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CONVOLUTION WITH MEASURES ON POLYNOMIAL CURVES

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This paper is concerned with convolution estimates for certain measures on degenerate curves in R^2 and R^3. Analogous estimates in R^n, $n \geq 4$, were recently obtained for the (nondegenerate) curve $(t, t^2, \ldots, t^n)$ in [4] – see also [9] and [10]. Here is some of the history of this problem. Ideas going back to [6] show, for example, that if μ is the measure given by dt on the circle $(\cos(t), \sin(t))$ or on the parabola (t, t^2), then

$$(1) \qquad \mu * L^{\frac{3}{2}}(\mathsf{R}^2) \subseteq L^3(\mathsf{R}^2).$$

And it is easy to see that these estimates are optimal – see [7] for more on this. The feature, common to these two curves, which in retrospect gives rise to (1) is the fact that on both of them the measure dt is a multiple of the measure $\kappa^{\frac{1}{3}}(s)ds$ where ds is arclength and κ is curvature. Drury [5] was the first to notice the importance of the measures μ given by $d\mu = \kappa^{\frac{1}{3}}(s)ds$ in the context of (1). In particular, it was Drury's idea to obtain (1) for the measure $d\mu = \kappa^{\frac{1}{3}}(s)ds$ on degenerate curves. His result (Theorem 1 in [5]) applies to curves of the form $(t, p(t))$, so that $d\mu = |p''(t)|^{\frac{1}{3}}dt$, where the convex function p satisfies certain regularity conditions. The paper [8] contains a similar result, valid for any real-valued polynomial p. And that estimate is uniform for polynomials of a fixed degree. Theorem 1 below generalizes this: the estimate (1) holds for curves $(p_1(t), p_2(t))$ with $d\mu = \kappa^{\frac{1}{3}}(s)ds$ if p_1 and p_2 are real-valued polynomials, and the convolution bounds are uniform in p_1 and p_2 if the degree of these polynomials is fixed.

Part of the motivation for the above-mentioned work of Drury stems from the fact that convolution estimates for curves in R^2 can be used to obtain convolution estimates for curves in R^3 – see [7]. The main result in [7] is the following: suppose that $p_1(t)$ and $p_2(t)$ are polynomials and that the two vectors $(p_1^{(j)}(t), p_2^{(j)}(t))$, $j = 1, 2$, are linearly independent for every $t \in$

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$[a, b]$. Then the measure μ given by $\chi_{[a,b]}dt$ on the curve $(t, p_1(t), p_2(t))$ satisfies

$$\mu * L^{\frac{3}{2}}(\mathsf{R}^3) \subseteq L^2(\mathsf{R}^3). \tag{2}$$

This result, and its proof, were generalized in several papers, e.g., [12], [13], [5], where the main emphasis was the study of the curves

$$(t, t^\alpha, t^\beta) \tag{3}$$

with the measures $t^{(1+\alpha+\beta)/6-1}dt$. (The method of [7] is not the only one applicable to the curves (3)– see [9] and, in particular, [15] where the definitive result is obtained by modifying a homogeneity argument of Christ [3].)

If $\gamma(t)$ is a curve in R^3 , we will write $D(t)$ for the absolute value of the determinant of the matrix

$$\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma^{(3)}(t) \end{pmatrix}.$$

When $\gamma(t)$ is given by (3), a computation shows that, up to a constant, $D(t) = t^{1+\alpha+\beta-6}$. The convolution results for these curves lead to the conjecture that, under mild additional hypotheses, the measure μ given by $D^{1/6}(t)\,dt$ on the curve $\gamma(t)$ will satisfy (2). Theorem 2 below shows that this conjecture is true for curves $\gamma(t) = (t, p_1(t), p_2(t))$ when p_1 and p_2 are real-valued polynomials.

The recent papers [1] and [2] contain, among other interesting results, special cases of our Theorems 1 and 2 obtained by specializing to compact or homogenenous curves.

