MATH. SCAND. 89 (2001), 201–216

KOSZUL PROPERTY FOR POINTS IN PROJECTIVE SPACES

ALDO CONCA, NGÔ VIÊT TRUNG and GIUSEPPE VALLA

Abstract

A graded K-algebra R is said to be Koszul if the minimal R-free graded resolution of K is linear. In this paper we study the Koszul property of the homogeneous coordinate ring R of a set of s points in the complex projective space P^n. Kempf proved that R is Koszul if $s \le 2n$ and the points are in general linear position. If the coordinates of the points are algebraically independent over Q, then we prove that R is Koszul if and only if $s \le 1 + n + n^2/4$. If $s \le 2n$ and the points are in linear general position, then we show that there exists a system of coordinates $x_0, \dots, x_n$ of P^n such that all the ideals $(x_0, x_1, \dots, x_i)$ with $0 \le i \le n$ have a linear R-free resolution.

Introduction

Let X be a set of s (distinct) points of the projective space P^n over the field C of complex numbers and let R denote the coordinate ring of X. Kempf proved that if $s \le 2n$ and the points are in general linear position then the ring R is Koszul, see [11]. In Section 2 and 4 we extend Kempf result in two directions. We first prove that if $s \le 2n$ and the points are in general linear position then there exists a system of generators $x_0, \dots, x_n$ of the maximal homogeneous ideal of R such that all the ideals $(x_0, \dots, x_j)$ with $0 \le j \le n$ have linear R-free resolution. Secondly we prove that if one takes points with generic coordinates (i.e. algebraically independent over Q), then R is Koszul if and only if $s \le 1 + n + n^2/4$. Note that our bound is quadratic in n while Kempf's bound is linear. On the other hand, we do not have a geometric description of the Koszul locus; we do not even know whether it is Zariski open.

Our results have nice applications to the theory of non-commutative graded algebras. Namely, we can compute the Hilbert series of non-commutative algebras defined by the squares of $s \le 1 + n + n^2/4$ linear forms with generic coefficients in $n + 1$ indeterminates.

The results of Section 4 are based on results of Section 3 which is devoted to the study of Artinian algebras. Given a series $H(z)$ one may ask whether there exists a Koszul algebra with Hilbert series $H(z)$. If such an algebra exists, then we will say that $H(z)$ is *Koszul admissible*. A necessary condition for $H(z)$ to

Received November 13, 1998.

be Koszul admissible is the positivity of the coefficients of the series $1/H(-z)$, while a sufficient condition is the existence of an algebra with Hilbert series $H(z)$ and whose relations are quadratic monomials. We show that if $H(z)$ is a polynomial of degree 2, say $H(z) = 1+nz+mz^2$, then the Koszul admissibility of $H(z)$, the positivity of $1/H(-z)$, the existence of an algebra with quadratic monomial relations and Hilbert series $H(z)$ are equivalent conditions and they are also equivalent to the condition $m \leq n^2/4$. This result has some interesting consequences like, for instance, Turán's theorem for triangles.

We would like to thank Ralf Fröberg and Clas Löfwall for helpful discussions and suggestions concerning the results of this paper.

1. Notation and generalities

In this paper K will denote the field of complex numbers. This assumption is not essential for most of the results of the paper but we adopt it for the sake of simplicity.

A graded commutative Noetherian K-algebra $R = \bigoplus_{i\in\mathbb{N}} R_i$ is said to be standard graded (or homogeneous) if $R_0 = K$ and R is generated (as a K-algebra) by elements of degree 1. A standard graded K-algebra R is said to be Koszul if the field K has a linear R-free resolution as an R-module. Equivalently, R is Koszul if and only if $\operatorname{Tor}_i^R(K, K)_j = 0$ for all $i \neq j$. Denote by $H_R(z)$ and by $P_R(z)$ respectively the Hilbert series and the Poincaré-Betti series of R, i.e.

$$H_R(z) = \sum_{i\geq 0} \dim_K R_i z^i$$

and

$$P_R(z) = \sum_{i\geq 0} \dim_K \operatorname{Tor}_i^R(K, K) z^i.$$

It is known that if R is Koszul then the two series satisfy the following relation:

$$P_R(z)H_R(-z) = 1 \tag{1}$$

Equation (1) is indeed equivalent to the Koszulness of R, see for instance [3, p. 87]. We may present R as a quotient of a polynomial ring $S = K[x_1, \dots, x_n]$ by a homogeneous ideal I. If R is Koszul then I is generated by quadrics (and linear forms if the presentation is not minimal). Not all the algebras defined by quadrics are Koszul; for instance the algebra $K[x, y, z, t]/(x^2, y^2, z^2, t^2, xy+zt)$ is not Koszul. It is a theorem of Fröberg that R is Koszul if its defining ideal I is generated by monomials of degree 2 [5]. More generally, R is Koszul if I has a Gröbner basis of quadrics (see for instance [4, Thm. 2.2]). For an updated survey on Koszul algebras we refer the reader to the paper of Fröberg [8].

We will define now *Koszul filtrations*. This notion was inspired by the work of Herzog, Hibi and Restuccia ([10]) on strongly Koszul algebras. This class of algebras was introduced to study mainly semigroup rings. Since semigroup rings have a specified system of generators (the semigroup generators), strongly Koszul algebras were defined in terms of a given system of generators. For the applications we have in mind we need to be more flexible.

