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2-GENERATOR ARITHMETIC KLEINIAN GROUPS III

M. D. E. CONDER, C. MACLACHLAN, G. J. MARTIN and E. A. O'BRIEN*

Abstract

This paper forms part of the program to identify all the 2-generator arithmetic Kleinian groups. Here we identify all conjugacy classes of such groups with one generator parabolic and the other generator elliptic. There are exactly 14 of these and exactly 5 Bianchi groups in their commensurability class, namely $\mathrm{PSL}(2, \mathcal{O}_d)$ for $d = 1, 2, 3, 7$ and 15. This complements our earlier identification of the 4 arithmetic Kleinian groups generated by two parabolic elements.

1. Introduction

In previous work [22], [10] we established the finiteness of the number of two-generator arithmetic Kleinian groups generated by a pair of elliptic or parabolic elements. Further, we found there are exactly 4 arithmetic Kleinian groups generated by two parabolic elements [10], which are all knot and link complements. Here we extend this result by identifying all the two-generator arithmetic Kleinian groups with one generator parabolic and the other generator elliptic.

There is a substantial literature on the topic of discrete groups generated by two parabolic elements, and in particular the question of when such groups are free. Numerical studies, particularly those of Riley [31], show that the space of all such groups (a one dimensional complex space) is very complicated. It consists of a "free" part with a highly fractal boundary and numerous isolated points clustering to this boundary. Among these points are the (infinitely many) hyperbolic 2-bridge knot and link complements. There is very little literature on the corresponding question of groups generated by parabolic and elliptic elements, or indeed other spaces of two-generator discrete groups; however see [8] and the recent innovative work of Gabai, Meyerhoff and Thurston [7].

There are infinitely many two-generator Kleinian groups of finite covolume with one generator parabolic and the other elliptic. For example, carrying out $(n, 0)$-Dehn filling on one component of a hyperbolic two-bridge link complement yields a hyperbolic orbifold for most values of n whose fundamental group is such a two-generator Kleinian group. If the group is to be arithmetic,

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then it must be commensurable with a Bianchi group $G_d = \mathrm{PSL}(2, O_d)$ where O_d is the ring of integers in a quadratic imaginary number field $\mathbb{Q}(\sqrt{-d}\,)$ for some square free positive integer d. An algorithm for determining the presentations of these Bianchi groups was developed by Swan [35].

Our method is to combine an elementary bound on the size of the relevant space of free products with some number theory to give a finite number of candidates for arithmetic Kleinian groups generated by a parabolic and an elliptic. In each candidate we identify a finite index subgroup which is a subgroup of a Bianchi group. This subgroup is identified in terms of a set of generators expressed as words in the generators of the ambient Bianchi group. The remaining problem is to decide whether or not this subgroup has finite index in the Bianchi group. To do this we use some computational group theoretic methods. The methods used to prove that (in many cases) the subgroups are of infinite index do not seem to be well-known but are clearly of importance for work in this area and may be of wider interest. The discussion of these is given independently in §7.

In stating the main theorem below, each of the 14 arithmetic groups which arises is described either by its relationship to the corresponding Bianchi group, or by a description of the related orbifold, or, in most cases, by both. The orbifold descriptions are given in terms of the two-bridge knot and link complements where we use the standard notation (p/q) for both the link complement and the corresponding group.

THEOREM 1.1. *Suppose that* $G = \langle f, g\rangle$ *is an arithmetic Kleinian group with* f *parabolic and* g *elliptic. Then the order of* g *is one of* 2, 3, 4, *or* 6 *and there are fourteen such groups.*

- *If g has order* 2*, then there are six groups:*
  a. *Two* $\mathbb{Z}_2$ *extensions of* $(5/3)$ *each with index* 6 *in* G_3*. One orbifold is obtained by* $(2, 0)$*-filling a component of* $(10/3)$*. The other is not a surgery on a link complement and is described below.*
  b. *A* $\mathbb{Z}_2$ *extension of* $(8/3)$ *with* $G \cap G_1$ *of index 2 in* G *and 12 in* G_1*. The orbifold is obtained by* $(2, 0)$*-filling a component of* $(16/5)$.
  c. *Two* $\mathbb{Z}_2$ *extensions of* $(10/3)$ *and for both* $G \cap G_3$ *has index 2 in* G *and 24 in* G_3*. One orbifold is obtained by* $(2, 0)$*-filling a component of* $(24/7)$*. The other by* $(2, 0)$*-filling a component of* $(21/5)$.
  d. *A* $\mathbb{Z}_2$ *extension of* $(12/5)$ *and* $G \cap G_7$ *has index 2 in* G *and 12 in* G_7*. The orbifold is obtained by* $(2, 0)$*-filling a component of* $(20/7)$.
- *If g has order* 3*, then there are three groups:*
  a. $[G_1 : G] = 8$ *and the orbifold is obtained by* $(3, 0)$*-filling a component of* $(10/3)$.
  b. $G = G_3$.

c. $[G_7 : G] = 2$ *and the orbifold is obtained by* $(3, 0)$*-filling a component of* $(8/3)$.

- *If g has order* 4*, then there are three groups:*
  a. $G = \mathrm{PGL}(2, O_1)$.
  b. $G \cap G_2$ *is of index 4 in G and 24 in* G_2 *and the orbifold is obtained by* $(4, 0)$*-filling a component of* $(24/7)$.
  c. $G \cap G_3$ *is of index 2 in G and 30 in* G_3 *and the orbifold is obtained by* $(4, 0)$*-filling a component of* $(8/3)$.
- *If g has order* 6*, then there are two groups:*
  a. $G = \mathrm{PGL}(2, O_3)$.
  b. $G \cap G_{15}$ *has index 6 in G and 6 in* G_{15} *and the orbifold is obtained by* $(6, 0)$*-filling a component of* $(8/3)$.

REMARK. Recall that $(5/3)$ is the figure-**8** knot complement and $(8/3)$ the Whitehead link. Arithmetic groups obtained by $(n, 0)$-filling a component of the Whitehead link were discussed in [25]. From Rolfsen's tables [32] of two bridge links we find $(24/7)$ is 8^2_4, $(16/5)$ is 8^2_2, $(20/7)$ is 9^2_1 and $(21/5)$ is 9^2_4.

2. Kleinian groups and arithmeticity

We begin with a few basic definitions and some notation. A *Kleinian group* is a discrete nonelementary subgroup of isometries of hyperbolic 3-space $\mathbb{H}^3$. (In this setting *nonelementary* means the group is not virtually abelian). Such groups are identified with (the Poincaré extensions of) discrete groups of Möbius or conformal transformations of the Riemann sphere $\overline{\mathbb{C}}$. The orbit spaces of Kleinian groups are the *hyperbolic 3-orbifolds* or, if the Kleinian group is torsion free, *hyperbolic 3-manifolds*. We use [1], [24], [27] and [37] as basic references for the theory of discrete groups and hyperbolic spaces.

