in (30) is a cycle.

This result can be generalized to higher Massey products (where as usual π denotes the universal twisting cochain $T^c(M) \longrightarrow M$). Let's begin by introducing the following notation:

$$\gamma_{i,j} = \sum_{r+s=j-i+1} \{\hat{\pi}_r \cup \hat{\pi}_s\}[a_i, \dots, a_j],$$

and set $\alpha_{i,j} = \phi\gamma_{i,j}$. Thus $\alpha_{i,j} = \hat{\pi}_{j-i+1}(a_i, a_{i+1}, \dots, a_j)$.

LEMMA 6.2. *$\gamma_{i,j}$ is a cycle.*

PROOF. evident since d is a derivation and since ∇ selects a cycle representative for each homology class.

THEOREM 6.3. *If $\nabla f(\gamma_{i,j}) = 0$ then the set $\{\alpha_{i,j} \mid n > j > i \geq 1\}$ is a defining system associated with $\langle a_1, a_2, \dots, a_n\rangle$.*

PROOF. Notice that it follows from 6.2 that if $\nabla(\gamma_{i,j}) = 0$ then $\gamma_{i,j}$ is a boundary. For the proof of the theorem apply the homotopy relation for ϕ to $\gamma_{i,j}$ to get the following calculation:

$$\begin{aligned} d\phi\gamma_{i,j} + \phi d\gamma_{i,j} &= \gamma_{i,j} - \nabla f \gamma_{i,j} \\ d\alpha_{i,j} &= \gamma_{i,j} - 0 - 0 \end{aligned}$$

where the last two zeros being $\nabla f \gamma_{i,j}$ and $\phi d\gamma_{i,j}$).

THEOREM 6.4. *If $\gamma_{i,j}$ is a boundary (i.e. $\nabla f \gamma_{i,j} = 0$) for $n > j > i \geq 1$ then the set $\{\alpha_{i,j} \mid n > j > i \geq 1\}$ constitutes a defining system for $\langle a_1, a_2, \dots, a_n\rangle$.*

PROOF. If one applies the chain homotopy relation for ϕ to $\gamma_{i,j}$ One gets $d\alpha_{i,j} = \gamma_{i,j} - \phi d\gamma_{i,j}$ but $d\gamma_{i,j} = 0$ since d is a derivation and by an induction over the number of elements in the defining system $(j - i)$. Thus we have the following relation.

$$d\alpha_{i,j} = \gamma_{i,j} \qquad \text{for} \quad n > j > i \geq 1. \tag{31}$$

Also note that $<a_1, a_2, \dots, a_n> = f\gamma_{1,n}$.

This relation between Massey products and the GM formula can be used to give an explicit reconstruction of the Golod resolution. When all Massey products vanish (in the sense of theorem 6.4) the A_∞-structure on M is the trivial one, but, of course, the twisting cochain $\hat{\pi}$ is non-trivial (indeed the induced coalgebra map gives an isomorphism in homology). An example of this was given in [14, 4.2.2] where M is the exterior algebra on one 1-dimensional element, $A = C^*(\mathsf{Z}; \mathsf{Z})$ is the functional cochain complex for the group of integers $\mathbb{Z}$ and $\hat{\pi}$ is the *binomial twisting cochain* whose n-th component in A^1 is given by the function

$$i \longmapsto (-1)^{n+1}\binom{i}{n}.$$

For the Golod resolution, consider the twisted tensor product $T^c H(A) \otimes_{\hat{\pi}} A$. When the Massey products vanish, then by the results in this section the A_∞-structure is trivial and it follows directly from the construction of the twisted differential that it fulfills conditions (i) and (ii) from [21, pp. 320]. That the complex is contractible follows from [14, lemma 4.1]. This twisted tensor product can therefore be identified with the Golod resolution. The key point is that once an SDR has been chosen no more choices are necessary for computing the Golod resolution.

The interested reader should see [26] for a correspondence between A_∞-structures and the differentials in Eilenberg-Moore type spectral sequences.

7. Co-A_∞ Structures

For the reader's convenience, we present the results dual to those in section 4 for (simply connected) coalgebras.