The remainder of this paper, then, is devoted to the proofs of the following results:

THEOREM 1. *Fix a positive integer N. There is a positive constant $C(N)$ such that if $p_1(t)$ and $p_2(t)$ are real-valued polynomials of degree not exceeding N and if μ is the measure on the curve $(p_1(t), p_2(t))$, $-\infty < t < \infty$, given by*

$$|\, p_1'(t)\, p_2''(t) - p_1''(t)\, p_2'(t)\,|^{\frac{1}{3}}\, dt,$$

then

$$\|\mu * f\|_{L^3(\mathsf{R}^2)} \leq C(N)\, \|f\|_{L^{\frac{3}{2}}(\mathsf{R}^2)}$$

for functions f on R^2.

THEOREM 2. *Suppose $p_1(t)$ and $p_2(t)$ are real-valued polynomials. Let μ be the measure on the curve $(t, p_1(t), p_2(t))$, $-\infty < t < \infty$, given by*

$$|p_1''(t)\, p_2^{(3)}(t) - p_1^{(3)}(t)\, p_2''(t)|^{\frac{1}{6}}\, dt.$$

Then there is a positive constant C such that

$$\|\mu * f\|_{L^2(\mathbb{R}^3)} \le C\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

for functions f on $\mathbb{R}^3$.

It seems likely that the convolution bound in Theorem 2 is, as in Theorem 1, a function only of the degrees of p_1 and p_2. A uniform version of Lemma 4 below would give this, but our current proof of that lemma does not seem to yield such an estimate.

The following lemma furnishes a Fourier transform estimate used in the proof of Theorem 1. It is an extension of the case $n = 2$ of Theorem 2 in [8] and we postpone its proof until after the proofs of our main results.

LEMMA 3. *Given $N = 2, 3, \ldots$ and $\lambda \in \mathbb{R}$ there is a constant $C(N, \lambda)$ such that if $s \in \mathbb{R}$ and if p and q are real-valued polynomials of degree not exceeding N, then we have*

$$\left| \int_a^b e^{ip(t)} |p''(t)|^{\frac{1}{2}+is} |q(t)|^{i\lambda s} dt \right| \le C(N, \lambda)(1 + |s|)^{\frac{1}{2}}$$

independently of $a, b \in \mathbb{R}$.

PROOF OF THEOREM 1. Let (a, b) be any interval on which both $p_1' p_2'' - p_1'' p_2'$ and
$p_1'' p_2^{(3)} - p_1^{(3)} p_2''$ are of constant sign. Write $\kappa(t)$ for $|(p_1' p_2'' - p_1'' p_2')(t)|$ and define

$$Tf(x_1, x_2) = \int_a^b f\big(x_1 - p_1(t), x_2 - p_2(t)\big)\, \kappa^{\frac{1}{3}}(t)\, dt.$$

It is enough to show that

$$\|Tf\|_3 \le C(N)\, \|f\|_{\frac{3}{2}}.$$

We will treat the case where the signs of $p_1' p_2'' - p_1'' p_2'$ and $p_1'' p_2^{(3)} - p_1^{(3)} p_2''$ are opposite. The other case is similar. Roughly following [5] (where, on p. 92, calculations similar to those which follow are done in more detail), we define an analytic family of operators by

$$T_z f(x_1, x_2) = \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_0^\infty \int_a^b f\big(x_1 - p_1(t) - u p_1''(t), x_2 - p_2(t) - u p_2''(t)\big) (\kappa(t))^{1+\frac{2z}{3}} dt\, |u|^z du.$$

Since T_{-1} is a multiple of T, it will suffice, by analytic interpolation, to observe that

(4) $$\|T_{is} f\|_\infty \le C(N)\, \|f\|_1$$

and

(5) $$\|T_{-\frac{3}{2}+is} f\|_2 \le C(N)\,(1+|s|)^{\frac{1}{2}}\, \|f\|_2.$$

To see (4), just observe that the absolute value of the Jacobian of the map

$$(t,u) \to (p_1(t), p_2(t)) + u(p_1''(t), p_2''(t))$$

is

$$\left|(p_1' p_2'' - p_1'' p_2') - u(p_1'' p_2^{(3)} - p_1^{(3)} p_2'')\right|$$