The elements of a Kleinian group, other than the identity, are either *loxodromic* (conjugate to $z \mapsto \lambda z$, $|\lambda| \neq 1$), *elliptic* (conjugate to $z \mapsto \lambda z$, $|\lambda| = 1$) or *parabolic* (conjugate to $z \mapsto z + 1$).

We associate with each Möbius transformation

$$f = \frac{az + b}{cz + d}, \qquad ad - bc = 1, \tag{1}$$

the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \tag{2}$$

and set $\mathrm{tr}(f) = \mathrm{tr}(X)$ where $\mathrm{tr}(X)$ denotes the trace of the matrix X. For each pair of Möbius transformations f and g we let $[f, g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$(3) \quad \beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4, \quad \gamma(f, g) = \mathrm{tr}([f, g]) - 2$$

the *parameters* of the two-generator group $\langle f, g\rangle$ and write

$$\mathrm{par}(\langle f, g\rangle) = (\gamma(f, g), \beta(f), \beta(g)). \tag{4}$$

These parameters are independent of the choice of matrix representatives for f and g in $\mathrm{SL}(2, \mathbb{C})$ and they determine $\langle f, g\rangle$ uniquely up to conjugacy whenever $\gamma(f, g) \neq 0$. If f is parabolic, then $\beta(f) = 0$; if g is elliptic, some power of g is primitive, and so we assume that $\beta(g) = -4\sin^2(\pi/n)$ where n is the order of g. Thus if $G = \langle f, g\rangle$ is a Kleinian group generated by a parabolic element and an elliptic element of order n, we have

$$\mathrm{par}(G) = (\gamma, 0, -4\sin^2(\pi/n)) \tag{5}$$

Thus, up to conjugacy, the space of all such discrete groups is determined uniquely by the one complex parameter $\gamma(f, g)$.

Note that when $n = 2$ the subgroup $\langle f, gfg\rangle$ is generated by a pair of parabolics and has parameters $(\gamma(f, g)^2, 0, 0)$ and is of index 2 in $\langle f, g\rangle$.

We recall some further notation and basic results from [11]. Let G be a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The *trace field* of G is the field generated over $\mathbb{Q}$ by the set $\mathrm{tr}(G) = \{\pm\,\mathrm{tr}(g) : g \in G\}$. Since G is finitely generated, the subgroup $G^{(2)} = \langle g^2 : g \in G\rangle$ is a normal subgroup of finite index with quotient group a finite abelian 2-group. Following [25] we call

$$kG = \mathbb{Q}(\mathrm{tr}(G^{(2)})) \tag{6}$$

the *invariant trace-field* of G. For any finite index subgroup G_1 of a nonelementary group G one can show that $\mathbb{Q}(\mathrm{tr}(G^{(2)})) \subset \mathbb{Q}(\mathrm{tr}(G_1))$; in [29] it is shown that kG is an invariant of the commensurability class. Furthermore

$$AG^{(2)} = \left\{\sum a_i\gamma_i \mid \gamma_i \in G^{(2)}, a_i \in kG\right\}$$

is a quaternion algebra which is also an invariant of the commensurability class of G [25], termed the *invariant quaternion algebra*.

We next recall some facts about quaternion algebras; see [38] for details. Let k be a number field, let ν be a place of k, i.e. an equivalence class of valuations on k and denote by k_ν the completion of k at ν. If B is a quaternion algebra over k, we say that B is *ramified* at ν if $B \otimes_k k_\nu$ is a division algebra of quaternions. Otherwise B is unramified at ν.

If ν is a place associated to a real embedding of k, B is ramified if and only if $B \otimes_k k_\nu \cong \mathscr{H}$, where $\mathscr{H}$ is the Hamiltonian division algebra of quaternions.

We now give the definition of an *arithmetic* Kleinian group. Let k be a number field with one complex place and A a quaternion algebra over k ramified at all real places. Let ρ be an embedding of A into $M(2, \mathbb{C})$, $\mathscr{O}$ an order of A, and $\mathscr{O}^1$ the elements of norm 1 in $\mathscr{O}$. Then $\rho(\mathscr{O}^1)$ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ and its projection, $P\rho(\mathscr{O}^1)$, to $\mathrm{PSL}(2, \mathbb{C})$ is an arithmetic Kleinian group. The commensurability classes of arithmetic Kleinian groups are obtained by considering all such $P\rho(\mathscr{O}^1)$, see [2] for further details.

In [23] it is shown that two arithmetic Kleinian groups are commensurable up to conjugacy if and only if their invariant quaternion algebras are isomorphic; see also [2]. We recall the following from [11].

THEOREM 2.1. *Let G be a finitely generated non-elementary subgroup of the group* $\mathrm{PSL}(2, \mathbb{C})$ *such that*

(1) *kG has exactly one complex place;*

(2) $\mathrm{tr}(G)$ *consists of algebraic integers;*

(3) *$AG^{(2)}$ is ramified at all real places of kG.*

Then G is a subgroup of an arithmetic Kleinian group.

Following [22] we define a Kleinian group G to be *nearly arithmetic* if G is a Kleinian subgroup of an arithmetic Kleinian group and G does not split as a nontrivial free product. Of course, an arithmetic Kleinian group is nearly arithmetic. We note the following well known result.

THEOREM 2.2. *If G is an arithmetic Kleinian group which contains a parabolic element, then G is commensurable with a Bianchi group. In particular the invariant trace field is a complex quadratic extension of* $\mathbb{Q}$.

3. Two-generator groups

Next we specialize to the case where G is a two-generator group with one generator, f, parabolic and the other generator, g, elliptic. Here both the invariant field and the invariant quaternion algebra are readily described in terms of the parameters of the group.

It is shown in [29] that the field kG coincides with the field

$$\mathbb{Q}(\{\mathrm{tr}^2(g) : g \in G\}) = \mathbb{Q}(\{\beta(g) : g \in G\}).$$

See also [17]. For two-generator groups this together with Theorem 2.2 has the following consequence, established in [10].

