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FAITHFUL REPRESENTATIONS OF CROSSED PRODUCTS BY ACTIONS OF $\mathbb{N}^k$

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Abstract

We study a family of semigroup crossed products arising from actions of $\mathbb{N}^k$ by endomorphisms of groups. These include the Hecke algebra arising in the Bost-Connes analysis of phase transitions in number theory, and other Hecke algebras considered by Brenken. Our main theorem is a characterisation of the faithful representations of these crossed products, and generalises a similar theorem for the Bost-Connes algebra due to Laca and Raeburn.

Crossed products of C^*-algebras by semigroups of endomorphisms were introduced to model Cuntz and Toeplitz algebras, and many of the main results concerning these algebras have been formulated as characterisations of faithful representations of semigroup crossed products [1], [10]. More recently, the Hecke algebra arising in the Bost-Connes analysis of phase-transition phenomena in number theory [3] has been identified as a crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by the semigroup $\mathbb{N}^*$ of positive integers under multiplication [11]. This immediately showed that two of the relations in the presentation of the Hecke algebra used in [3] were redundant, and the techniques developed in [10] for studying Toeplitz algebras carried over to this crossed product without substantial difficulty. The resulting characterisation of faithful representations of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ improves a fundamental result used by Bost and Connes to switch between different Hilbert-space realisations.

Several generalisations of the Bost-Connes algebra have been considered: the field $\mathbb{Q}$ has been replaced by other number fields [2], [8], [9], and Brenken has realised a wider class of Hecke algebras as semigroup crossed products [5]. The analysis of [11] was extended to number fields in [2], and used extensively by Laca in [9]; while the techniques of [2] should also work for the algebras in [8], it is not so clear how they apply to the general situation of [5]. We shall formulate a theorem which covers the algebras of [3], [11], [8] and the most important of the other examples in [5], which derive from Brenken's work in topological dynamics [4].

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To prove results of this sort there is a standard path, going back at least to Douglas [7] and Cuntz [6]. Following this path is not always easy: there are several technical variations, and it can be surprisingly hard to formulate general results which encompass different applications. Here we are looking for a theorem which is general enough to cover the main examples, transparent enough to add insight to these examples, and specific enough to avoid nasty technical hypotheses.

We have found that the examples in [5], [8], [11] share a common structure: the underlying semigroup is a direct sum $\mathbb{N}^k$ of copies of the additive semigroup $\mathbb{N}$ (in the case of [11], prime factorisation gives an isomorphism of $\mathbb{N}^*$ onto $\mathbb{N}^\infty$), and it acts on the C^*-algebra of an abelian group which is a direct limit over the same partially ordered set $\mathbb{N}^k$. This direct limit structure allows us to construct actions in which there is interaction between the different copies of $\mathbb{N}$ of the sort important in [3], [9]. And because the semigroup $\mathbb{N}^k$ has minimal elements, we can avoid some of the technical difficulties encountered in [10, §3].

In our first section, we describe our class of dynamical systems $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$. We start with an action of $\mathbb{N}^k$ by injective endomorphisms of a group G, and by forming a direct limit G_∞ convert these to automorphisms (for recent applications of this standard technique to C^*-algebras, see [15], [12]). The resulting automorphic action on G_∞ leaves the canonical image of G invariant, and hence induces an action β of $\mathbb{N}^k$ on G_∞/G by surjective endomorphisms; the action α is obtained by averaging in the group algebra over the solutions of equations $\beta(s) = r$ in G_∞/G (see Proposition 1.3). The notation set up in this section will be used throughout the paper; to help keep it consistent, we have resisted temptations to generalise basic lemmas.

In Section 2, we discuss our main theorem. The proof follows the standard path, but is basically self-contained. The key ingredient is an estimate which says that killing off-diagonal terms in a sum decreases the norm, and whose proof requires an analysis of $(G_\infty/G)^\wedge$. The necessary properties of $(G_\infty/G)^\wedge$ hold because it is an inverse limit over $\mathbb{N}^k$; the relative simplicity of this argument compared with those in [2, §3] or [8, §5.1] allows us to claim that our level of generality adds insight.

We finish by showing how the situations of [11], [8] and [5] fit our model. In particular, we prove using the results of [5] that each of our semigroup crossed products is the enveloping C^*-algebra of a Hecke algebra, and apply our theorem to the algebra of [5, §4.5]. Not all the situations in [2] or [5] do fit: some involve actions of semigroups with invertible elements, and our techniques would require substantial modification to deal with these.

Conventions. By $\mathbb{N}^k$ we mean the direct sum of k copies of the additive semigroup $(\mathbb{N}, +)$, where $0 \in \mathbb{N}$ and we allow either $k \in \mathbb{N}$ or $k = \infty$. We write a typical element m of $\mathbb{N}^k$ as $m = (m_1, m_2, \dots, m_{|m|}, 0, \dots)$. The semigroup $\mathbb{N}^k$ is partially ordered by $m \le n \iff m_i \le n_i$ for all i; we denote the maximum of $m, n \in \mathbb{N}^k$ by $m \vee n$, so that $(m \vee n)_i := \max(m_i, n_i)$, and their minimum by $m \wedge n$. We denote by $\{e_i\}$ the usual basis elements for $\mathbb{N}^k$, so that $m = \sum_i m_i e_i$ for $m \in \mathbb{N}^k$.