DEFINITION 1.1. Let R be a standard graded K-algebra. A family $\mathbf{F}$ of ideals of R is said to be a *Koszul filtration* of R if:

1) Every ideal $I \in \mathbf{F}$ is generated by linear forms,
2) The ideal 0 and the maximal homogeneous ideal $\mathcal{M}$ of R belong to $\mathbf{F}$.
3) For every $I \in \mathbf{F}$ different from 0 there exists $J \in \mathbf{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathbf{F}$.

One has:

PROPOSITION 1.2. *Let* $\mathbf{F}$ *be a Koszul filtration of* R. *Then* $\operatorname{Tor}_i^R(R/I, K)_j = 0$ *for all* $i \neq j$ *and for all* $I \in \mathbf{F}$. *In particular, the homogeneous maximal ideal* $\mathcal{M}$ *of* R *has a system of generators* $x_1, \dots, x_n$ *such that all the ideals* $(x_1, \dots, x_j)$ *with* $j = 1, \dots, n$ *have a linear* R*-free resolution and* R *is Koszul.*

PROOF. The second statement follows immediately from the first. To prove that $\operatorname{Tor}_i^R(R/I, K)_j = 0$ for all $i \neq j$ and for all $I \in \mathbf{F}$ we may argue by induction on i and on I (by inclusion). If $i = 0$ or if $I = 0$ then the assertion clearly holds. So assume $i > 0$ and $I \neq 0$. Then there exists $J \in \mathbf{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathbf{F}$. Set $H = J : I$. The short exact sequence

$$0 \to R/H[-1] \simeq I/J \to R/J \to R/I \to 0$$

yields the exact sequence

$$\operatorname{Tor}_i^R(R/J, K)_j \to \operatorname{Tor}_i^R(R/I, K)_j \to \operatorname{Tor}_{i-1}^R(R/H, K)_{j-1}$$

By induction the first and the third term of the sequence vanish for all $j \neq i$. Then the middle term vanishes for all $j \neq i$.

An important class of rings with a Koszul filtration are rings defined by quadratic monomial relations. If $R = K[x_1, \dots, x_n]/I$ where I is generated by monomials of degree 2 then it is easy to see that the family of all the ideals generated by subsets of $\{x_1, \dots, x_n\}$ is a Koszul filtration. This was observed already in [4, Sect. 2].

2. Koszul filtration for points

The goal of this section is to show that the coordinate ring R of a set X of s points of $\mathbb{P}^n$ in general linear position has a Koszul filtration provided $s \leq 2n$. In particular we have:

THEOREM 2.1. *Let X be a set of points in $\mathbb{P}^n$ and let R be the coordinate ring of X. Assume that $|X| \leq 2n$ and that the points of X are in general linear position. Then there exists a system of generators $x_0, x_1, \ldots, x_n$ of the maximal ideal $\mathscr{M}$ of R such that the ideals $(x_0, x_1, \ldots, x_j)$ have a linear R-free resolution for all $j = 0, 1, \ldots, n$.*

We need some preliminary results. To this end we introduce a piece of notation. Let S be the coordinate ring of $\mathbb{P}^n$. Set $s = |X|$, and denote by $P_1, \ldots, P_s \in \mathbb{P}^n$ the points in X. Further denote by $\wp_1, \ldots, \wp_s$ the corresponding prime ideals of S. The defining ideal of X is the ideal $I = \cap_{i=1}^s \wp_i$ and its coordinate ring is $R = S/I$. In this section we will always assume that:

i) The points of X are in general linear position: if $s \leq n$ then it means that the points span a $\mathbb{P}^{s-1}$, while if $s \geq n+1$ then it means that no subset of $n+1$ points of X is contained in a hyperplane of $\mathbb{P}^n$.
ii) $n+1 \leq s \leq 2n$.

The assumption $n+1 \leq s$ is not restrictive: if $s \leq n$, then we may assume that the points are indeed coordinate points so that their defining ideal is generated by quadratic monomials and some variables. It is well-known that a set of s points in general linear position in $\mathbb{P}^n$ have maximal Hilbert function provided $s \leq 2n+1$; this means that for all $i \in \mathbb{N}$

$$\dim_K R_i = \min\left\{\binom{n+i}{n}, s\right\}.$$

It is clear that we can find an hyperplane $L = 0$ passing through the points $P_1, \ldots, P_n$ and avoiding the points $P_{n+1}, \ldots, P_s$ and another hyperplane $M = 0$ passing through the points $P_{s-n+1}, \ldots, P_s$ and avoiding the points $P_1, \ldots, P_{s-n}$.

LEMMA 2.2. *With the above notation we have*

$$I + (L) = \cap_{i=1}^n \wp_i.$$

PROOF. The inclusion $I + (L) \subseteq \cap_{i=1}^n \wp_i$ is obvious. So it is enough to prove that the two ideals have the same Hilbert series. The Hilbert series of $S/\cap_{i=1}^n \wp_i$ is

$$\frac{1+(n-1)z}{1-z}.$$

One has the short exact sequence

$$0 \to (S/I : L)[-1] \to R \to S/I + (L) \to 0$$

Note that

$$I : L = (\cap_{i=1}^{s} \wp_i) : L = \cap_{i=1}^{s} (\wp_i : L) = \cap_{i=n+1}^{s} \wp_i .$$

It follows that the Hilbert series of $S/I : L$ is

$$\frac{1 + (s - n - 1)z}{1 - z}.$$

Hence the Hilbert series of $S/I + (L)$ is

$$\frac{1 + nz + (s - n - 1)z^2}{1 - z} - \frac{z(1 + (s - n - 1)z)}{1 - z} = \frac{1 + (n - 1)z}{1 - z}.$$

Let $T \in S_1$ be a linear form which is a non-zerodivisor on $R = S/I$. We denote by x, y and z the residue classes of T, L and M respectively in S/I. Since $LM \in I$, we have $yz = 0$ in R.