THEOREM 3.1. *Let $G = \langle f, g \rangle$ be a nearly arithmetic Kleinian group with f parabolic and g either parabolic or elliptic of order n. Then $n = 2, 3, 4, 6$. Further, $\gamma = \gamma(f, g)$ is an algebraic integer. If γ is complex, then $kG = \mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt{-d})$. The field kG is real if and only if $n = 2$ and $\gamma \in \mathbb{Z}$. In this case, $kG = \mathbb{Q}$ and G contains a Fuchsian subgroup of index 2 which is a free product of cyclic groups. If $n \neq 2$ and γ is real, then γ is a negative integer and $kG = \mathbb{Q}(\sqrt{t_n \gamma})$ where $t_n = 1, 2, 3, 4$ for $n = 3, 4, 6, \infty$ (when g is parabolic) respectively.*

The converse of this result is almost true in the sense that a group G with parameters $(\gamma, 0, -4\sin^2(\pi/n))$, $n = 2, 3, 4, 6$ is a Kleinian subgroup of an arithmetic group whenever $\gamma \neq 0$ is a rational or quadratic integer, see [8], [11]. However, as we will see, it is most often true that G splits as a free product of cyclic groups and so is not nearly arithmetic.

We now give a fairly general criterion to determine when a group generated by a parabolic element and an elliptic element of order 3, 4, 6 is discrete and free on its two generators. It extends earlier results of [9] and [21].

Given a closed and bounded set $\Omega \subset \mathbb{C}$ we define the maximal horizontal width, $\delta(\Omega)$, of Ω to be the maximum of the distances of pairs of points in Ω with the same imaginary part; that is

$$\delta(\Omega) = \max\{|z - w| : z, w \in \Omega, \Im(z) = \Im(w)\}. \tag{7}$$

Let Ω consist of two discs of the same radii which overlap. It is a simple geometric exercise to show that the maximal horizontal width is either the diameter of a disc or is achieved by the horizontal line through the mid-point of the line joining the centres of the discs.

LEMMA 3.2. *Let $0 \le \lambda \le 2$ and $\omega = x + iy \in \mathbb{C}$ with $x, y \ge 0$. Let Ω be the region bounded by the two circles*

$$|z| < 1/|\omega|, \qquad |z + \lambda/\omega| < 1/|\omega|. \tag{8}$$

Then $\delta(\Omega) \le 1$ if and only if $|w| \ge 2$ and $|w - \lambda| \ge 2$.

PROOF. The part of the horizontal line through the mid-point of the line joining the centres of these circles which lies inside the circles has length

$$|w|^{-2}\big(\sqrt{(4|w|^2 - \lambda^2 y^2)} + \lambda x\big).$$

The result follows a simple calculation.

THEOREM 3.3. *Let f be parabolic and g elliptic of order $n \ge 2$ and suppose that $G = \langle f, g \rangle$ is non-elementary. Then G is conjugate to the subgroup of*

$\mathrm{PSL}(2, \mathbb{C})$ *generated by the images of the two matrices*

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -1/\omega \\ \omega & \lambda_n \end{pmatrix} \tag{9}$$

where $\lambda_n = 2\cos(\pi/n)$. *If* $\Re(\omega) \geq 0$ *and*

$$|\omega| \geq 2 \qquad \text{and} \qquad |\omega - \lambda_n| \geq 2 \tag{10}$$

then G *is discrete and* $G \cong \mathbb{Z} * \mathbb{Z}_n$ *splits as a nontrivial free product of cyclic groups.*

PROOF. We can conjugate G so that $f(z) = z + 1$ [1]. Since G is non-elementary we must have $g(\infty) \neq \infty$. We then may conjugate G by a translation commuting with f so that $g(\infty) = 0$. Since $g(z) = (az + b)/(cz + d)$ with $ad - bc = 1$, $\mathrm{tr}(g) = \pm\lambda_n$ and $g(\infty) = a/c = 0$, a matrix representative for g in $\mathrm{PSL}(2, \mathbb{C})$ is completely determined and has the indicated form. Next we note that the isometric circles of g are the two circles

$$|\omega z + \lambda_n| = 1, \qquad |\omega z| = 1. \tag{11}$$

By the above lemma, the maximum horizontal width of the region bounded by these two isometric circles is at most 1. Therefore this region lies inside a family of horizontal segments $\{I_y\}_{y \in \mathbb{R}}$ of width 1. Such a family of horizontal strips forms a fundamental domain for the action of $\langle f \rangle$ on $\mathbb{C}$. The exterior of the isometric circles of g are a fundamental domain for the action of $\langle g \rangle$ on $\overline{\mathbb{C}}$. The Klein combination theorem [24] now implies that the group generated by f and g is discrete and isomorphic to the free product of the cyclic groups $\langle f \rangle$ and $\langle g \rangle$.

4. Candidates

Here we discuss the possible values for ω such that the group $\langle f, g \rangle$ is arithmetic, f parabolic and g elliptic of order n. Theorem 3.1 implies $n \in \{2, 3, 4, 6\}$. Here we normalise so that

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -1/\omega \\ \omega & 2\cos(\pi/n) \end{pmatrix} \tag{12}$$

Further it is an easy matter to see that we can restrict our attention to those values of ω lying in the positive quadrant $\Re(\omega) \geq 0$ and $\Im(\omega) \geq 0$.

If $G = \langle f, g \rangle$ is an arithmetic Kleinian group (where f and g are the images in $\mathrm{PSL}(2, \mathbb{C})$ of X and Y above), then kG is a quadratic imaginary field and all traces of elements of G are algebraic integers.

Then

$$kG = \mathbb{Q}\big(\mathrm{tr}^2(X), \mathrm{tr}^2(Y), \mathrm{tr}(X)\,\mathrm{tr}(Y)\,\mathrm{tr}(XY)\big) \tag{13}$$

see [22]. In particular the elements X^2Y^2, $[X, Y]$ lie in $G^{(2)}$ and so their traces are integers in kG. That is, with the notation above,

$$\mathrm{tr}(X^2Y^2) = 2\big(2\omega\cos(\pi/n) + \cos(2\pi/n)\big) \quad \text{and} \quad \mathrm{tr}[X, Y] = 2 + \omega^2 \tag{14}$$

are integers in $kG = \mathbb{Q}(\sqrt{-d})$ for some positive square free integer d.

4.1. *f parabolic, g elliptic of order 2*

This case is basically covered by the results of [10]. If f is parabolic and g is elliptic of order 2, then the group $\langle f, g\rangle$ has the index 2 subgroup $\langle f, h\rangle$, with $h = gfg^{-1}$ generated by two parabolics. Both groups must be simultaneously arithmetic. In [10] we showed there are 4 arithmetic Kleinian groups generated by two parabolics. These are the four two bridge knots and links (5/3), (8, 3), (10/3) and (12/5). As explained in [8] there are at most two such $\mathbb{Z}_2$ extensions since $\gamma(f, g)^2 = \gamma(f, h)$. The values of $\gamma(f, h)$ are given in [10] and we deduce the two possible values for $\gamma(f, g)$. The element g of order 2 must appear in the symmetry group of the knot or link and conjugate one parabolic generator to the other (or its inverse). The symmetry group can be found using SNAPPEA [39] as well as a description of the action on the cusps. Checking the various possibilities yields our list which is subsequently easily verified. One of the symmetries of the figure-**8** has fixed point set meeting the knot and thus the orbifold is not surgery on a link complement. This symmetry (as well those in the other cases) can be seen in most drawings of the knot, see eg. [32], and the associated orbifold described accordingly.