Our dynamical systems will consist of an action α of the semigroup $\mathbb{N}^k$ by endomorphisms of a C^*-algebra A with identity. The crossed products appearing here are those of [10], [11]: a *covariant representation* of $(A, \mathbb{N}^k, \alpha)$ consists of a unital representation π of A and a representation W of $\mathbb{N}^k$ by isometries on the same space such that $\pi(\alpha_m(a)) = W_m \pi(a) W_m^*$ for $a \in A$, $m \in \mathbb{N}^k$, and the crossed product $A \rtimes_\alpha \mathbb{N}^k$ is the C^*-algebra generated by a universal covariant pair. Thus there is a bijection $(\pi, W) \mapsto \pi \times W$ between the covariant representations of the system and the representations of $A \rtimes_\alpha \mathbb{N}^k$.

1. The dynamical system $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$

Our construction begins with an action η of $\mathbb{N}^k$ by injective endomorphisms of a (discrete, additive) abelian group G satisfying

$$|G : \eta_m(G)| < \infty, \tag{1.1}$$

and

$$\eta_m(G) + \eta_n(G) = \eta_{m \wedge n}(G) \tag{1.2}$$

for all $m, n \in \mathbb{N}^k$. We can form a direct system $(G^{(m)}, \eta_m^n)$ over the directed set $\mathbb{N}^k$ by taking $G^{(m)} := G$ and $\eta_m^n := \eta_{n-m} : G^{(m)} \to G^{(n)}$ for $m \le n \in \mathbb{N}^k$; let

$$G_\infty := \varinjlim (G^{(m)}, \eta_m^n) \tag{1.3}$$

denote the direct limit. Since all the bonding maps η_{n-m} are injective, so are the canonical maps i_m of $G^{(m)}$ into G_∞; we view the direct limit as a union by writing $G_m = i_m(G^{(m)})$, so that $G_\infty = \bigcup G_m$. Notice that passing to this direct limit has converted the endomorphisms η_m into inclusions; indeed, since $i_n \circ \eta_m^n = i_m$ and η_m^n is really just η_{n-m}, the index of G_m in G_n is precisely $|G : \eta_{n-m}(G)|$. Similarly, Equation (1.2) translates into:

LEMMA 1.1. *For all $m, n \in \mathbb{N}^k$, we have $G_m + G_n = G_{m \vee n}$.*

PROOF. We just need to apply $i_{m \vee n}$ to the equation

$$\eta_{(m \vee n)-m}(G) + \eta_{(m \vee n)-n}(G) = G,$$

which follows from (1.2) because $\big((m \vee n) - m\big) \wedge \big((m \vee n) - n\big) = 0$.

For fixed m, the maps $\eta_m : G = G^{(p)} \to G = G^{(p)}$ are compatible with the bonding maps η_p^q:

$$\eta_m \circ \eta_p^q = \eta_m \circ \eta_{q-p} = \eta_{m+q-p} = \eta_{q-p} \circ \eta_m = \eta_p^q \circ \eta_m.$$

Thus there is a well-defined endomorphism η_m^∞ of G_∞ such that

$$\eta_m^\infty(i_p(g)) = i_p(\eta_m(g)) \text{ for } g \in G^{(p)} = G. \tag{1.4}$$

In fact, each η_m^∞ is an automorphism: the identity maps $\zeta_m : G = G^{(p)} \to G = G^{(p+m)}$ induce an endomorphism ζ_m^∞ which is an inverse for η_m^∞. To see this, we compute that

$$\zeta_m^\infty\big(\eta_m^\infty(i_p(g))\big) = \zeta_m^\infty\big(i_p(\eta_m(g))\big) = i_{p+m}(\eta_m(g)) = i_p(g),$$

and similarly that $\eta_m^\infty\big(\zeta_m^\infty(i_p(g))\big) = i_p(g)$. It follows immediately from (1.4) that $\eta_m^\infty \circ \eta_n^\infty = \eta_{m+n}^\infty$, so that η^∞ is an action of $\mathbb{N}^k$ by automorphisms of G_∞.

Equation (1.4) implies that each η_m^∞ leaves the subgroup $G = i_0(G^{(0)})$ invariant, and hence induces an endomorphism β_m of the quotient G_∞/G such that

$$\beta_m(g + G) = \eta_m^\infty(g) + G \text{ for } g \in G_\infty.$$

This in turn induces an action β of $\mathbb{N}^k$ by endomorphisms of the group C^*-algebra $C^*(G_\infty/G)$, which is characterised on the canonical generators $\{\delta_r : r \in G_\infty/G\}$ by $\beta_m(\delta_r) = \delta_{\beta_m(r)}$.

EXAMPLE 1.2. Let p and q be distinct prime numbers, and define $\eta : \mathbb{N}^2 \to \operatorname{End} \mathbb{Z}$ by $\eta_{m,n}(x) := p^m q^n x$. We have $|\mathbb{Z} : \eta_{m,n}(\mathbb{Z})| = p^m q^n$, and (1.2) holds because $x\mathbb{Z} + y\mathbb{Z}$ is the set $(x, y)\mathbb{Z}$ of multiples of the g.c.d. (x, y), and $(p^m q^n, p^k q^\ell) = p^{m\wedge k} q^{n\wedge\ell}$. The maps $\phi_{m,n}$ of $G^{(m,n)} = \mathbb{Z}$ into $\mathbb{Q}$ defined by $\phi_{m,n}(x) = p^{-m} q^{-n} x$ satisfy $\phi_{m,n} \circ \eta_{m-k,n-\ell} = \phi_{k,\ell}$, and induce an isomorphism ϕ of G_∞ onto the additive group H of rational numbers with denominators $p^m q^n$, which converts the maps $i_{m,n}$ into the inclusions of $G_{m,n} = p^{-m} q^{-n}\mathbb{Z}$ in H. The automorphism $\eta_{m,n}^\infty$ is multiplication by $p^m q^n$ on H, and the induced endomorphism $\beta_{m,n}$ of $G_\infty/G = H/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ is multiplication by $p^m q^n$ modulo $\mathbb{Z}$.