LEMMA 2.3. *With the above notation the Hilbert series of* $R/(x, y)$ *is* $1 + (n - 1)z$.

PROOF. By assumption $T \notin \cup_{i=1}^{s} \wp_i$ and hence T is also a non-zerodivisor modulo $I + (L) = \cap_{i=1}^{n} \wp_i$. Since the points of X are in general linear position it follows that the Hilbert series of $R/(x, y)$ is $1 + (n - 1)z$.

As a corollary we have:

COROLLARY 2.4. *Every homogeneous ideal of* R *containing* x *and* y *is generated by linear forms.*

PROOF. Let J be an ideal of R containing x and y. By virtue of Lemma 2.3 the ideal (x, y) contains R_2, hence the minimal generators of J have degree 1.

Now we are in the position to define a family $\mathbf{F}$ of ideals of linear forms of R and to show that it is a Koszul filtration for R. We will denote by $\mathscr{M}$ the homogeneous maximal ideal of R.

Since by construction $yz = 0$, we have $(x, y) \subseteq (x) : (z)$. By virtue of Corollary 2.4, the ideal $(x) : (z)$ is generated by linear forms. Hence there exist linear forms $y_2, \dots, y_k$ such that

$$(x) : (z) = (x, y, y_2, \dots, y_k).$$

Exchanging the role between y and z, there exist linear forms $z_2, \dots, z_h$ such that

$$(x) : (y) = (x, z, z_2, \dots, z_h).$$

Indeed $h = k = s - 2n + 1$ but we do not need this. We may now extend $x, z, z_2, \dots, z_h$ to a system of generators of $\mathcal{M}$ by adding linear forms, say $z_{k+1}, \dots, z_n$. Then define

$$\mathbf{F} = \left\{ \begin{matrix} 0, (x), \\ (x, y), (x, y, y_2), \dots, (x, y, y_2, \dots, y_h), \\ (x, z), (x, z, z_2), \dots, \dots, (x, z, z_2, \dots, z_{n-1}), \\ \mathcal{M} = (x, z, z_2, \dots, z_n) \end{matrix} \right\}$$

PROPOSITION 2.5. *The family* $\mathbf{F}$ *is a Koszul filtration for* R.

PROOF. Conditions 1) and 2) of Definition 1.1 are clearly satisfied. One notes that:

$*)$ $\quad 0 : (x) = 0,$

1.1)	$(x) : (y) = (x, z, z_2, \dots, z_h)$,	2.1)	$(x) : (z) = (x, y, y_2, \dots, y_k)$,
1.2)	$(x, y) : (y_2) = \mathcal{M}$,	2.2)	$(x, z) : (z_2) = \mathcal{M}$,
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.h)	$(x, y, y_2, \dots, y_{h-1}) : (y_k) = \mathcal{M}$,	2.h)	$(x, z, z_2, \dots, z_{h-1}) : (z_h) = \mathcal{M}$,
		$\vdots$	$\vdots$
		2.n)	$(x, z, z_2, \dots, z_{n-1}) : (z_n) = \mathcal{M}$

where $*)$ holds because x is a non-zerodivisor, 1.1) and 2.1) hold by construction, 1.2), 1.3), $\dots$, 1.h) hold because $R_2 \subset (x, y)$, 2.2), 2.3),. . . ,2.$n - 1$), 2.n) hold because $R_2 \subseteq (x, z)$. This shows that condition 3) of Definition 1.1 is satisfied.

Now Theorem 2.1 is a consequence of Proposition 2.5 and Proposition 1.2.

3. Short Hilbert series compatible with the Koszul property

In this section we consider (a special case of) the following problem: Given a series $H(z)$ does there exist a Koszul algebra R whose Hilbert series is $H(z)$? If yes, then we will say that $H(z)$ is *Koszul admissible*.

It follows from Equation (1) of Section 1 that a necessary condition for $H(z)$ to be Koszul admissible is that the series $1/H(-z)$ has positive coefficients. For Artinian algebras of socle degree 2 this condition is also sufficient.

THEOREM 3.1. *Let n, m be non-negative integers and let $H(z) = 1 + nz + mz^2$. Set $A = K[x_1, \dots, x_n]$. The following conditions are equivalent:*

1) *There exists an ideal $I \subset A$ generated by monomials of degree 2 such that the Hilbert series of A/I is $H(z)$.*
2) *$H(z)$ is Koszul admissible.*
3) *The coefficients of the series $1/H(-z)$ are positive.*
4) *$m \le n^2/4$.*

PROOF. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) hold by virtue of Fröberg's theorem [5] and equation (1) in Section 1 (no matter what $H(z)$ is).