4.2. *f parabolic, g elliptic of order 3*

Here $2\cos(\pi/n) = 1$ and $\mathrm{tr}(XY) = 1 + w$. Thus from (13), $kG = \mathbb{Q}(w) = \mathbb{Q}\big(\sqrt{-d}\,\big)$ for some square free positive d. Note that $\mathrm{tr}(X^2Y^2) = -1 + 2w$ and $\mathrm{tr}[X, Y] = 2 + w^2$. Thus w is an algebraic integer such that $2w \in \mathbb{Q}\big(\sqrt{-d}\,\big)$. Since number fields are integrally closed, it follows that w is an algebraic integer in $\mathcal{O}_d$, the ring of integers in $\mathbb{Q}\big(\sqrt{-d}\,\big)$. We thus need to find those w such that at least one of the inequalities $|w| < 2$, $|w - 1| < 2$ holds. See Table 1.

4.3. *f parabolic, g elliptic of order 4*

In this case, $\mathrm{tr}(XY) = w + \sqrt{2}$, $\mathrm{tr}(X^2Y^2) = 2\sqrt{2}w$, $\mathrm{tr}[X, Y] = 2 + w^2$. Thus w is an algebraic integer such that $2\sqrt{2}w \in \mathcal{O}_d$. Thus $\sqrt{2}w \in \mathbb{Q}\big(\sqrt{-d}\,\big)$ and so as before $\sqrt{2}w \in \mathcal{O}_d$. Also $w^2 \in \mathcal{O}_d$.

TABLE 1. Possible ω values when $|g| = 3$.

d	ω
-1	$i, 1+i, 2+i$
-2	$i\sqrt{2}, 1+i\sqrt{2}, 2+i\sqrt{2}$
-3	$(1+i\sqrt{3})/2, (3+i\sqrt{3})/2, (5+i\sqrt{3})/2, i\sqrt{3}, 1+i\sqrt{3}$
-7	$(1+i\sqrt{7})/2, (3+i\sqrt{7})/2$
-11	$(1+i\sqrt{11})/2, (3+i\sqrt{11})/2$

LEMMA 4.1. *If $\sqrt{2}w, w^2 \in \mathcal{O}_d$, then*

- $w = \frac{x+y\sqrt{-d}}{\sqrt{2}}$ *where $x, y \in \mathbb{Z}$ and $x \equiv y \pmod 2$ if $d \equiv 1 \pmod 4$.*
- $w = \sqrt{2}x + y\sqrt{\frac{-d}{2}}$ *where $x, y \in \mathbb{Z}$ if $d \equiv 2 \pmod 4$.*
- $w = \frac{x+y\sqrt{-d}}{\sqrt{2}}$ *where $x, y \in \mathbb{Z}$ and $x \equiv y \pmod 2$ if $d \equiv 3 \pmod 4$.*

PROOF. If $d \not\equiv 3 \pmod 4$ then $\sqrt{2}w = a + b\sqrt{-d}$. Then $w^2 = \big(a^2 - db^2 + 2ab\sqrt{-d}\big)/2$.

(i) $d \equiv 1 \pmod 4$. Since w^2 is an integer we must have $a \equiv b \pmod 2$ and hence w has the given form.

(ii) $d \equiv 2 \pmod 4$. Then $a \equiv 0 \pmod 2$ and again w has the given form.

(iii) $d \equiv 3 \pmod 4$. Then $\sqrt{2}w = \frac{a+b\sqrt{-d}}{2}$ where a, b have the same parity. But then

$$w^2 = \frac{\frac{a^2-db^2}{4} + \frac{ab}{2}\sqrt{-d}}{2}.$$

If w^2 is an integer, then either a or b must be even and hence they must both be even. But then w^2 has the form $\big((a^2 - db^2) + 2ab\sqrt{-d}\big)/2$. If that is to be an integer as well, then this new a, b must have the same parity. The result now follows.

TABLE 2. Possible ω values when $|g| = 4$.

d	ω
-1	$(1+i)/\sqrt{2}, (3+i)/\sqrt{2}, i\sqrt{2}, \sqrt{2}+i\sqrt{2}$
-2	$i, \sqrt{2}+i, 2\sqrt{2}+i$
-3	$(1+i\sqrt{3})/\sqrt{2}, (3+i\sqrt{3})/\sqrt{2}$
-5	$(1+i\sqrt{5})/\sqrt{2}, (3+i\sqrt{5})/\sqrt{2}$
-6	$i\sqrt{3}, \sqrt{2}+i\sqrt{3}$

In this case we need to find all w as described in the above lemma such that at least one of the inequalities $|w| < 2$, $\left|w - \sqrt{2}\right| < 2$ holds. See Table 2.

4.4. f parabolic, g elliptic of order 6

In this case, $\mathrm{tr}(XY) = w+\sqrt{3}$, $\mathrm{tr}(X^2Y^2) = 1+2\sqrt{3}w$, and $\mathrm{tr}[X, Y] = 2+w^2$. Thus w is an algebraic integer such that $2\sqrt{3}w$ and $w^2 \in \mathcal{O}_d$. Thus $\sqrt{3}w \in \mathcal{O}_d$. An analysis, similar to the last lemma, gives the following result.

LEMMA 4.2. *Let $\sqrt{3}w$, $w^2 \in \mathcal{O}_d$. Then*

- *for all d, we have $w = \sqrt{3}v$, where $v \in \mathcal{O}_d$.*
- *if $d \equiv 0 \pmod 3$ we have $w = \left(\sqrt{3}x + y\sqrt{-d/3}\right)/2$ where $x, y \in \mathbb{Z}$ with $x \equiv y \pmod 2$ and $x \equiv y \equiv 0 \pmod 2$ if $d \equiv 1, 2 \pmod 4$.*

In this case we need to find all w as described in the above lemma such that at least one of the inequalities $|w| < 2$, $|w - \sqrt{3}| < 2$ holds. See Table 3.

TABLE 3. Possible ω values when $|g| = 6$.

d	ω
-1	$i\sqrt{3}$, $\sqrt{3}+i\sqrt{3}$
-3	i, $\sqrt{3}+i$, $(\sqrt{3}+i)/2$, $(\sqrt{3}+i3)/2$, $(3\sqrt{3}+i)/2$
-6	$i\sqrt{2}$, $\sqrt{3}+i\sqrt{2}$
-15	$(\sqrt{3}+i\sqrt{5})/2$, $(3\sqrt{3}+i\sqrt{5})/2$

5. Bianchi Groups

The candidate groups given in Tables 1 to 3 show that we need to consider the eight Bianchi groups G_d for $d = 1, 2, 3, 5, 6, 7, 11, 15$. To further study these, we use the presentations of these groups in terms of the images of convenient matrices [13], [35].