A listing $p_1, p_2, \ldots$ of the prime numbers gives an isomorphism $m \mapsto \prod p_i^{m_i}$ of $\mathbb{N}^\infty$ with the multiplicative semigroup $\mathbb{N}^*$, which converts $\wedge$ to the g.c.d. and $\vee$ to the l.c.m. Thus the action η of $\mathbb{N}^*$ on $\mathbb{Z}$ defined by $\eta_k(x) = kx$ also satisfies our hypotheses. As in the previous paragraph, we can identify G_∞/G with $\mathbb{Q}/\mathbb{Z}$, and β_k is the map $r \mapsto kr$.

The action of particular interest to us will be a right inverse for $\beta \in \operatorname{End} C^*(G_\infty/G)$ obtained by averaging over the solutions of $\beta_m(s) = r$ in the group G_∞/G.

PROPOSITION 1.3. *There is an action α of $\mathbb{N}^k$ by endomorphisms of $C^*(G_\infty/G)$ such that*

$$\alpha_m(\delta_r) = \frac{1}{|G : \eta_m(G)|} \sum_{\{s \in G_\infty/G \,:\, \beta_m(s)=r\}} \delta_s \tag{1.5}$$

for $r \in G_\infty/G$ and $m \in \mathbb{N}^k$. The projections $\alpha_m(1) = \alpha_m(\delta_0)$ satisfy

$$\alpha_m(1)\alpha_n(1) = \alpha_{m\vee n}(1); \tag{1.6}$$

the endomorphism β_m of $C^*(G_\infty/G)$ is a left inverse for α_m, and $\alpha_m \circ \beta_m$ is multiplication by $\alpha_m(1)$.

This proposition can be deduced from the analysis of [5, §1] and [5, Proposition 3.2]; see the proof of Proposition 3.5 below. It can also be proved directly by following the argument of [11, Proposition 2.1], whereby one deduces the existence of α_m by showing that the map $r \mapsto \alpha_m(\delta_r)$ defined in (1.5) is a unitary representation of G_∞/G in a corner of $C^*(G_\infty/G)$ and invoking the universal property of $C^*(G_\infty/G)$. The only tricky bit is then to verify (1.6). However, as in [11], one can reduce to the case $m \wedge n = 0$; then a counting argument using Lemma 1.1 shows that $(r, s) \mapsto r + s$ is a surjection of $\ker \beta_m \times \ker \beta_n$ onto $\ker \beta_{m\vee n}$, and (1.6) follows.

REMARK 1.4. The identity $\alpha_m(1)\alpha_n(1) = \alpha_{m\vee n}(1)$ implies that, whenever (π, W) is a covariant representation of $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$, W is itself covariant in the sense of Nica [13]:

$$W_m W_m^* W_n W_n^* = W_{m\vee n} W_{m\vee n}^*, \tag{1.7}$$

or equivalently

$$W_m W_n^* W_p W_q^* = W_{m-n+n\vee p} W_{q-p+n\vee p}^*, \tag{1.8}$$

for m, n, p, q in $\mathbb{N}^k$.

LEMMA 1.5. *There is an isometric representation L of $\mathbb{N}^k$ on $l^2(G_\infty/G)$ such that*

$$L_m(\epsilon_r) = \frac{1}{|G : \eta_m(G)|^{1/2}} \sum_{\{s \,:\, \beta_m(s)=r\}} \epsilon_s,$$

where $\{\epsilon_r : r \in G_\infty/G\}$ is the usual basis of $l^2(G_\infty/G)$. Together with the regular representation λ of $C^*(G_\infty/G)$, L forms a covariant representation (λ, L) of $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$.

PROOF. As in [11, Example 2.3], one sees that L_m is an isometry by verifying that $\{L_m(\epsilon_r)\}$ is orthonormal, and $L_m^*(\epsilon_t) = |G : \eta_m(G)|^{-1/2}\epsilon_{\beta_m(t)}$, and the proof of $\alpha_m \circ \alpha_n = \alpha_{m+n}$ shows that $L_m L_n = L_{m+n}$. To verify covariance, we let $m \in \mathbb{N}^k$, $r, t \in G_\infty/G$ and compute:

$$L_m\lambda(\delta_r)L_m^*(\epsilon_t) = \frac{1}{|G : \eta_m(G)|} \sum_{\{u\,:\,\beta_m(u)=r+\beta_m(t)\}} \epsilon_u$$
$$= \frac{1}{|G : \eta_m(G)|} \sum_{\{s\,:\,\beta_m(s)=r\}} \epsilon_{t+s}.$$

Thus

$$\lambda(\alpha_m(\delta_r))(\epsilon_t) = \frac{1}{|G : \eta_m(G)|} \sum_{\{s\,:\,\beta_m(s)=r\}} \lambda(\delta_s)(\epsilon_t) = L_m\lambda(\delta_r)L_m^*,$$

from which covariance follows by linearity and continuity.

2. Faithful representations

Since the system $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$ has a nontrivial covariant representation by Lemma 1.5, it has a crossed product $C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k$, which is generated by a universal covariant pair (ι, V) and is unique up to isomorphism [10, Proposition 2.1]. Our main theorem characterises the faithful representations of this crossed product.