To show that 3) implies 4) we consider the roots, say α_1 and α_2 of the polynomial $H(-z)$. Then $H(-z) = (1 - z/\alpha_1)(1 - z/\alpha_2)$ and hence

$$\frac{1}{H(-z)} = \frac{1}{(1 - z/\alpha_1)} \frac{1}{(1 - z/\alpha_2)} = \sum_{i \ge 0} \frac{z^i}{\alpha_1^i} \sum_{j \ge 0} \frac{z^j}{\alpha_2^j}$$

It follows that the k-th coefficient, say β_k, of $1/H(-z)$ is given by

$$\beta_k = \sum_{i=0}^{k} \frac{1}{\alpha_1^i \alpha_2^{k-i}} = \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1^k \alpha_2^k (\alpha_1 - \alpha_2)}$$

By contradiction, assume that $m > n^2/4$. Then α_1 and $\alpha_2 = \bar{\alpha}_1$ are non-real complex numbers. Then

$$\beta_k = \frac{\operatorname{Im}(\alpha_1^{k+1})}{N(\alpha_1)^k \operatorname{Im}(\alpha_1)}$$

where $\operatorname{Im}(w)$ denotes the imaginary part of a complex number w and $N(w) = w\bar{w}$ denotes its norm. It is easy to see that $\operatorname{Im}(\alpha_1^{k+1})$ changes sign as k vary, and hence β_k cannot be positive for all k.

It remains to show that 4) implies 1). To this end assume that $m \le n^2/4$ and let us first consider the case when n is even, say $n = 2k$. Take J to be the ideal

$$J = (x_1, \dots, x_k)^2 + (x_{k+1}, \dots, x_{2k})^2.$$

Then J is generated by $2\binom{k+1}{2} = k(k+1)$ monomials and $J_3 = A_3$.

Hence the Hilbert series of A/J is $1 + nz + (n^2/4)z^2$. Now let H be the ideal generated by any set of $n^2/4 - m$ distinct monomials of degree 2 not in J, and set $I = J + H$. By construction I is generated by monomials of degree 2 and the Hilbert series of A/I is $1 + nz + mz^2$.

If n is odd, say $n = 2k + 1$, then let J be the ideal

$$J = (x_1, \dots, x_k)^2 + (x_{k+1}, \dots, x_{2k+1})^2.$$

The conclusion follows by using the same arguments as before.

REMARK 3.2. The relationship between conditions 2), 3) and 4) of the theorem has been observed also by Anick, see [2, Lemma 5.10]

REMARK 3.3. A (surprising) corollary of the above theorem is Turán's theorem for triangles (see for instance [12, Sect. 4]). Turán's theorem for triangles says that the maximum number of edges that a graph with n vertices and without triangles can have is $[n^2/4]$. That the number is at least $[n^2/4]$ follows immediately from the fact that the complete bipartite graphs $K_{a,a}$ and $K_{a,a+1}$ do not contain triangles.

Let now G be a graph with n vertices $\{1, 2, \dots, n\}$, m edges and no triangles. Let I be the ideal of $S = k[x_1, \dots, x_n]$ generated by $x_1^2, \dots, x_n^2$ and $x_i x_j$ where $i \neq j$ and (i, j) is not an edge of G.

Then I is minimally generated by $n + \binom{n}{2} - m = \binom{n+1}{2} - m$ monomials. Since G does not contain triangles, $I_3 = S_3$ so that the Hilbert series of S/I is $1 + nz + mz^2$.

By virtue of Theorem 3.1 we have $m \leq n^2/4$ and this proves the theorem.

The connection between Turán's theorem and the positivity of a series related to $1/H(-z)$ was observed also by Stanley [15, 2.5, 2.6], see also [16, p. 481].

Let now $I = (f_1, \dots, f_h)$ be an ideal generated by quadratic forms of the polynomial ring $A = K[x_1, \dots, x_n]$, say $f_k = \sum a_{ij}^{(k)} x_i x_j$. We set $R = A/I$.

We will say that the forms $f_1, \dots, f_h$ have *generic coefficients* if the coefficients $a_{ij}^{(k)}$ of the quadrics are algebraically independent over $\mathbb{Q}$. Further we may consider the sequence of forms $(f_1, \dots, f_h)$ as a point in the affine space $\mathscr{P} = \mathbf{A}_K^m$ where $m = h\binom{n+1}{2}$.

Hence for every $T = (f_1, \dots, f_h) \in \mathscr{P}$ there is a corresponding algebra $R = K[x_1, \dots, x_n]/I$ where I is the homogeneous ideal generated by $f_1, \dots, f_h$.

As a corollary of the above theorem we have part 1) and 2) of the following:

PROPOSITION 3.4. 1) *Let $c \in \mathbb{N}$ be an integer, $c \geq 2$. If $h \geq \binom{n+1}{2} - n^2/4$ then there exists a non-empty Zariski open subset $\mathscr{U}_c$ of $\mathscr{P}$ such that for every point in $\mathscr{U}_c$ the corresponding algebra R has the following property: $Tor_i^R(K, K)_j = 0$ for all $i \leq c$ and all $j \neq i$.*

2) *Let $I = (f_1, \ldots, f_h)$ be an ideal generated by quadrics in $A = K[x_1, \ldots, x_n]$ and set $R = A/I$. If $h \geq \binom{n+1}{2} - n^2/4$ and the quadrics $f_1, \ldots, f_h$ have generic coefficients then R is Koszul.*