Given a simply connected coalgebra C and a module M, suppose that

$$M \underset{f}{\overset{\nabla}{\rightleftarrows}} (C, \phi). \tag{32}$$

is an SDR. Let $T^a(M)$ be the tensor algebra on the desuspension $s^{-1}(\bar{M})$ of the elements of positive degree in M. We write elements of $T^a(M)$ as $<c_1|\dots|c_n>$. Let $M \overset{\pi}{\hookrightarrow} T^a(M)$ be the map given by $\pi(m) = <s^{-1}m>$. The generalized GM-formula in this case is given by the inductive formula

$$\widehat{\pi} = (\widehat{\pi} \cup \widehat{\pi})\phi + \pi f$$

so that

$$\begin{tikzcd} C \arrow[d, shift right, "f"'] \arrow[dr, "\widehat{\pi}"] & \\ M \arrow[u, shift right, "\nabla"'] \arrow[r, hook, "\pi"] & T^a(M). \end{tikzcd}$$

with $\widehat{\pi}\nabla = \pi$.

Using the Heyneman-Sweedler (H-S) notation for coproducts,

$$\Delta c = c_{(1)} \otimes c_{(2)},$$

we have

$$\begin{aligned}
\widehat{\pi}_0(c) &= 0\\
\widehat{\pi}_1(c) &= <s^{-1} f(c)>\\
\widehat{\pi}_n(c) &= \sum_{k=1}^{n-1} <\widehat{\pi}_k(\overline{\phi(c)}_{(1)}) \otimes \widehat{\pi}_{n-k}(\phi(c)_{(2)})>.
\end{aligned}$$

We introduce the non-coassociative operation $\Delta\phi$ on C and write

$$c_{<1>} \otimes c_{<2>} = \phi(c)_{(1)} \otimes \phi(c)_{(2)}$$

to see that the first few terms of the generalized GM-formula are

$$\begin{aligned}
\widehat{\pi}_1(c) &= <s^{-1} f(c)>\\
\widehat{\pi}_2(c) &= <s^{-1} f(\overline{c}_{<1>})> \otimes <f(c_{<2>})>\\
\widehat{\pi}_3(c) &= <s^{-1} f(\overline{c}_{<1>})> \otimes <s^{-1} f(\overline{c}_{<2><1>})|c_{<2><2>}>\\
&\quad + <s^{-1} f(c_{<1><1>})|s^{-1} f(\overline{c}_{<1><2>})> \otimes <s^{-1} f(c_{<2>})>.
\end{aligned}$$

Dual to our observation in section (3.1), we see that these terms are made up from obstructions to the non-coassociativity of $\Delta\phi$. Once again, the point of the main theorem is that these higher obstructions are exactly reproduced in the tensor trick and the double induction. We can think of these obstructions as creating the associated co-A_∞ structure.

Now for a map $X \xrightarrow{f} T^a(X)$, let $T^a(X) \xrightarrow{\omega(f)} T^a(X)$ be the extension of f as a derivation. Let $t = \omega(\tilde{\Delta})$ where $\tilde{\Delta}$ is the composite

$$C \xrightarrow{\Delta} C \otimes C \hookrightarrow T^a(C).$$

For the proof of the dual theorem, observe that in the case of the tensor trick we want to show that

$$T^a(f)_\infty \pi = \widehat{\pi},$$

but

$$T^a(f)_\infty \pi = T^a(f)\pi + T^a(f)(t + (tT^a(\phi))t + \cdots)T^a(\phi)\pi$$

and clearly $T^a(f)_\infty \pi(c) = <f(c)>$ so that the main result again amounts to showing that

$$T^a(f)((tT^a(\phi))^n)\pi = \widehat{\pi}_n$$

for $n \geq 2$. The proof of this uses formulæ for D and Q analogous to those used in section 4.1.

The results dual to those in section 4.2 are also true by analogous proofs using the following expression for π and δ:

$$
\begin{aligned}
\pi_2 &= \pi f \\
\pi_{n+1} &= \tau_n - (D_n(\tau_n) - \tau_n \cup \tau_n)\phi \\
\partial_2 &= \omega(-\pi d_M) \\
\partial_{n+1} &= \partial_n - \omega((D_n(\tau_n) - \tau_n \cup \tau_n)\nabla).
\end{aligned} \tag{33}
$$

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DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
S-106 91 STOCKHOLM
SWEDEN
E-mail: leifj@matematik.su.se, lambe@matematik.su.se