THEOREM 2.1. *Let (π, W) be a covariant representation of the dynamical system $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$ of Section 1. Then the representation $\pi \times W$ of $C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k$ is faithful if and only if π is faithful.*

The canonical homomorphism ι of $C^*(G_\infty/G)$ into the crossed product is faithful, because λ is and $(\lambda \times L) \circ \iota = \lambda$. This immediately gives one direction of the Theorem: if $\pi \times W$ is faithful, so is $\pi = (\pi \times W) \circ \iota$.

For the other direction, we shall follow the strategy developed in [10]. To do this, we need a simple spanning set for the crossed product.

LEMMA 2.2. *We have*

$$C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k = \overline{\operatorname{span}}\{\iota(\delta_r)V_m V_n^* : r \in G_\infty/G, m, n \in \mathbb{N}^k\}.$$

PROOF. It is enough to show that $\{\iota(\delta_r)V_m V_n^*\}$ is closed under multiplication and adjoints, since it contains the generators $\iota(\delta_r)$ and V_m. We first notice that

$$V_m\iota(\beta_m(\delta_r)) = \iota(\alpha_m(\beta_m(\delta_r)))V_m = \iota(\delta_r)\iota(\alpha_m(1))V_m = \iota(\delta_r)V_m. \tag{2.1}$$

Using this and the covariance property (1.8), we find

$$\begin{aligned}(\iota(\delta_r)V_mV_n^*)(\iota(\delta_s)V_pV_q^*) &= \iota(\delta_r)V_m\iota(\beta_n(\delta_s))V_n^*V_pV_q^* \\ &= \iota(\delta_r)V_m\iota(\beta_n(\delta_s))V_m^*V_mV_n^*V_pV_q^* \\ &= \iota\big(\delta_r\alpha_m\circ\beta_n(\delta_s)\big)V_{m-n+n\vee p}V^*_{q-p+n\vee p}.\end{aligned}$$

Another calculation using (2.1) gives $\big(\iota(\delta_r)V_mV_n^*\big)^* = \iota\big(\alpha_n\circ\beta_m(\delta_{-r})\big)V_nV_m^*$.

The uniqueness of the crossed product implies that there is a strongly continuous dual action $\widehat{\alpha}$ of $\widehat{\mathbb{Z}^k}$ on $C^*(G_\infty/G)\rtimes_\alpha\mathbb{N}^k$ such that

$$\widehat{\alpha}_\gamma\big(\iota(\delta_r)V_mV_n^*\big) = \gamma(m-n)\iota(\delta_r)V_mV_n^* \text{ for } \gamma\in\widehat{\mathbb{Z}^k}$$

(see [10, Remark 3.6]). Averaging over this dual action gives a faithful positive linear map Φ of $C^*(G_\infty/G)\rtimes_\alpha\mathbb{N}^k$ onto $\iota\big(C^*(G_\infty/G)\big)$ such that

$$\Phi\big(\iota(\delta_r)V_mV_n^*\big) = \begin{cases}\iota(\delta_r)V_mV_m^* = \iota\big(\delta_r\alpha_m(1)\big) & \text{if } m=n\\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Now suppose (π, W) is covariant and π is faithful. We aim to prove Theorem 2.1 by constructing a positive contraction ϕ of $\pi\times W\big(C^*(G_\infty/G)\rtimes_\alpha\mathbb{N}^k\big)$ onto $\pi(C^*(G_\infty/G))$ such that $\pi\circ\Phi = \phi\circ(\pi\times W)$, and running the argument of [10, §3]. As in [11] and [2], this depends on an analysis of the dual of the abelian group G_∞/G. In this analysis we write $\gamma|_{G_m}\equiv 1$ to mean that γ is trivial on G_m/G.

Lemma 2.3. *Let $\gamma\in\widehat{G_\infty/G}$ and define $S := \{m\in\mathbb{N}^k : \gamma|_{G_m}\equiv 1\}$. Then there exist $n_i\in\mathbb{N}\cup\{\infty\}$ such that*

$$S = \{m\in\mathbb{N}^k : m_i\le n_i \ \textit{ for all } \ i\ge 1\}. \tag{2.3}$$

Proof. Since $0\in S$, S is nonempty. Let

$$n_i := \max\{n\in\mathbb{N} : n = q_i \text{ for some } q\in S\},$$

or $n_i := \infty$ if the set is unbounded. Then by definition every m in S belongs to the right-hand side of (2.3). Next, note that if m and n are in S, then so is $m\vee n$; indeed, since γ is a character, this follows immediately from the decomposition $G_{m\vee n} = G_m + G_n$ of Lemma 1.1. Now suppose m belongs to the right-hand side of (2.3). Let $J := \{i : i\le |m| \text{ and } n_i<\infty\}$. For each $i\in J$, there exists $p^{(i)}\in S$ such that $p_i^{(i)} = n_i$, and for each $i\notin J$ there exists $q^{(i)}\in S$ such that $q_i^{(i)}\ge m_i$. Then $p := \bigvee_{i\in J}p^{(i)}\vee\bigvee_{i\notin J}q^{(i)}$ belongs to S. But $m\le p$, so $G_m\subset G_p$, and $\gamma|_{G_p}\equiv 1$ implies $\gamma|_{G_m}\equiv 1$. Thus $m\in S$.