3) *If $h \geq \binom{n}{2} + 1$, then there exists a non-empty Zariski open subset $\mathscr{U}$ of $\mathscr{P}$ such that for every point in $\mathscr{U}$ the corresponding algebra R has a Koszul filtration.*

PROOF. 1) Set $m = \binom{n+1}{2} - h$. By assumption $m \leq n^2/4$. For a point $I = (f_1, \ldots, f_h) \in \mathscr{P}$ note that the Hilbert series of $R = A/I$ is $1 + nz + mz^2$ if and only if $\dim(I_2) = h$ and $I_3 = A_3$. These conditions can be expressed by maximal rank conditions on matrices whose entries are (linear) forms in the coefficients of the f_i's. Since $1 + nz + mz^2$ actually occurs as Hilbert series for some point in $\mathscr{P}$, namely the one corresponding to the h quadratic forms (indeed monomials) which generate the ideal I of Theorem 3.1, we may conclude that there is a non-empty Zariski open subset of $\mathscr{P}$ on which the Hilbert series for the corresponding algebras is $1 + nz + mz^2$.

Assume now that we are given a point $I = (f_1, \ldots, f_h) \in \mathscr{P}$ such that the corresponding algebra R has Hilbert series $1 + nz + mz^2$. It is clear that K has linear 1-syzygies over R. We are going to prove that if K has linear $(c-1)$-syzygies over R, then the condition of having also linear c-syzygies can be translated into a maximal rank condition. From this and from the knowledge of an example (again the one of Theorem 3.1) of a Koszul algebra with Hilbert series $1 + nz + mz^2$, one deduces 1) by induction on c.

Let $\Omega_c(K)$ be the c-th syzygy module of K. Denote by $\mathscr{M}$ the maximal ideal of R and by β_i the i-th coefficient of the series $1/(1 - nz + mz^2)$. In other words, $\beta_i = n\beta_{i-1} - m\beta_{i-2}$ with $\beta_0 = 1$ and $\beta_1 = n$.

By assumption we have an exact sequence of R-modules

$$\begin{aligned} 0 \to \Omega^c(K) \to R(-c+1)^{\beta_{c-1}} \to R(-c+2)^{\beta_{c-2}} \\ \to \ldots \to R(-1)^{\beta_1} \to R \to K \to 0 \end{aligned}$$

The Hilbert function of the syzygy module $\Omega^c(K)$ can be read off from the exact sequence. One has $\dim \Omega^c(K)_c = n\beta_{c-1} - m\beta_{c-2} = \beta_c$ while $\dim \Omega^c(K)_{c+1} = m\beta_{c-1}$. Now K has linear c-syzygies if and only if $\Omega^c(K)$ is generated (as an R-module) by $\Omega^c(K)_c$, that is $R_1\Omega^c(K)_c = \Omega^c(K)_{c+1}$. The last condition can be expressed by saying that the rank of the $m\beta_{c-1} \times n\beta_c$ matrix describing $R_1\Omega^c(K)_c$ is $m\beta_{c-1}$. Note that $n\beta_c > m\beta_{c-1}$ because by virtue of Theorem 3.1 the number $n\beta_c - m\beta_{c-1} = \beta_{c+1}$ is strictly positive.

One notes that the entries of the matrices in the resolution can be chosen to be polynomials whose coefficients are polynomial functions of the coefficients of the f_i's, and hence the above maximal rank condition can be expressed in terms of the coefficients of the f_i's.

2) It follows immediately from 1).

3) Set $m = \binom{n+1}{2} - h$. By assumption $m \leq n-1$. As in 1) one proves that there is a non-empty Zariski open subset of $\mathscr{P}$ on which the Hilbert series for the corresponding algebra is $1 + nz + mz^2$. Further it is clear that on another non-empty Zariski open subset of $\mathscr{P}$ the corresponding algebra R has the property $R_2 \subset (x_1)$.

So assume that $R_2 \subset (x_1)$. Since $n > m$ the multiplication by x_1 from R_1 to R_2 is not injective. Hence there exists a linear form say $y_1 \in R_1$ such that $x_1 y_1 = 0$. The form $y_1 = a_1 x_1 + \ldots + a_n x_n$ can be chosen such that its coefficients a_i are rational functions in the coefficients of the f_i's. The condition $R_2 \subset (y_1)$ is also open and we will show later that it is not empty (for one of the linear forms in $0 : (x_1)$). The ideal $0 : (x_1)$ contains y_1 and, since $R_2 \subset (y_1)$, it is generated by linear forms, say $0 : (x_1) = (y_1, y_2, \ldots, y_{n-m})$ with $y_i \in R_1$. We may complete $y_1, \ldots, y_{n-m}$ to a basis of R_1 by adding some linear forms, say, $y_{n-m+1}, \ldots, y_n$. By the same reason there exists a basis $x_1, z_2, \ldots, z_n$ of R_1 such that $0 : y_1 = (x_1, z_2, \ldots, z_{n-m})$. Then one proves as in Proposition 2.5 that the family

$$\mathbf{F} = \left\{ \begin{array}{l} 0, \\ (y_1), (y_1, y_2), \ldots, \ldots, (y_1, y_2 \ldots, y_{n-1}), \\ (x_1), (x_1, z_2), \ldots, (x_1, z_2, \ldots, z_{n-1}), \mathscr{M} = (y_1, y_2 \ldots, y_n) \end{array} \right\}$$

is a Koszul filtration of R.