For all groups, let a, b, c be the images of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

where $\tau = \left(1 + \sqrt{-d}\right)/2$ or $\sqrt{-d}$ according as $d \equiv 3 \pmod 4$ or not.

$d = 1$

$$\langle a, b, c \mid b^2 = (ab)^3 = [a, c] = (c^2bc^{-1}b)^2 = (cbc^{-1}bcb)^2 = (acbc^{-1}bcb)^2 = 1\rangle.$$

$d = 2$

$$\langle a, b, c \mid b^2 = (ab)^3 = [a, c] = [b, c]^2 = 1\rangle.$$

$d = 3$

$$\begin{aligned}\langle a, b, c | b^2 = (ab)^3 = [a, c] = (cbc^{-1}ac^{-1}ab)^2 = (cbc^{-1}ab)^3 \\ = a^{-1}c^{-1}ba^{-1}cbac^{-1}bac^{-1}ba^{-1}cb = 1\rangle.\end{aligned}$$

$d = 5$

$$\begin{aligned}\langle a, b, c, e_1, e_2 | b^2 = (ab)^3 = [a, c] = e_2^2 = (be_2)^2 = (bce_2c^{-1})^2 \\ = be_1bae_1^{-1}a^{-1} = ce_2c^{-1}e_1e_2ae_1^{-1}a^{-1} = 1\rangle\end{aligned}$$

where e_1, e_2 are the images of the matrices

$$E_1 = \begin{pmatrix} 4 + i\sqrt{5} & 2i\sqrt{5} \\ -2i\sqrt{5} & 4 - i\sqrt{5} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} -i\sqrt{5} & 2 \\ 2 & i\sqrt{5} \end{pmatrix}.$$

$d = 6$

$$\begin{aligned}\langle a, b, c, e_1, e_2 | b^2 = (ab)^3 = [a, c] = e_2^2 = [b, e_1] = (bae_2)^3 \\ = (bace_2c^{-1})^3 = a^{-1}e_1ace_2c^{-1}e_1^{-1}e_2 = 1\rangle\end{aligned}$$

where e_1, e_2 are the images of the matrices

$$E_1 = \begin{pmatrix} 5 & -2i\sqrt{6} \\ 2i\sqrt{6} & 5 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} -1 - i\sqrt{6} & 2 - i\sqrt{6} \\ 2 & 1 + i\sqrt{6} \end{pmatrix}.$$

$d = 7$

$$\langle a, b, c | b^2 = (ab)^3 = [a, c] = (bac^{-1}bc)^2 = 1\rangle.$$

$d = 11$

$$\langle a, b, c | b^2 = (ab)^3 = [a, c] = (bac^{-1}bc)^3 = 1\rangle.$$

$d = 15$

$$\langle a, b, c, e | b^2 = (ab)^3 = [a, c] = [b, e] = cecbac^{-1}e^{-1}c^{-1}ba^{-1} = 1\rangle$$

where e is the image of the matrix

$$E = \begin{pmatrix} 4 & -i\sqrt{15} \\ i\sqrt{15} & 4 \end{pmatrix}.$$

6. Subgroups

Now for each possible value of ω in Tables 1–3 we want to identify a subgroup of a Bianchi group which is commensurable with the appropriate two-generator subgroup. For $n = 3$ all the groups in question are easily identifiable subgroups of Bianchi groups. This is not so straightforward for elliptics of order 4 and 6 since these elements do not lie in the Bianchi groups. We overcome this problem by looking at the finite index subgroup $\langle f, gfg^{-1}, g^2fg^{-2}, g^{-1}fg\rangle$ when g has order 4 and the finite index subgroup $\langle f, gfg^{-1}, g^2fg^{-2}, g^3fg^{-3}, g^{-2}fg^2, g^{-1}fg\rangle$ when g has order 6. This procedure is straightforward for the Euclidean Bianchi groups (those where O_d has a Euclidean Algorithm). However the identification of these matrices in the Bianchi groups which are not Euclidean is quite tricky. We were reduced to solving certain systems of nonlinear equations to identify various conjugates. Of course, once the matrices are in hand, it is a trivial matter to verify their correctness.

TABLE 4. Parabolic and elliptic of order 3.

d	ω	Subgroup generators	I
-1	i	ba, c	∞
-1	$1+i$	ba, ac	8
-1	$2+i$	ba, a^2c	∞
-2	$i\sqrt{2}$	ba, c	∞
-2	$1+i\sqrt{2}$	ba, ac	∞
-2	$2+i\sqrt{2}$	ba, a^2c	∞
-3	$(1+i\sqrt{3})/2$	ba, c	1
-3	$(3+i\sqrt{3})/2$	ba, ac	∞
-3	$(5+i\sqrt{3})/2$	ba, a^2c	∞
-3	$i\sqrt{3}$	ba, ac^2a^{-2}	∞
-3	$1+i\sqrt{3}$	ba, ac^2a^{-1}	∞
-7	$(1+i\sqrt{7})/2$	ba, c	2
-7	$(3+i\sqrt{7})/2$	ba, ac	∞
-11	$(1+i\sqrt{11})/2$	ba, c	∞
-11	$(3+i\sqrt{11})/2$	ba, ac	∞

At this point the only obstruction to proving that the groups we are considering are arithmetic is showing they have finite covolume. Equivalently, we may show that the finite index subgroup of our two-generator group which lies in a Bianchi group has finite index in that Bianchi group. In each table below we give the appropriate subgroup of our two-generator group and its index in the respective Bianchi group. How this index is derived is discussed in the next section.

6.1. Parabolic and order 3

A conjugation by a diagonal matrix of the representatives given at (12) allows us to assume the matrix representatives have the form

$$X = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos(\pi/n) \end{pmatrix} \tag{15}$$

The subgroup $\langle f, g\rangle$ is identified with a subgroup of the Bianchi group in terms of the generators of the Bianchi group given in §5. The generators of the subgroup and its index I are given in Table 4.

6.2. Parabolic and order 4

Returning to the representation at (12) the group generated by four parabolics $\langle f, gfg^{-1}, g^2fg^{-2}, g^{-1}fg\rangle$ is the group generated by the images of the following matrices:

$$X_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 & 0 \\ -\omega^2 & 1 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1+\sqrt{2}\omega & 1 \\ -2\omega^2 & 1-\sqrt{2}\omega \end{pmatrix} \qquad X_4 = \begin{pmatrix} 1+\sqrt{2}\omega & 2 \\ -\omega^2 & 1-\sqrt{2}\omega \end{pmatrix}$$

This subgroup is identified as a subgroup of the corresponding Bianchi group in Table 5.