PROPOSITION 2.4. *Suppose $\gamma \in (G_\infty/G)^\wedge$, $N \in \mathbb{N}$ and U is a neighbourhood of γ. Then there exist $\chi \in U$ and $m \in \mathbb{N}^k$ such that $\chi \in G_m^\perp$ and $\chi \notin G_{m+e_i}^\perp$ for $i \leq N$.*

PROOF. Since G_∞/G is the union of the subgroups G_m/G,

$$(G_\infty/G)^\wedge = \varprojlim (G_m/G)^\wedge :$$

the canonical maps of $(G_\infty/G)^\wedge$ into $(G_m/G)^\wedge$ are just restriction. Since the groups $(G_m/G)^\wedge$ are finite and hence discrete, we deduce that there exists $p \in \mathbb{N}^k$ such that

$$\{\chi \in (G_\infty/G)^\wedge : \chi|_{G_p} = \gamma|_{G_p}\} \subset U.$$

Let $m_i := \min\{n_i, p_i\}$ and $m(i) := (m_1, m_2, \dots, m_i, 0, \dots)$. We shall prove by induction over i that there exist $\gamma_i \in (G_\infty/G)^\wedge$ such that $\gamma_i|_{G_p} = \gamma|_{G_p}$ and $\gamma_i \in G_{m(i)}^\perp \setminus G_{m(i)+e_j}^\perp$ for $1 \leq j \leq i$.

If $n_1 \leq p_1$, take $\gamma_1 = \gamma$. Then we trivially have $\gamma_1|_{G_p} = \gamma|_{G_p}$, and $\gamma_1 \in G_{m(1)}^\perp \setminus G_{m(1)+e_1}^\perp$ by Lemma 2.3. If $n_1 > p_1$, choose γ' in $G_p^\perp \setminus G_{p+e_1}^\perp$ and take $\gamma_1 = \gamma'\gamma$; the identity $G_{m(1)+e_1} + G_p = G_{p+e_1}$ means γ_1 cannot be trivial on $G_{m(1)+e_1}^\perp$. Suppose now that we have γ_i with the stated properties.

If $n_{i+1} < p_{i+1}$, take $\gamma_{i+1} = \gamma_i$. Then $\gamma_{i+1}|_{G_p} = \gamma_i|_{G_p} = \gamma|_{G_p}$. For $1 \leq j \leq i$, we have $G_{m(i)+e_j} \subset G_{m(i+1)+e_j}$, so $\gamma_i \notin G_{m(i+1)+e_j}^\perp$. Since $m(i+1) \leq p$ and $m(i+1) \leq n$, we have $\gamma_{i+1}|_{G_{m(i+1)}} = \gamma|_{G_{m(i+1)}} \equiv 1$; on the other hand, since $m(i+1) + e_{i+1} \leq p$ and $m(i+1)_{i+1} = n_{i+1}$, we have

$$\gamma_{i+1}|_{G_{m(i+1)+e_{i+1}}} = \gamma|_{G_{m(i+1)+e_{i+1}}} \not\equiv 1.$$

Next suppose $n_{i+1} \geq p_{i+1}$. If $\gamma_i \notin G_{m(i+1)+e_{i+1}}^\perp$, then $\gamma_{i+1} := \gamma_i$ will do. If $\gamma_i \equiv 1$ on $G_{m(i+1)+e_{i+1}}$, pick

$$\gamma' \in G_{p+\sum_{j=1}^{i} e_j}^\perp \setminus G_{p+\sum_{j=1}^{i+1} e_j}^\perp$$

and take $\gamma_{i+1} = \gamma'\gamma_i$. Since

$$G_{m(i+1)+e_{i+1}} + G_{p+\sum_{j=1}^{i} e_j} = G_{p+\sum_{j=1}^{i+1} e_j}$$

by Lemma 1.1, we must have $\gamma' \notin G_{m(i+1)+e_{i+1}}^\perp$, and hence $\gamma_{i+1} \notin G_{m(i+1)+e_{i+1}}^\perp$. Because $\gamma' \equiv 1$ on $G_{p+\sum_{j=1}^{i} e_j}$ and $m(i+1) \leq p$, we have $\gamma_{i+1} = \gamma_i$ on $G_{m(i+1)}$ and $G_{m(i+1)+e_j}$ whenever $j \leq i$. In particular, this implies that $\gamma_{i+1} \equiv \gamma_i \equiv 1$ on $G_{m(i+1)}$; because $G_{m(i+1)} + G_{m(i)+e_j} = G_{m(i+1)+e_j}$ and γ_i does not annihilate $G_{m(i)+e_j}$, it also implies

$$\gamma_{i+1} \in G_{m(i+1)}^\perp \setminus G_{m(i+1)+e_j}^\perp \qquad \text{for} \quad 1 \leq j \leq i.$$

Thus γ_{i+1} has the desired properties, and we have proved the claim.

To finish off, let $i = \max(|p|, N)$, and take $\chi = \gamma_i$.

Since the expectation Φ kills off-diagonal terms in our spanning set, to construct ϕ we have to do this spatially without increasing the norm of the sum. Usually, we fix a finite sum and kill these terms by compressing with a suitable projection (cf. [1, Theorem 2.4] or [10, Lemma 3.2]). Here, as in [11] and [2], we have to allow small changes in the norm of the diagonal terms.