It remains to prove that this open set is not empty. To this end it suffices to exhibit a ring R with Hilbert series $1 + nz + mz^2$ and two linear forms x_1, y_1 in R with $R_2 \subset (x_1)$, $R_2 \subset (y_1)$ and $x_1 y_1 = 0$.

One can consider the ring $R = K[x_1, \ldots, x_n]/I$ where I is an ideal minimally generated by h quadratic forms and containing the following $\binom{n}{2} + 1$ quadrics:

$$\begin{array}{lll} x_1 x_j - x_{j+1} x_n & \text{for} & 1 \leq j \leq n-1 \\ x_i x_j & \text{for} & 2 \leq i \leq j \leq n-1 \\ x_1 x_n & & \end{array}$$

It is easy to see that the Hilbert series of R is $1 + nz + mz^2$ and that in R one has $x_1 x_n = 0$, $(x_1) \supset R_2$, and $(x_n) \supset R_2$.

REMARK 3.5. Löfwall [13] has shown that if I is the ideal generated by h sufficiently "generic" quadrics of $A = K[x_1, \ldots, x_n]$, then A/I is Koszul if and only if either $h \le n$ (the complete intersection case) or $h \ge \binom{n+1}{2} - n^2/4$.

The above Proposition gives the “if" part of this result. However our method does not apply for proving the converse. For instance if one takes seven quadrics with generic coefficients in $K[x_1, \ldots, x_6]$, then the socle degree of the quotient algebra is greater than 2, so that one cannot use the criterion given in Theorem 3.1.

REMARK 3.6. With the notation of Proposition 3.4, we have proved that for $h \ge \binom{n+1}{2} - n^2/4$ there is an intersection of countable many non-empty Zariski open subsets of $\mathscr{P}$ on which the corresponding algebra is Koszul. One can ask whether this intersection is a Zariski open subset of $\mathscr{P}$. We do not know the answer to this question but we note that the matter is quite subtle because Roos [14, Thm. 1′] has shown that if R is a graded K-algebra and K has linear c-syzygies then K need not to have linear $(c+1)$-syzygies, no matter what c is. Indeed, for every integer $c \ge 2$ he presented an Artinian algebra R_c with Hilbert series $1 + 6z + 8z^2$ such that $\operatorname{Tor}_i^{R_c}(K, K)_j = 0$ for all $i \ne j$ and $i \le c$ while $\operatorname{Tor}_{c+1}^{R_c}(K, K)_{c+2} \ne 0$.

4. Koszul property for generic points

In this section we deal with the Koszul property of ideals of points in the projective space P^n over the field K. As in the second section we denote by X a set of distinct points in P^n, by s the cardinality of X and by R the coordinate ring of X.

Let $P_1 = (a_{10}, \ldots, a_{1n})$, $P_2 = (a_{20}, \ldots, a_{2n})$, $\ldots$, $P_s = (a_{s0}, \ldots, a_{sn})$ be the points of X. We say that the points of X have *generic coordinates* if the numbers $\{a_{ij}\}$ are algebraically independent over Q. Further we may consider the sequence $(P_1, \ldots, P_s)$ as a point in

$$\mathscr{Q} = \underbrace{\mathsf{P}^n \times \mathsf{P}^n \times \ldots \times \mathsf{P}^n}_{s \text{ times}}.$$

Hence for every point $T = (P_1, \ldots, P_s) \in \mathscr{Q}$ there is a corresponding algebra $R = K[x_0, \ldots, x_n]/I$ where I is the homogeneous ideal defining the set of points $\{P_1, \ldots, P_s\}$ in P^n.

We have:

THEOREM 4.1. *The following condition are equivalent:*

1) *There exists a set X of s points in P^n with maximal Hilbert function such that R is Koszul,*

2) $s \le 1 + n + n^2/4$

PROOF. 1) $\Rightarrow$ 2): Let X be a set of s distinct points of P^n with maximal Hilbert function such that R is Koszul. We may assume that $s \geq n+1$. Since the ideal of X must be generated by quadrics, the Hilbert function $H(n)$ of R is $H(0) = 1$, $H(1) = n+1$, and $H(i) = s$ for $i \geq 2$. Then take a linear form x which is a non-zerodivisor on R and set $B = R/(x)$. The Hilbert series of B is $1 + nz + (s-n-1)z^2$ and, by [3, Thm. 4], B is Koszul. It follows from Theorem 3.1 that $(s-n-1) \leq n^2/4$ and hence $s \leq 1 + n + n^2/4$.

2) $\Rightarrow$ 1): Assume $s \leq 1 + n + n^2/4$. If $s \leq n+1$, then any set X of s coordinate points has a Koszul coordinate ring. So assume that $s \geq n+1$. Set $m = s - n - 1$. Since $m \leq n^2/4$ by Theorem 3.1 there exists an ideal J in $K[x_1, \dots, x_n]$ generated by monomials of degree 2 with Hilbert series $1 + nz + mz^2$. Any monomial ideal can be lifted to a radical ideal by means of the Hartshorne deformation, see for instance [9, Sect. 2]. Hence, by lifting J, we obtain an ideal of points I of $R = K[x_0, \dots, x_n]$ such that x_0 is a non-zerodivisor modulo I and $I + (x_0) = J + (x_0)$. It follows that R/I is Koszul and has Hilbert series $(1 + nz + mz^2)/(1-z)$ as desired.