6.3. Parabolic and order 6

In this case we consider the group generated by the images of the following six parabolic matrices:

$$X_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 & 0 \\ -\omega^2 & 1 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 1+\sqrt{3}\omega & 1 \\ -3\omega^2 & 1-\sqrt{3}\omega \end{pmatrix} \qquad X_4 = \begin{pmatrix} 1+2\sqrt{3}\omega & 3 \\ -4\omega^2 & 1-2\sqrt{3}\omega \end{pmatrix}$$

$$X_5 = \begin{pmatrix} 1+2\sqrt{3}\omega & 4 \\ -3\omega^2 & 1-2\sqrt{3}\omega \end{pmatrix} \qquad X_6 = \begin{pmatrix} 1+\sqrt{3}\omega & 3 \\ -\omega^2 & 1-\sqrt{3}\omega \end{pmatrix}$$

TABLE 5. Groups generated by 4 parabolics.

d	ω	Subgroup generators	I
-1	$\frac{1+i}{\sqrt{2}}$	$a, bcb^{-1}, a^{-1}cbcb^{-1}c^{-1}a, a^{-1}cba^{-1}c^{-1}babcabc^{-1}a$	1
-1	$\frac{3+i}{\sqrt{2}}$	$a, ba^4c^3b^{-1}, ba^3cb^{-1}ab^{-1}a^{-3}c^{-1}b, a^{-1}ba^{-2}cbcb^{-1}a^2c^{-1}b^{-1}a$	∞
-1	$i\sqrt{2}$	$a, ba^{-2}b^{-1}, cbc^{-2}babc^2bc^{-1}, cba^{-2}b^{-1}c^{-1}$	∞
-1	$\sqrt{2}+i\sqrt{2}$	$a, bc^4b^{-1}, b^{-1}c^2ba^{-1}b^{-1}c^{-2}a^{-4}b^{-1}, cba^2bc^{-1}b^{-1}a^{-2}b$	∞
-2	i	$a, ba^{-1}b^{-1}, cbc^{-1}babcbc^{-1}, cba^{-1}b^{-1}c^{-1}$	∞
-2	$\sqrt{2}+i$	$a, bac^2b^{-1}, b^{-1}acba^{-3}c^{-1}b, a^{-1}cba^{-1}c^{-1}ba^{-2}b$	24
-2	$2\sqrt{2}+i$	$a, ba^7c^4b^{-1}, b^{-1}a^3cba^{-5}c^{-1}b, bab^{-1}a^{-1}cba^{-1}c^{-1}b^{-1}a^{-3}b^{-1}$	∞
-3	$\frac{1+i\sqrt{3}}{\sqrt{2}}$	$a, bc^2a^{-2}b^{-1}, bc^2b^{-1}abc^{-2}b^{-1}, bcb^{-1}a^2bc^{-1}b^{-1}$	60
-3	$\frac{3+i\sqrt{3}}{\sqrt{2}}$	$a, bc^6b^{-1}, ba^2c^2b^{-1}aba^{-2}c^{-2}b^{-1}, bacb^{-1}a^2ba^{-1}c^{-1}b^{-1}$	∞
-5	$\frac{1+i\sqrt{5}}{\sqrt{2}}$	$a, bc^2a^{-2}b^{-1}, bcb^{-1}a^{-2}c^{-1}b, bce_2^{-1}a^{-1}c^{-1}b^{-1}$	∞
-5	$\frac{3+i\sqrt{5}}{\sqrt{2}}$	$a, ba^{-3}c^{-3}b^{-1}, bace_2^{-1}a^{-2}c^{-1}b^{-1}, ba^2cba^{-1}b^{-1}abab^{-1}a^{-2}c^{-1}b^{-1}$	∞
-6	$i\sqrt{3}$	$a, ba^{-3}b^{-1}, bce_2^{-1}a^{-1}c^{-1}b^{-1}, ba^{-1}cba^{-1}b^{-1}abab^{-1}ac^{-1}b^{-1}$	∞
-6	$\sqrt{2}+i\sqrt{3}$	$a, ba^{-1}c^2b^{-1}, bcae_2^{-1}c^{-1}a^{-2}b^{-1}, bacba^{-1}b^{-1}abab^{-1}a^{-1}c^{-1}b^{-1}$	∞

Next, for the various parameters ω we identify this subgroup of the appropriate Bianchi group.

Only two of the values of ω in this case give a subgroup of finite index. In the interests of avoiding unnecessary tedium, we suppress the details of all the cases except these two given in Table 6.

TABLE 6. Groups generated by 6 parabolics.

d	ω	Subgroup generators	I
-3	$\frac{\sqrt{3}+i}{2}$	$a, bcb^{-1}, bacb^{-1}aba^{-1}c^{-1}b^{-1}, bcbc^{-2}b^{-1}c^{-1}bc^2b^{-1}c^{-1}b^{-1}$, $a^{-2}cb^{-1}cba^2c^{-1}, bacba^2c^{-1}b^{-1}cba^{-2}cb^{-1}a^{-1}c^{-1}b^{-1}$	1
-15	$\frac{\sqrt{3}+i\sqrt{5}}{2}$	$a, ba^{-1}cb, bcba^{-2}c^{-1}b, bcbc^{-1}bcbabc^{-1}b$, $bceba^{-1}b, a^{-1}cbc^{-1}ba^{-1}b$	6

COMPLETION OF THE PROOF OF THE MAIN THEOREM: We have established that there are fourteen groups. The Dehn filling descriptions of the groups are obtained using SNAPPEA [39].

For $|g| = 2$, the groups are $\mathbb{Z}_2$-extensions of the two bridge knot and link complements (5/3), (8/3), (10/3), (12/5) (see §5.1 and [10]) and are generated by the images of the matrices X and $\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}$ where z is given in [10]. Up to conjugacy, this group will lie in $\mathrm{PGL}(2, O_d)$ if and only if $z \in O_d^*$. This only occurs in the case of (5/3) where $z = (1 + i\sqrt{3})/2$ and in that case the group G lies in G_d.

For $|g| = 3$, the results follow immediately from §6.1.