LEMMA 2.5. *Let E be a finite subset of $\mathbb{N}^k$, and $\{f_{n,p} : n, p \in E\}$ a subset of $C^*(G_\infty/G)$. Then for each $\varepsilon > 0$, there is a projection $q = q_\varepsilon$ in $C^*(G_\infty/G)$ satisfying*

$$\iota(q)\iota(f_{n,p})V_nV_p^*\iota(q) = 0 \tag{2.4}$$

if $n \neq p$ in E, and

$$\Big\| q\Big(\sum_{n\in E} f_{n,n}V_nV_n^*\Big)q\Big\| \geq \Big\|\sum_{n\in E} f_{n,n}V_nV_n^*\Big\| - \varepsilon. \tag{2.5}$$

PROOF. Let f be the element $\sum_{n\in E} f_{n,n}\alpha_n(1)$ of $C^*(G_\infty/G)$, which is isomorphic to $C((G_\infty/G)^\wedge)$ via the Fourier transform $g \mapsto \hat{g}$. Since $(G_\infty/G)^\wedge$ is compact, there exists $\gamma \in (G_\infty/G)^\wedge$ such that $|\hat{f}(\gamma)| = \|\hat{f}\|_\infty = \|f\|$. Let U be a neighbourhood of γ such that $|\hat{f}(\chi)| \geq \|f\| - \varepsilon$ for $\chi \in U$, and let $N = \max\{|n| : n \in E\}$.

By Proposition 2.4, there exist $\chi \in U$ and $m \in \mathbb{N}^k$ such that $\chi \in G_m^\perp$ and $\chi \notin G_{m+e_i}^\perp$ for $i \leq N$. Let

$$q := \alpha_m\Big(\prod_{i=1}^{N}\big(1-\alpha_{e_i}(1)\big)\Big) = \prod_{i=1}^{N}\big(\alpha_m(1) - \alpha_{m+e_i}(1)\big);$$

notice that q is a product of projections in the commutative C^*-algebra $C^*(G_\infty/G)$, and hence is itself a projection.

We claim that $\iota(q)V_nV_p^*\iota(q) = 0$ if $n \neq p$; since q commutes with each $f_{n,p}$, this will establish (2.4). If $m \ngeq n$, say $n_1 > m_1$, then from (1.8) we have

$$V_m^*V_n = V_{-m+m\vee n}V_{-n+m\vee n}^* = V_{e_1}V_{-m+m\vee n-e_1}V_{-n+m\vee n}^*,$$

and $\iota(q)V_n = \prod_{i=1}^{N} V_m(1 - V_{e_i}V_{e_i}^*)V_m^*V_n$ vanishes because it contains $(1 - V_{e_1}V_{e_1}^*)V_{e_1}$ as a factor. Similarly, $V_p^*\iota(q) = 0$ if $m \ngeq p$. If $m \geq n$ and $m \geq p$, (1.8) gives

$$\begin{aligned}(V_m^*V_n)(V_p^*V_m) &= V_{-m+m\vee n}V_{-n+m\vee n}^*V_{-p+p\vee m}V_{-m+p\vee m}^* = V_{m-n}^*V_{m-p}\\ &= V_{-(m-n)+(m-n)\vee(m-p)}V_{-(m-p)+(m-n)\vee(m-p)}^*.\end{aligned}$$

Since $n \neq p$, there exists j such that $n_j \neq p_j$, say $n_j > p_j$. Then

$$\big(-(m-n)+(m-n)\vee(m-p)\big)_j = -m_j + n_j + m_j - p_j = n_j - p_j > 0,$$

so that $-(m-n)+(m-n)\vee(m-p)-e_j$ still belongs to $\mathbb{N}^k$. Thus $\iota(q)V_nV_p^*\iota(q)$ contains a term of the form $(1-V_{e_j}V_{e_j}^*)V_{e_j}$, which is zero. Similarly, if $p_j > n_j$, it contains a term of the form $V_{e_j}^*(1-V_{e_j}V_{e_j}^*)$. This justifies our claim that q annihilates the off-diagonal terms.

We now claim that $\hat{q}(\chi) = 1$. This is equivalent to

$$\prod_{i=1}^{N}\big(\widehat{\alpha_m(1)} - \widehat{\alpha_{m+e_i}(1)}\big)(\chi) = 1,$$

which is further equivalent to

$$\widehat{\alpha_m(1)}(\chi) = 1 \quad \text{and} \quad \widehat{\alpha_{m+e_i}(1)}(\chi) = 0 \qquad \text{for} \quad 1 \le i \le N, \tag{2.6}$$

because the Fourier transforms of projections only attain the values 0 and 1. For $\mu \in (G_\infty/G)^\wedge$ and $n \in \mathbb{N}^k$, we compute:

$$\widehat{\alpha_n(1)}(\mu) = \frac{1}{|G : \eta_n(G)|}\sum_{\{s\,:\,\beta_n(s)=0\}}\hat{\delta}_s(\mu) = \frac{1}{|G : \eta_n(G)|}\sum_{\{s\,:\,\beta_n(s)=0\}}\mu(s).$$

If $\mu \not\equiv 1$ on $\ker\beta_n = G_n/G$, the set $\{\mu(s) : \beta_n(s) = 0\}$ is a finite subgroup of $\mathbb{T}$, and hence sums to 0. Because $|\ker\beta_n| = |G : \eta_n(G)|$, we therefore have

$$\widehat{\alpha_n(1)}(\mu) = \begin{cases} 1 & \text{if } \mu|_{G_n} \equiv 1 \\ 0 & \text{otherwise.}\end{cases}$$

Thus the properties $\chi \in G_m^\perp$ and $\chi \notin G_{m+e_i}^\perp$ for $i \le N$ imply (2.6), and $\hat{q}(\chi) = 1$, as claimed. This is enough to finish off:

$$\Big\| q\Big(\sum_{n\in E} f_{n,n}V_nV_n^*\Big)q\Big\| \ge |\widehat{qfq}(\chi)| = |\hat{f}(\chi)| \ge \|f\| - \varepsilon,$$

so q satisfies (2.5).