The construction of the set of points with Koszul coordinate ring can be carried out explicitly. For instance, suppose we want to construct 9 points in P^4 with maximal Hilbert function and with Koszul coordinate ring. The Hilbert series of 9 points in P^4 with maximal Hilbert function is $1 + 4z + 4z^2/(1-z)$. Hence we start with an ideal J in $K[x_1, x_2, x_3, x_4]$ generated by monomials of degree 2 and with Hilbert series $1 + 4z + 4z^2$. For instance one can take $J = (x_1, x_2)^2 + (x_3, x_4)^2$. Indeed, up to permutation this is the only possibility (and it corresponds to the graph $K_{2,2}$ in Turán's theorem). Then we lift J following Hartshorne's method, and we get $I = (x_1(x_1 - x_0), x_1x_2, x_2(x_2 - x_0), x_3(x_3 - x_0), x_3x_4, x_4(x_4 - x_0))$. The corresponding points are

$$\begin{array}{lll} P_1 = (1,0,0,0,0), & P_2 = (1,1,0,0,0), & P_3 = (1,0,1,0,0), \\ P_4 = (1,0,0,1,0), & P_5 = (1,0,0,0,1), & P_6 = (1,1,0,1,0), \\ P_7 = (1,1,0,0,1), & P_8 = (1,0,1,1,0), & P_9 = (1,0,1,0,1) \end{array}$$

and this is a set of 9 points in P^4 with maximal Hilbert function and with Koszul coordinate ring.

THEOREM 4.2. *Let X be a set of s points in P^n with generic coordinates. Then R is Koszul if and only if $s \leq 1 + n + n^2/4$.*

PROOF. Assume that R is Koszul. Since points with generic coordinates have maximal Hilbert function, by virtue of Theorem 4.1, we have that $s \leq 1 + n + n^2/4$.

Assume now that $s \leq 1 + n + n^2/4$. Then by Theorem 4.1, there exists a set of s points with maximal Hilbert function and with Koszul coordinate ring. It suffices now to show that the condition of having linear i-th syzygies for K can be translated into the non-vanishing of certain polynomials in the coordinates of the points. One first notes that the generators of the ideal of X can be chosen so that their coefficients are polynomial functions of the coordinates of the points. Then one can argue as in the proof of Proposition 3.4.

The same argument can be used to show the following:

THEOREM 4.3. *Let $s \leq 1 + n + n^2/4$ and let c be a positive number. Then there is a non-empty Zariski open subset $\mathcal{U}_c$ of $\mathcal{Q}$ such that for every point in $\mathcal{U}_c$ the corresponding algebra R has the following property:* $\operatorname{Tor}_i^R(K, K)_j = 0$ *for all $i \leq c$ and all $j \neq i$.*

REMARK 4.4. We do not know whether a stronger version of the above theorem holds. For example we do not know whether for $s \leq 1 + n + n^2/4$ points in P^n the Koszul locus is open.

The above results has a nice application on Hilbert series of non-commutative graded algebras. The Hilbert series of non-commutative graded algebras is itself a fascinating topic, and few concrete examples are known where the Hilbert series can be computed explicitly (see e.g. [1], [2], [7]).

Let R be a *quadratic algebra*, that is $R = K\langle X\rangle/I$, where

$$K\langle X\rangle = K\langle X_0, \dots, X_n\rangle$$

denotes the free associative non-commutative algebra generated over K by the variables $X_0, \dots, X_n$ and I is a two-sided ideal generated by a subspace W of $K\langle X\rangle_2$. Then one can associate with R an algebra R^*, called the *dual algebra* of R as follows. We consider in the vector space $K\langle X\rangle_2$ the scalar product induced by the assignment

$$\langle X_iX_j, X_rX_s\rangle = \begin{cases} 1 & \text{if } (i, j) = (r, s), \\ 0 & \text{otherwise.} \end{cases}$$

Then the algebra R^* is the algebra $R^* = K\langle X\rangle/J$ where J is the two-sided ideal generated by the orthogonal space $W^{\perp}$ of W.

It is known that R is a Koszul algebra if and only the Hilbert series of R^* is equal to the Poincare series of R (see [8], Theorem 1).

If $X = \{P_1, \dots, P_s\}$ is a set of $s < 1 + n + n^2/4$ points with generic coordinates in P^n, its coordinate ring R is a quadratic algebra defined by the commutators $X_iX_j - X_jX_i$ and $\binom{n+2}{2} - s$ quadratic forms $F_t = \sum_{0\leq i,j\leq n} \alpha_{ij}^{(t)} X_iX_j$, $t = 1, \dots, \binom{n+2}{2} - s$. If $P_h = (a_{h0}, a_{h1}, \dots, a_{hn})$, $h = 1, \dots, s$, the coefficients $\alpha_{ij}^{(t)}$ must satisfy the equations

$$\sum_{0\le i,j\le n} \alpha_{ij}^{(t)} a_{hi}a_{hj} = 0,\ h = 1,\dots,s.$$

From this it follows that the dual algebra R^* is defined by the relations

$$G_h = \sum_{0\le i,j\le n} a_{hi}a_{hj}X_iX_j = \Big(\sum_{i=0}^{n} a_{hi}X_i\Big)^2,\ h = 1,\dots,s.$$