For $|g| = 4$, let H be the subgroup generated by the 4 parabolic elements whose index in the Bianchi group is given in Table 5, so that H is normal in G and $|G/H| \mid 4$. When $\omega = (1+i)/\sqrt{2}$, note that g is the image of $\begin{pmatrix} 0 & -1 \\ i & 1+i \end{pmatrix}$ so that G is a subgroup of $\mathrm{PGL}(2, O_1)$. Now G properly contains G_1 and since $\mathrm{PGL}(2, O_1)$ is a maximal discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, it follows that $G = \mathrm{PGL}(2, O_1)$. For $\omega = \sqrt{2} + i$, it is easy to check that $g^2 \notin G_2$ so that $|G/H| = 4$. With a bit of extra calculation, one can show that no conjugate of $\langle f, g^2 \rangle$ can lie inside $\mathrm{PGL}(2, O_2)$. For $\omega = \big(1 + i\sqrt{3}\big)/\sqrt{2}$, note that H lies in the congruence subgroup

$$G_3(1 - i\sqrt{3}) = P\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O_3) \mid c \equiv 0 \ \big(\mathrm{mod}\big(1 - i\sqrt{3}\big)O_3\big)\right\}.$$

But then one can show that $g^2 \notin G_3\big(1 - i\sqrt{3}\big)$ so that $|G/H| = 4$. Now

$$g^2 = P\begin{pmatrix} -1 & \frac{-1+i\sqrt{3}}{2} \\ 1 + i\sqrt{3} & 1 \end{pmatrix}, \qquad gfg^{-1} = P\begin{pmatrix} 1 & 0 \\ 1 - i\sqrt{3} & 1 \end{pmatrix}$$

so that there is a subgroup of index 2 which lies inside G_3.

For $|g| = 6$, $\omega = (\sqrt{3} + i)/2$, then

$$g = P\begin{pmatrix} 0 & \big(1 - i\sqrt{3}\big)/2 \\ \big(-1 + i\sqrt{3}\big)/2 & i\sqrt{3} \end{pmatrix} \in \mathrm{PGL}(2, O_3).$$

As above for $\omega = (1+i)/\sqrt{2}$, we obtain that $G = \mathrm{PGL}(2, O_3)$. The case $\omega = \big(\sqrt{3} + i\sqrt{5}\big)2$ is of arithmetic interest. In contrast to the cases $d = 1, 2, 3, 7$, there are two conjugacy classes of maximal orders in $M_2\big(\mathbb{Q}\big(\sqrt{-15}\big)\big)$ as the class number of $\mathbb{Q}\big(\sqrt{-15}\big)$ is two. One is represented by $\Lambda = M_2(O_{15})$, so that $G_{15} = P\Lambda^1$. The other is represented by

$$\Omega = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\big(\mathbb{Q}\big(\sqrt{-15}\big)\big) \mid a, d \in O_{15}, c \in I, b \in I^{-1}\right\}$$

where $I = \langle 2, (3 + i\sqrt{15})/2\rangle$. Let $G'_{15} = P\Omega^1$. Now G_{15}, G'_{15} are commensurable, have the same covolume but are not isomorphic. One can show that $G \cap G'_{15}$ has index 2 in both G and G'_{15}. Furthermore, $G \cap G'_{15}$ is not conjugate to a subgroup of G_{15}. Examples of link complements in G'_{15} which are not conjugate to subgroups of G_{15} were given in [34].

7. Infinite index subgroups

A central task in obtaining the results presented in §6 is to decide whether or not a given subgroup of a finitely-presented group has infinite index.

As a first step, we sought to prove that a subgroup H has finite index in a group G. We used the coset enumeration process of Havas [15] as implemented in the computational algebra system MAGMA [3] to carry out these enumerations. If the number of cosets defined exceeded some pre-assigned limit, we aborted this process, and then attempted to prove that the subgroup has infinite index by using one of the two techniques presented below.

7.1. A low-index strategy

This strategy relies on the following result.

THEOREM 7.1. *If H is an m-generator subgroup of finite index in a group G, then for any intermediate subgroup K of G containing H the abelianisation $K/K' = K/[K, K]$ has rank at most m.*

PROOF. Since H has finite index in G, it also has finite index in K and therefore $H[K, K]/[K, K]$ has finite index in $K/[K, K]$. It follows that if $K/[K, K]$ is isomorphic to $\mathbb{Z}^n \times A$ where A is finite, then $H[K, K]/[K, K]$ must be isomorphic to $\mathbb{Z}^n \times B$ where B is finite. But H is an m-generator group and can therefore have no abelian quotients of rank greater than m, so this implies $n \leq m$.

How can we exploit this result to prove that a given m-generator subgroup H of a group G has infinite index? It suggests the following approach:

(1) Search in G for some subgroups K which contain H.

(2) For each such subgroup K, determine its abelian quotient invariants. If K has more than m infinite cycles in its abelianisation, then H has infinite index in G.

How can we find intermediate subgroups K of G which contain H? One approach is to use the *low-index subgroup* algorithm. It constructs some or all of those subgroups of G of index up to a selected value which contain a given subgroup. A description of the algorithm can be found in [33, Chapter 5].

We used the algorithm in [16] to determine the abelian quotient invariants of each subgroup produced.

Again, we used MAGMA to carry out these computations. In particular, we implemented a procedure in the MAGMA language, which searched for a limited number of subgroups of index lying between prescribed bounds which contain the given subgroup H. The abelian quotient invariants of the resulting subgroups were then computed. If no subgroup having the desired number of infinite cycles was found, we then resumed our search, after adjusting the search parameters.

This general approach was applied successfully to the subgroups in G_d for $d = 2, 5, 6, 7, 11$.

On occasion, the low-index algorithm failed to construct any useful subgroups of G containing H. In these cases, we used an alternative approach to find subgroups K of G which contain H: namely, we chose K to be the subgroup generated by H and some random words in the generators of G. This random subgroup approach was necessary when we considered G_d for $d = 3, 15$.

These approaches allowed us to determine the index of all subgroups in all families, excluding G_1.

7.2. *Automatic coset systems*

The concept of an *automatic coset system* was introduced in [28]. Its origins lie in the theory of automatic groups. It is potentially a powerful tool for exploring the subgroups of an automatic group.

Here we provide only the briefest summary. We refer the interested reader to [6] for a general discussion of automatic groups and to [18] for a detailed discussion of automatic coset systems.

Let G be a group with monoid generating set A and let H be a subgroup of G. For our present limited purpose, the central relevant component of an automatic coset system is a finite state automata W called the *coset word acceptor*. The alphabet of W is A.

For each right coset of H in G, the coset word acceptor W has the property that it accepts the unique minimal word (under some specific ordering) that lies in each right coset of H in G. The size of the language accepted by W is the index of H in G; if the language accepted is infinite, then H has infinite index in G.

If an automatic coset system containing such a W exists for a specified subgroup H of a group G, then G is *coset automatic* with respect to the subgroup H. As one example, it is shown in [28] that the quasiconvex subgroups of word-hyperbolic groups have automatic coset systems.