which, by our assumption on the signs of $p_1' p_2'' - p_1'' p_2'$ and $p_1'' p_2^{(3)} - p_1^{(3)} p_2''$, exceeds κ. For (5) we must estimate the Fourier transform of $T_{-3/2+is}$ at $\xi \in \mathbb{R}^2$. If we write $p(t) = \xi \cdot (p_1(t), p_2(t))$ and $q(t) = (p_1' p_2'' - p_1'' p_2')(t)$, then a well-known calculation shows that this Fourier transform is a multiple of

$$\int_a^b e^{ip(t)} |p''(t)|^{\frac{1}{2}-is} |q(t)|^{\frac{2is}{3}}\, dt.$$

This integral is controlled by Lemma 3, and so the proof of Theorem 1 is complete.

The proof of Theorem 2 is an adaptation of the proof in [7]. It depends on Theorem 1 and on Lemma 4 below. The proof of Lemma 4 is elementary but tedious, and we postpone it until the end of the paper.

LEMMA 4. *Suppose f and g are real-valued polynomials on $\mathbb{R}$. Define*

$$G(a,b) = (f'g'' - f''g')(a)(f'g'' - f''g')(b),$$

$$F(a,b) = \frac{(f(b)-f(a))(g'(b)-g'(a)) - (f'(b)-f'(a))(g(b)-g(a))}{(b-a)^2}$$

if $a, b \in \mathbb{R}$, $a \neq b$, and

$$F(a,a) = (f'g'' - f''g')(a).$$

Then there are a finite partition of $\mathbb{R}$ into a union of intervals I_j and a positive constant M such that

$$|G(a,b)|^{\frac{1}{2}} \le M |F(a,b)|$$

whenever a and b are both in the same I_j.

PROOF OF THEOREM 2. Fix polynomials p_1 and p_2, take $f = p_1'$ and $g = p_2'$, and let I_j be as in Lemma 4. If $I_j = [a, b]$, $\gamma(t) = (t, p_1(t), p_2(t))$, and $D(t) = |p_1''(t)p_2^{(3)}(t) - p_1^{(3)}(t)p_2''(t)|$, define

$$Tf(x) = \int_a^b f(x - \gamma(t))D^{\frac{1}{6}}(t)\,dt$$

for $x \in \mathsf{R}^3$ and functions f on R^3. It is enough to prove that T maps $L^{3/2}(\mathsf{R}^3)$ into $L^2(\mathsf{R}^3)$. We will do this by applying Theorem 1 in conjunction with the method of [7]. By the "method of T^*T", it is enough to show that, if S is the operator given by

$$Sf(x) = \int_a^b \int_a^b f(x - \gamma(t) + \gamma(s)\,)D^{\frac{1}{6}}(t)D^{\frac{1}{6}}(s)\,dt\,ds,$$

then S maps $L^{3/2}(\mathsf{R}^3)$ into $L^3(\mathsf{R}^3)$. Writing $x = (x_1, x')$ for $x \in \mathsf{R}^3 = \mathsf{R} \times \mathsf{R}^2$ and $\phi(t) = (p_1(t), p_2(t))$ and then changing variables leads to

$$Sf(x_1, x') = \int_{a-b}^{b-a} \int_{I_u} f(x_1 - u, x' - \phi(s+u) + \phi(s)\,)D^{\frac{1}{6}}(s+u)D^{\frac{1}{6}}(s)\,ds\,du,$$

where I_u is the appropriate subinterval of $[a - b, b - a]$. Writing

$$p_{1,u}(s) = p_1(s + u) - p_1(s)$$

and similarly for $p_{2,u}$, the conclusion of Lemma 4 shows that $|Sf(x)|$ is majorized by

$$Pf(x_1, x') = \int_{a-b}^{b-a}\int_{I_u} |f|\big(x_1 - u, x' - (p_{1,u}(s), p_{2,u}(s))\big)\big|(p_{1,u}'p_{2,u}'' - p_{1,u}''p_{2,u}')(s)\big|^{\frac{1}{3}}ds\,|u|^{-\frac{2}{3}}du.$$