COROLLARY 2.6. *Let (π, W) be a covariant representation of $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$, and suppose that π is faithful. Then there is a well-defined contractive linear map ϕ of the range of $\pi \times W$ onto the range of π such that*

$$\phi\Big(\sum_{n,p\in E}\pi(f_{n,p})W_nW_p^*\Big) = \sum_{n\in E}\pi(f_{n,n})W_nW_n^*. \tag{2.7}$$

PROOF. Given a finite sum $a := \sum f_{n,p} V_n V_p^*$ and $\varepsilon > 0$, we choose q as in Lemma 2.5. Then compressing with q gives the diagonal f, and hence compressing $\pi \times W(a)$ with $\pi(q)$ gives $\pi(f)$. Now the estimate (2.5) implies that $\|\pi \times W(a)\| \geq \|f\| = \|\pi(f)\|$, so $\pi \times W(a) \mapsto \pi(f)$ is norm-decreasing. (See the proof of [11, Lemma 3.6] for further details.)

PROOF OF THEOREM 2.1. Suppose (π, W) is a covariant representation with π faithful, and $\pi \times W(b) = 0$. Then $\phi \circ (\pi \times W)(b^*b) = 0$. The formulas (2.7) and (2.2) imply that $\phi \circ (\pi \times W) = \pi \circ \Phi$, so we have $\pi(\Phi(b^*b)) = 0$. Since π is faithful, and Φ is faithful on positive elements, we deduce that $b = 0$.

3. Examples and Applications

EXAMPLE 3.1. Since the dynamical system $(C^*(\mathbb{Q}/\mathbb{Z}), \mathbb{N}^*, \alpha)$ of [11] arises by applying our construction to the action η of $\mathbb{N}^*$ by multiplication on $G = \mathbb{Z}$ (see Example 1.2), Theorem 3.7 of [11] is a special case of our Theorem 2.1.

EXAMPLE 3.2. For the system $(C^*(H/\mathbb{Z}), \mathbb{N}^2, \alpha)$ obtained by applying Proposition 1.3 to the action of $\mathbb{N}^2$ discussed in Example 1.2, our theorem is technically new. However, it was obvious when writing [11] that the techniques would work here too.

EXAMPLE 3.3. Let $\mathcal{O}$ be the ring of integers in a number field K of class number $h_K = 1$, and let $\mathcal{O}^\times$ denote the multiplicative semigroup of nonzero elements in $\mathcal{O}$. Choosing a sequence of generators $\{p_i\}$ for the prime ideals of $\mathcal{O}$ gives an embedding $m \mapsto \prod p_i^{m_i}$ of $\mathbb{N}^\infty$ in $\mathcal{O}^\times$ whose image S consists of a generator for each ideal. (The hypothesis $h_K = 1$ says precisely that each ideal in $\mathcal{O}$ is principal.) Let η be the action of $\mathbb{N}^\infty$ by multiplication on the additive group $\mathcal{O}$. Since everything takes place in a field, these endomorphisms are injective; the index $|\mathcal{O} : n\mathcal{O}|$ is the norm $N(n)$ of the ideal $n\mathcal{O}$; and since the identification of $\mathbb{N}^\infty$ with ideals in $\mathcal{O}$ reflects the prime decomposition of ideals, the identity (1.2) follows from the formula for the prime decomposition of the g.c.d. $(m\mathcal{O}, n\mathcal{O})$. As in Example 1.2, the maps $\mathcal{O} \to \prod p_i^{-m_i}\mathcal{O}$ induce an isomorphism of $\mathcal{O}_\infty$ onto K, our construction yields the dynamical system $(C^*(K/\mathcal{O}), S, \alpha)$ considered in [2, §5], and Theorem 2.1 gives [2, Theorem 5.1]. It is curious that we have apparently replaced the number-theoretic analysis of $(K/\mathcal{O})^\wedge$ used in [2, §5] with a general analysis of $(G_\infty/G)^\wedge$; it is intriguing to wonder if there is a parallel framework which encompasses the main results of [2] on arbitrary number fields.

EXAMPLE 3.4. Let $\mathcal{O}$ be one of the principal subrings of a global field K considered in [8]. Thus if K has class number $h_K = 1$, $\mathcal{O}$ could be the ring of integers; in general, $\mathcal{O}$ is a principal localisation of the ring of integers. As above, a choice of generators for the prime ideals of $\mathcal{O}$ identifies N^∞ with a subsemigroup of $\mathcal{O}^\times$ comprising a generator for each ideal of $\mathcal{O}$, and we define $\eta : \mathsf{N}^\infty \to \operatorname{End}\mathcal{O}$ by $\eta_n(a) = na$, which satisfies our basic hypotheses by the argument of the previous example. The maps $i_n : \mathcal{O} \to \frac{1}{n}\mathcal{O}$ identify $\mathcal{O}_\infty$ with K, and our construction yields an action α of N^∞ on $C^*(K/\mathcal{O})$. One can see from the presentation in [8, Proposition 3.1], or from the Proposition below, that the crossed product $C^*(K/\mathcal{O}) \rtimes_\alpha \mathsf{N}^\infty$ is the C^*-enveloping algebra of the Hecke algebra $\mathscr{H}(P_K^+; P_\mathcal{O}^+)$ of [8]; our theorem thus extends [2, Theorem 5.1] to the situation considered in [8]. It implies in particular that the reduced Hecke C^*-algebra used there is universal for representations of $\mathscr{H}(P_K^+; P_\mathcal{O}^+)$ [8, Proposition 3.2].