Thus, R^* is the non-commutative algebra defined by the squares of s generic linear relations. Now, applying Theorem 4.2 we obtain:

COROLLARY 4.5. *Let $n+1 \le s \le 1+n+n^2/4$ and let T be the non-commutative graded algebra defined by the squares of s linear forms in $K\langle X\rangle$ with generic coefficients. Put $m = s-n-1$. Then*

$$H_T(z) = (1+z)/(1-nz+mz^2).$$

PROOF. Let $L_1,\dots,L_s$ be the given linear forms and $P_1,\dots,P_s$ the corresponding points in P^n. If R is the coordinate ring of this set of points, we have

$$H_R(z) = (1+nz+mz^2)/(1-z).$$

Since R is a Koszul algebra, the Poincare series of R is determined by the formula:

$$P_R(z) = H_R(-z)^{-1} = (1+z)/(1-nz+mz^2).$$

Since $H_T(z) = H_{R^*}(z) = P_R(z)$, we obtain the conclusion.

It should be mentioned that the Hilbert series of the non-commutative graded algebra defined by the squares of s linear forms with generic coefficients is not the same as the one of the non-commutative graded algebras defined by s quadratic forms with generic coefficients, whereas they may be the same in the commutative case [6]. In fact, using the same technique we can prove the following result:

COROLLARY 4.6. *Let $s \le (n+1)^2/4$ and T be the non-commutative graded algebra defined by s quadratic forms in $K\langle X\rangle$ with generic coefficients. Then*

$$H_T(z) = 1/\big(1-(n+1)z+sz^2\big).$$

PROOF. By [2, Theorem 2.6, Lemma 1.2], we have

$$H_T(z) \ge 1/\big(1-(n+1)z+sz^2\big).$$

Equality holds above if we can find a non-commutative graded algebra defined by s quadratic forms in $n+1$ variables with the right side as Hilbert series. Such an algebra can be chosen to be the dual algebra of a commutative algebra defined by $r = \binom{n+2}{2} - s$ quadratic relations. Indeed, by Theorem 3.1 there is a commutative Koszul algebra R defined by r quadratic monomials with

$$H_R(z) = 1 + (n+1)z + sz^2.$$

This implies

$$H_{R^*}(z) = P_R(z) = H_R(-z)^{-1} = 1/\big(1 - (n+1)z + sz^2\big).$$

Corollary 4.6 follows also from Lemma 5.6 and Lemma 5.10 in Anick's paper [2].

REFERENCES

1. Anick, D., *Non-commutative graded algebras and their Hilbert series*, J. Algebra 78 (1982), 120–140.
2. Anick, D., *Generic algebras and CW complexes*, in: J. D. Stasheff ed., Proceedings of the 1983 Conference on Algebraic Topology and K-theory, Princeton, Ann. of Math. Stud. 113 (1987), 247–321.
3. Backelin, J., Fröberg, R., *Koszul algebras, Veronese subrings and rings with linear resolution*, Rev. Roumaine Math. Pures Appl. 30 (1985), 85–97.
4. Bruns, W., Herzog, J., Vetter, U., *Syzygies and walks*, ICTP Proceedings 'Commutative Algebra', Eds. A. Simis, N. V. Trung, G. Valla, World Scientific 1994, 36–57.
5. Fröberg, R., *Determination of a class of Poincare series*, Math. Scand. 37 (1975), 29–39.
6. Fröberg, R., Hollman, J., *Hilbert series for ideals generated by generic forms*, J. Symbolic Comput. 17 (1994), 149–157.
7. Fröberg, R., Löfwall, C., *On Hilbert series for commutative and non commutative graded algebras*, J. Pure Appl. Algebra 76 (1991), 33–38.
8. Fröberg, R., *Koszul algebras*, Adv. in Commutative Ring Theory (Fez, 1997) 337–350, Lecture Notes in Pure and Appl. Math. 205, 1999.
9. Geramita, A., Gregory, D., Roberts, L., *Monomial ideals and points in projective space*, J. Pure Appl. Algebra 40 (1986), 33–62.
10. Herzog, J., Hibi, T., Restuccia, G., *Strongly Koszul algebras*, Math. Scand. 86 (2000), 161–178.
11. Kempf, G., *Syzygies for points in projective space*, J. Algebra 145 (1992), 219–223.
12. van Lint, J., Wilson, R., *A course in Combinatorics*, Cambridge University Press 1992.
13. Löfwall, C., personal communication.
14. Roos, J. E., *Commutative non-Koszul algebras having a linear resolution of arbitrarily high order. Applications to torsion in loop space homology*, C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), 1123–1128.
15. Stanley, R., *Graph colorings and related symmetric functions: ideas and applications*, Discrete Math. 193 (1998), 267–286.
16. Shelton, B., Yuzvinsky, S., *Koszul algebras from graphs and hyperplane arrangements*, J. London Math. Soc. (2) 56 (1997), 477–490.

CONCA & VALLA:
DIPARTIMENTO DI MATEMATICA
UNIVERSITA' DI GENOVA
VIA DODECANESO 35
I-16146 GENOVA
ITALY
E-mail: conca@dima.unige.it, valla@dima.unige.it

TRUNG:
INSTITUTE OF MATHEMATICS
BOX 631
BO HÔ, HANOI
VIETNAM
E-mail: nvtrung@thevinh.ncst.ac.vn