For fixed x_1 and u, Theorem 1 shows that

$$\left\| \int_{I_u} |f|\big(x_1 - u, x' - (p_{1,u}(s), p_{2,u}(s))\big)\big|(p_{1,u}'p_{2,u}'' - p_{1,u}''p_{2,u}')(s)\big|^{\frac{1}{3}}ds \right\|_{3,x'}$$

is bounded by a constant times $\|f(x_1 - u, \cdot)\|_{3/2}$, and so

$$\|Pf\|_3 \le C \left\| \int_{a-b}^{b-a} \|f(x_1 - u, \cdot)\|_{3/2}|u|^{-\frac{2}{3}}du \right\|_{3,x_1}.$$

The boundedness of the one-dimensional Riesz potential of order $\frac{1}{3}$ as a mapping of $L^{3/2}(\mathsf{R})$ into $L^3(\mathsf{R})$ now completes the proof of Theorem 2.

The two lemmas which follow are used in the proof of Lemma 3. The first is Lemma 3 in [8].

LEMMA 5. *Fix a positive integer N. There are positive constants $K = K(N)$ and $L = L(N)$ such that if*

$$r(t) = \prod_{j=1}^{J_1}(t - a_j) \prod_{j=J_1+1}^{J_2} [(t - a_j)^2 + b_j{}^2]$$

is a monic polynomial of degree not exceeding N with the a_j's distinct and each b_j real, then there exists a collection $\{I_l\}_{l=1}^{L_1}$, with $L_1 \leq L$, of pairwise disjoint subintervals of R *satisfying*

$$\int_{\mathsf{R}\sim\cup I_l} \left|\frac{r'}{r}\right| \leq K$$

and such that for each l there are $C = C(l) \in (0, \infty)$, $j = j(l) \in \{1, 2, \dots, J_2\}$, and a nonnegative integer $n = n(l)$ with

$$\frac{C}{K}|t - a_j|^n \leq |r(t)| \leq CK|t - a_j|^n, \qquad t \in I_l,$$

and

$$\frac{1}{K|t - a_j|} \leq \left|\frac{r'}{r}\right| \leq \frac{K}{|t - a_j|}, \qquad t \in I_l.$$

LEMMA 6. *Given a positive integer N, there is a positive constant $C = C(N)$ such that if $p(t)$ is a real-valued polynomial of degree not exceeding N, then, for any $\rho > 1$, $K > 0$,*

$$\int_{\{K\leq|tp(t)|\leq\rho K\}} \frac{dt}{|t|} \leq C \cdot (\log(\rho) + 1).$$

PROOF OF LEMMA 6. Without loss of generality we can write

$$tp(t) = t^{l_1} \prod(t - a_j) \prod((t - b_j)^2 + c_j^2) \prod(t^2 + d_j^2) \doteq t^{l_1} \prod p_j(t)$$

where the number of factors p_j does not exceed N. Let C be a constant depending only on N, but which may not be the same at each occurrence. For nonnegative numbers A and B, we will write $A \sim B$ if $B/C \leq A \leq CB$. We begin by observing that for each p_j there is a partition

$$\mathsf{R} = I_j \cup \left(\bigcup_l I_j^l\right)$$

of R into at most ten intervals such that

$$\int_{I_j} \frac{dt}{|t|} \leq C$$

and such that on each I_l^j either $|p_j| \sim c_l^j$ for some positive constant c_l^j or $|p_j(t)| \sim |t|$ or $|p_j(t)| \sim t^2$. (For example, if $p_j(t) = t - a_j$ with $a_j > 0$, then

$$|p_j(t)| \sim |t| \quad \text{if } t \leq \frac{-a_j}{2}, \qquad |p_j(t)| \sim |a_j| \quad \text{if } \frac{-a_j}{2} \leq t \leq \frac{a_j}{2},$$

$$\int_{\frac{a_j}{2}}^{\frac{3a_j}{2}} \frac{dt}{|t|} \leq \log(3),$$

and

$$|p_j(t)| \sim |t| \text{ if } \frac{3a_j}{2} \leq t.)$$

It is a consequence of this observation that the complement of $\cup I_j$ can be represented as a union of at most C disjoint intervals J_l on each of which $|tp(t)| \sim c_l|t|^{n_l}$ for some positive c_l and some nonnegative integer n_l. Then

$$\int_{\{K \leq |tp(t)| \leq \rho K\} \cap J_l} \frac{dt}{|t|} \leq \int_{\left\{\frac{K}{(Cc_l)} \leq |t|^{n_l} \leq \frac{\rho K C}{c_l}\right\}} \frac{dt}{|t|} \leq C(\log(\rho) + 1).$$