In [18] an algorithm is presented for computing the finite state automata that constitute an automatic coset system. Holt has implemented this algorithm and it is distributed as part of his package KBMAG.

Two of the subgroups in $\mathrm{PSL}(2, O_1)$ have finite index. We used KBMAG to construct automatic coset systems for the remainder; in all cases, the language accepted was infinite, and hence these subgroups have infinite index.

The list of subgroups used in establishing infinite index in each of the cases above is quite long, and some of the generating sets are quite complicated. Thus we have not listed these groups. However the interested reader can obtain this list from the authors.

REFERENCES

1. Beardon, A. F., *The Geometry of Discrete Groups*, Springer-Verlag 1983.
2. Borel, A., *Commensurability classes and volumes of hyperbolic 3-manifolds*, Ann. Scuola Norm. Sup. Pisa 8 (1981), 1–33.
3. Bosma, W., Cannon, J. and Playoust, C., *The Magma Algebra System I: The User Language*, J. Symbolic Comput. 24 (1997), 235–265.
4. Bowditch, B., Maclachlan, C. and Reid, A. W., *Arithmetic hyperbolic surface bundles*, Math. Ann. 302 (1995), 31–60.
5. Burde, G. and Zieschang, H., *Knots*, de Gruyter, 1985.
6. Epstein, D. B. A. et al., *Word Processing in Groups*, Jones and Bartlett, Boston, 1992.
7. Gabai, D., Meyerhoff, R. and Thurston, N., *Homotopy hyperbolic manifolds are hyperbolic*, Ann. of Math., to appear.
8. Gehring, F. W. and Martin, G. J., *Commutators, collars and the geometry of Möbius groups*, J. Analyse Math. 63 (1994), 175–219.
9. Gehring, F. W., Maclachlan, C. and Martin, G. J., *On discrete free products of cyclic groups*, Mitt. Math. Sem. Giessen 228 (1996), 9–15.
10. Gehring, F. W., Maclachlan, C. and Martin, G. J., *2-generator arithmetic Kleinian groups* II, Bull. London Math. Soc. 30 (1998), 258–266.
11. Gehring, F. W., Maclachlan, C., Martin, G. J. and Reid, A. W., *Arithmeticity, Discreteness and Volume*, Trans. Amer. Math. Soc. 349 (1997), 3611–3643.
12. Gehring, F. W., Marshall, T. H. and Martin, G. J., *On the spectrum of axial distances in Kleinian groups*, Indiana Math. J. 47 (1998), 1–10.
13. Grunewald, F. and Schwermer, J., *Subgroups of Bianchi groups and arithmetic quotients of hyperbolic 3-space*, Trans. Amer. Math. Soc. 335 (1993), 47–78.
14. Hagelberg, M., Maclachlan, C. and Rosenberger, G., *On discrete generalised triangle groups*, Proc. Edinburgh Math. Soc. 38 (1995), 397–412.
15. Havas, G., *Coset Enumeration Strategies*, Proc. 1991 International Symposium on Symbolic and Algebraic Computation, ACM Press, New York (1991), 191–199.
16. Havas, G. and Majewski, B. S., *Integer matrix diagonalization*, J. Symbolic Comput. 24 (1997), 399–408.
17. Hilden, H. M., Lozano, M-T. and Montesinos, J. M., *A characterization of arithmetic subgroups of* $\mathrm{SL}(2, \mathbf{R})$ *and* $\mathrm{SL}(2, \mathbf{C})$, Math. Nachr. 159 (1992), 245–270.
18. Holt, D. F. and Hurt, D. F., *Automatic coset systems and subgroup presentations*, J. Symbolic Comput. 27 (1999), 1–19.

19. Ignatov, J. I., *Free and nonfree subgroups of* PSL(2, $\mathbb{C}$) *generated by two parabolic elements*, Math. USSR Sbornik 35 (1979), 49–55.
20. Keen, L. and Series, C., *The Riley slice of Schottky space*, Proc. London Math. Soc. 69 (1994), 72–90.
21. Lyndon, R. C. and Ullman, J. L., *Groups generated by two parabolic linear fractional transformations*, Canad. J. Math. XXI (1969), 1388–1403.
22. Maclachlan, C. and Martin, G. J., *2-generator arithmetic Kleinian Groups*, J. Reine Angew. Math. 511 (1999), 95–117.
23. Maclachlan, C. and Reid, A. W., *Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups*, Math. Proc. Cambridge Philos. Soc. 102 (1987), 251–257.
24. Maskit, B., *Kleinian Groups*, Springer-Verlag, 1987.
25. Neumann, W. D. and A. W. Reid, *Arithmetic of hyperbolic manifolds*, Topology '90 (Columbus, OH, 1990), 273–310, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.
26. Parker, J. R., *Kleinian groups with two parabolic generators*, Preprint.
27. Ratcliffe, J. G., *Foundations of Hyperbolic Manifolds*, Springer-Verlag, 1994.
28. Redfern, I. D., *Automatic Coset Systems*, PhD thesis. University of Warwick (1993).
29. Reid, A. W., *A note on trace-fields of Kleinian groups*, Bull. London Math. Soc. 22 (1990), 349–352.
30. Reid, A. W., *Arithmeticity of Knot Complements*, J. London Math. Soc. (2) 43 (1991), 171–184.
31. Riley, R., *Applications of a computer implementation of Poincarés theorem on fundamental polyhedra*, Math. Comp. 40 (1983), 607–632.
32. Rolfsen, D., *Knots and Links*, Publish or Perish, 1976.
33. Sims, C. C., *Computation with Finitely Presented Groups*, Cambridge University Press, 1994.
34. Stephan, J., *Complémentaires d'entrelacs dans S^3 et ordres maximaux des algèbres de quaternions $M_2(\mathbb{Q}[i\sqrt{d}\,])$*, C. R. Acad. Sci. Paris 322 (1996), 685–688.
35. Swan, R., *Generators and relations for certain special linear groups*, Adv. Math. 6 (1971), 1–78.
36. Takeuchi, K., *Arithmetic triangle groups*, J. Math. Soc. Japan 29 (1977), 91–106.
37. Thurston, W. P., *The geometry and Topology of Three-manifolds*, Princeton University Lecture Notes, 1976.
38. Vignéras, M-F., *Arithmétique des algèbres de quaternions*, Lecture Notes in Math. 800 (1980).
39. G. Weeks, *SNAPPEA, a computational geometry program*, available from `weeks@geom.umn.edu`.

M. D. E. CONDER, G. J. MARTIN AND E. A. O'BRIEN
UNIVERSITY OF AUCKLAND,
AUCKLAND,
NEW ZEALAND

C. MACLACHLAN
UNIVERSITY OF ABERDEEN,
ABERDEEN,
SCOTLAND