Before applying our theorem to some of the examples discussed in [5], we deduce from the results of [5] that our semigroup crossed products can be realised as Hecke algebras. Since each η_m^∞ is an automorphism, we can define an action of Z^k on G_∞ by $\psi_m = (\eta_m^\infty)^{-1}$, and form the semi-direct product $G_\infty \rtimes_\psi \mathsf{Z}^k$ in which $(g, m)(h, n) := (g + \psi_m(h), m + n)$. We denote by ν the splitting $m \mapsto (0, m)$ for the quotient map $(g, m) \mapsto m$.

PROPOSITION 3.5. *Let* $\Gamma := G_\infty \rtimes_\psi \mathsf{Z}^k$ *and* $\Gamma_0 := \{(g, 0) : g \in G = i_0(G)\}$. *Then* (Γ, Γ_0) *is an almost normal subgroup pair, and the enveloping* C^**-algebra of the Hecke algebra* $\mathscr{H}(\Gamma, \Gamma_0)$ *is isomorphic to* $C^*(G_\infty/G) \rtimes_\alpha \mathsf{N}^k$.

PROOF. The right cosets of Γ_0 are the sets $(g + G_m, m)$ for $m \in \mathsf{N}^k$, and the sets $\big(g + i_0(\eta_n(G)), -n\big)$ for $n \in \mathsf{N}^k$. Because $G = i_0(G) \subset G_m = i_m(G)$, the left action of Γ_0 fixes the cosets $(g + G_m, m)$. On the other hand, if $h, k \in G \cong \Gamma_0$ then

$$h \cdot \big(g + i_0(\eta_n(G)), -n\big) = k \cdot \big(g + i_0(\eta_p(G)), -p\big)$$

iff $n = p$ and $h - k \in \eta_n(G)$, so the orbit of $\big(g + i_0(\eta_n(G)), -n\big)$ has $|G : \eta_n(G)|$ elements. Thus the hypothesis (1.1) implies that (Γ, Γ_0) is almost normal, with, in Brenken's notation, $R(\nu(-n)) = |G : \eta_n(G)|$.

We now verify the hypotheses of [5, Theorem 3.12]. The elements of Z^k which leave Γ_0 invariant are precisely those in N^k, so $G_\infty = \{(g, 0) : g \in G_\infty\}$ is the normaliser N_{Γ_0} of Γ_0 and the semigroup T of [5] is N^k. The action $\operatorname{ad}\circ\nu$ of N^k on $N = G_\infty$ is just ψ, and the equation $G_m + G_n = G_{m\vee n}$ of Lemma 1.1 (which is a restatement of (1.2)) says that $\psi_m(\Gamma_0) + \psi_n(\Gamma_0) = \psi_{m\vee n}(\Gamma_0)$, so $m \vee n$ is a solvable upper bound for m and n in the sense of [5, §2]. We trivially have $\mathsf{Z}^k = \mathsf{N}^k - \mathsf{N}^k$. Thus Theorem 3.12 of [5] applies, and it only remains to identify our action of N^k with Brenken's. But since ψ_m is $(\eta_m^\infty)^{-1}$, the action on $G_\infty/G = N/\Gamma_0$ induced by $\operatorname{ad}\circ\nu(-m) = \eta_m^\infty$ is β_m. Since we have already seen that $R(\nu(-n)) = |G : \eta_n(G)|$, this implies that the endomorphism $\tilde{\Theta}(-n) = \Theta(\nu(-n))$ described following Remark 1.6 in [5] is our α_n.

EXAMPLE 3.6. Let F and M be commuting matrices in $GL_d(\mathbb{Z})$ such that $\det F \neq 1$, $\det M \neq 1$ and $(\det F, \det M) = 1$, and define $\eta : \mathbb{N}^2 \to \operatorname{End} \mathbb{Z}^d$ by $\eta_{m,n} = F^m M^n$. The identity $|\mathbb{Z}^d : T\mathbb{Z}^d| = \det T$ for $T \in GL_d(\mathbb{Z})$ (from, for example, [14, p. 49]) immediately implies that η satisfies (1.1), and it is proved in [5, §4.4] that η satisfies (1.2). As in the number-theoretic examples, we can identify the direct limit G_∞ with the additive subgroup $N := \bigcup_{m,n} F^{-m}M^{-n}\mathbb{Z}^d$ of $\mathbb{Q}^d$ and $G_{m,n}$ with $F^{-m}M^{-n}\mathbb{Z}^d$, and we obtain a dynamical system $(C^*(N/\mathbb{Z}^d), \mathbb{N}^2, \alpha)$ to which our theorem applies. Proposition 3.5 says that, modulo replacing F and M by their transposes, the crossed product $C^*(N/\mathbb{Z}^d) \rtimes_\alpha \mathbb{N}^2$ is the Hecke C^*-algebra $C^*(\Gamma, \Gamma_0)$ considered by Brenken in [5, §4.5], in which $\Gamma := N \rtimes_\phi \mathbb{Z}^2$ is the semidirect product for the action $\phi_{m,n}(x) := F^{-m}M^{-n}x$, and $\Gamma_0 := G_{0,0} = \mathbb{Z}^d \subset N$. We therefore have a characterisation of the faithful representations of these algebras.

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