Proof of Lemma 3. This is similar to, but more complicated than, the proof of Theorem 2 in [8]. We begin with some reductions: replacing q by a power of q shows that we can assume $0 < \lambda \leq 1$. It is clear that we may asssume that $q(t)$ is monic, and a scaling argument shows that we may assume $p'(t)$ to be monic. Then an approximation argument shows that it is enough to prove Lemma 3 under the additional hypothesis that both $r(t) \doteq p'(t)$ and $r(t) \doteq q(t)$ meet the other requirements of Lemma 5. Finally, it will suffice to show that the conclusion of Lemma 3 holds if p', p'', and

$$\left|\frac{p''}{(p')^2}\right| - \frac{1}{10(1+|s|)}$$

are of constant sign on $(a, b) \doteq I$.

Case 1: $\frac{1}{10(1+|s|)} \leq \left|\frac{p''}{(p')^2}\right|$ on I. The argument here is identical to that for Case II in the proof of Theorem 2 in [8].

Case 2: $\left|\frac{p''}{(p')^2}\right| \leq \frac{1}{10(1+|s|)}$ on I. After making the change of variables $u = p(t)$, we need to estimate an integral of the form

$$\int_J e^{i(u+2s\log|p'(p^{-1}(u))|+\lambda s\log|q(p^{-1}(u))|)} \left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}+is} du, \tag{6}$$

where $J = p(I)$. The derivative of the phase function is

$$1 + 2s\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2} + \lambda s\frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}. \tag{7}$$

For any subinterval J' of J we have

$$\int_{J'}\left|\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}+is}\right|du = 2\left|\tfrac{1}{2}+is\right|\int_{J'}\left|\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}}\right|du$$
$$\leq C(N)\left|\tfrac{1}{2}+is\right|\sup\left\{\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}} : u \in J'\right\} \leq C(N)(1+|s|)^{\frac{1}{2}}.$$

Here the first inequality follows from the fact that, since p is a polynomial of degree not exceeding N,

$$\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}}$$

will have at most $C(N)$ sign changes on J'. The second inequality is a consequence of the Case 2 assumption. It follows from a variant of van der Corput's lemma ([16], p. 334), that if J' is a subinterval of J on which the absolute value of (7) exceeds, say, $\frac{1}{10}$, then the part of (6) corresponding to J' is bounded by $C(N)(1+|s|)^{\frac{1}{2}}$. Since J is a union of at most $C(N)$ intervals on each of which either $|(7)| > \frac{1}{10}$ or $|(7)| \leq \frac{1}{10}$, it suffices to estimate (6) with J replaced by some J' on which $|(7)| \leq \frac{1}{10}$. From the Case 2 assumption it follows that then

$$\frac{7}{10|s|\lambda} \leq \left|\frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}\right| \leq \frac{13}{10|s|\lambda} \tag{8}$$

on J' and again that

$$\left|\frac{p''(p^{-1}(u))}{(p'(p^{-1}(u)))^2}\right| \leq \frac{1}{10(1+|s|)} \leq \frac{7}{10|s|\lambda} \leq \left|\frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}\right| \tag{9}$$

on J'. Now take $r = q$ in Lemma 5 and let the intervals I'_l be such that

$$\int_{\mathbb{R}\sim\cup I'_l}\left|\frac{q'}{q}\right| \leq C. \tag{10}$$

Let $I' = p^{-1}(J')$ so that on I' we have the inequalities

$$\frac{7}{10|s|\lambda} \leq \left|\frac{q'(t)}{q(t)p'(t)}\right| \leq \frac{13}{10|s|\lambda} \tag{8$'$}$$

and

$$\left|\frac{p''(t)}{(p'(t))^2}\right| \le \left|\frac{q'(t)}{q(t)p'(t)}\right| \tag{9'}$$

From (9′) and (8′) it follows that on I' we have

$$|p''| \le C|p'|\left|\frac{q'}{q}\right|, \qquad |p'| \le C|s|\left|\frac{q'}{q}\right|,$$

and so

$$|p''|^{\frac{1}{2}} \le C|s|^{\frac{1}{2}}\left|\frac{q'}{q}\right|. \tag{11}$$

Now (10) and (11) give

$$\int_{I'\sim\cup I'_l} |p''|^{\frac{1}{2}} \le C|s|^{\frac{1}{2}}.$$

On the other hand, on an I'_l we have, by Lemma 5,

$$\left|\frac{q'(t)}{q(t)}\right| \sim \frac{1}{|t-c|} \tag{12}$$

for some c. With (8′) this gives the inequalities

$$\frac{1}{C|s|\lambda} \le \frac{1}{|p'(t)||t-c|} \le \frac{C}{|s|\lambda}$$

on $I' \cap I'_l$. And with (11) and (12) this gives

$$\int_{I'\cap I'_l} |p''|^{\frac{1}{2}} \le C|s|^{\frac{1}{2}} \int_{I'\cap I'_l} \left|\frac{q'}{q}\right| \le C|s|^{\frac{1}{2}} \int_{\left\{\frac{1}{C|s|\lambda} \le \frac{1}{|(t-c)p'(t)|} \le \frac{C}{|s|\lambda}\right\}} \frac{dt}{|t-c|}.$$

Thus Lemma 6 completes the proof of Lemma 3.

Proof of Lemma 4. This is a consequence of the following two facts:

Sublemma A. *If $x_0 \in \mathrm{R}$ then there are $\delta > 0$ and $M < \infty$ such that the inequality*

$$|G(a,b)|^{\frac{1}{2}} \le M\,|F(a,b)| \tag{13}$$

holds if $a, b \in (x_0 - \delta, x_0)$ or if $a, b \in (x_0, x_0 + \delta)$.

Sublemma B. *There are positive constants P and M such that (13) holds if $a, b \ge P$ or $a, b \le -P$.*

PROOF OF SUBLEMMAS A AND B. Without loss of generality we will take $x_0 = 0$. Let n be the maximum of the degrees of f and g. Write

$$f(x) = \sum_{j=0}^{n} c_j x^j, \qquad g(x) = \sum_{j=0}^{n} d_j x^j.$$

Letting T_k stand for the sum

$$\sum_{l=0}^{k} a^{k-l} b^l,$$

we see that

$$\frac{f(b) - f(a)}{b - a} = \sum_{j=1}^{n} c_j T_{j-1} \qquad \text{and} \qquad \frac{g'(b) - g'(a)}{b - a} = \sum_{j=2}^{n} j d_j T_{j-2}.$$

With similar expressions for

$$\frac{f'(b) - f'(a)}{b - a} \qquad \text{and} \qquad \frac{g(b) - g(a)}{b - a}$$

this leads to

$$F(a, b) = \sum_{j_1=1,\ j_2=2}^{n,n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) j_2 T_{j_1-1} T_{j_2-2}. \tag{14}$$

Let $n(j_1, j_2, l)$ stand for the cardinality of the set

$$\big\{(l_1, l_2) : 0 \le l_1 \le j_1 - 1,\ 0 \le l_2 \le j_2 - 2,\ l_1 + l_2 = l\big\}.$$

Then the coefficient of $a^j b^l$ in (14) is

$$\sum_{\substack{j_1=1,\ j_2=2 \\ j_1+j_2-3=j+l}}^{n,n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) j_2 n(j_1, j_2, l) \doteq \sum_{\substack{j_1,\ j_2=1 \\ j_1<j_2 \\ j_1+j_2-3=j+l}}^{n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) m(j_1, j_2, l).$$

Thus

$$F(a, b) = \sum_{J\ge 0} \sum_{j+l=J} a^j b^l \sum_{\substack{j_1,\ j_2=1 \\ j_1<j_2 \\ j_1+j_2-3=J}}^{n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) m(j_1, j_2, l). \tag{15}$$

We will need to know that the term $m(j_1, j_2, l)$ is positive if $0 \le l \le j_1 + j_2 - 3$. Since $m(j_1, j_2, l) = j_2 n(j_1, j_2, l) - j_1 n(j_2, j_1, l)$ and $j_2 > j_1$, it is enough to check that $n(j_1, j_2, l) \ge n(j_2, j_1, l) > 0$. But, by definition,

$$n(j_1, j_2, l) = \left|\left\{(l_1, l_2) : l_1 + l_2 = l,\ 0 \le l_1 \le j_1 - 1,\ 0 \le l_2 \le j_2 - 2\right\}\right|$$

and so

$$n(j_2, j_1, l) = \left|\left\{(l_1, l_2) : l_1 + l_2 = l,\ 0 \le l_1 \le j_1 - 2,\ 0 \le l_2 \le j_2 - 1\right\}\right|.$$

A picture in the $l_1 l_2$-plane now shows that $n(j_1, j_2, l) \ge n(j_2, j_1, l) > 0$ as desired.

In addition to (15) we will use

$$(16) \qquad (f'g'' - f''g')(x) = \sum_{J \ge 0} x^J \sum_{\substack{j_1,\ j_2 = 1 \\ j_1 < j_2 \\ j_1 + j_2 - 3 = J}}^{n} j_1 j_2 (j_2 - j_1)(c_{j_1} d_{j_2} - d_{j_1} c_{j_2}).$$

We will also need the following fact:

SUBLEMMA C. *For some fixed J suppose that either*

$$(17) \qquad c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0 \qquad \textit{whenever} \quad j_1 + j_2 - 3 < J,$$

or

$$(18) \qquad c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0 \qquad \textit{whenever} \quad j_1 + j_2 - 3 > J.$$

Then there is at most one pair (j_1, j_2) with $1 \le j_1 < j_2 \le n$ and $j_1 + j_2 - 3 = J$ such that $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} \ne 0$.

PROOF OF SUBLEMMA C. Suppose that (17) holds (the proof under the hypothesis (18) is similar) and that $c_1 d_{J+2} - c_{J+2} d_1 \ne 0$. Then either $c_1 \ne 0$ or $d_1 \ne 0$. Without loss of generality, assume $c_1 \ne 0$. Suppose also that $1 < j_1 < j_2 \le n$ and that $j_1 + j_2 - 3 = J$. We will start by observing that $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0$. Since $1 + j_1 - 3 < 1 + j_2 - 3 < J$, we have $c_1 d_{j_2} - c_{j_2} d_1 = 0$ and $c_1 d_{j_1} - c_{j_1} d_1 = 0$ by assumption. Multiplying the first of these by c_{j_1} and the second by c_{j_2} and subtracting leads to $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0$ as desired. Thus our conclusion holds if $c_1 d_{J+2} - c_{J+2} d_1 \ne 0$. The next case, $c_2 d_{J+1} - c_{J+1} d_2 \ne 0$, and all subsequent cases, are handled similarly.

CONCLUSION OF PROOF OF SUBLEMMA A. Let J_1 be the first J such there are j_1 and j_2 with $j_1 + j_2 - 3 = J$ and $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} \ne 0$. Then, by (15) and

Sublemma C, there are j_1 and j_2 with $j_1 + j_2 - 3 = J_1$ and

$$F(a,b) = (c_{j_1}d_{j_2} - d_{j_1}c_{j_2}) \sum_{j+l=J_1} a^j b^l m(j_1, j_2, l) + O\Bigg(\sum_{j+l=J_1+1} a^j b^l \Bigg).$$

It follows from (16) that

$$(f'g'' - f''g')(x) = O(|x|^{J_1}).$$

Thus Sublemma A follows from the fact that the $m(j_1, j_2, l)$'s are positive along with the inequality

$$|ab|^{\frac{J_1}{2}} \leq \sum_{j+l=J_1} a^j b^l.$$

The proof of Sublemma B is similar, starting with the choice of J_1 as the greatest J such that there are j_1 and j_2 with $j_1 + j_2 - 3 = J$ and $c_{j_1}d_{j_2} - d_{j_1}c_{j_2} \neq 0$.

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