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SMOOTH CURVES ON PROJECTIVE $K3$ SURFACES

ANDREAS LEOPOLD KNUTSEN

Abstract

In this paper we give for all $n \geq 2$, $d > 0$, $g \geq 0$ necessary and sufficient conditions for the existence of a pair (X, C), where X is a $K3$ surface of degree $2n$ in P^{n+1} and C is a smooth (reduced and irreducible) curve of degree d and genus g on X. The surfaces constructed have Picard group of minimal rank possible (being either 1 or 2), and in each case we specify a set of generators. For $n \geq 4$ we also determine when X can be chosen to be an intersection of quadrics (in all other cases X has to be an intersection of both quadrics and cubics). Finally, we give necessary and sufficient conditions for $\mathscr{O}_C(k)$ to be non-special, for any integer $k \geq 1$.

1. Introduction

In recent years the interest for $K3$ surfaces and Calabi-Yau threefolds has increased because of their importance in theoretical physics and string theory in particular. The study of curves on $K3$ surfaces is interesting not only in its own right, but also because one can use $K3$ surfaces containing particular curves to constuct $K3$ fibered Calabi-Yau threefolds containing the same curves as rigid curves (see [4], [13], [2], [5] and [6]).

The problem of determining the possible pairs (d, g) of degree d and genus g of curves contained in certain ambient varieties is rather fascinating. A fundamental result of L. Gruson and C. Peskine in [3] determines all such pairs for which there exists a smooth irreducible nondegenerate curve of degree d and genus g in P^3. To solve the problem, the authors need curves on some rational quartic surface with a double line.

S. Mori proved in [9] that essentially the same degrees and genera as those found by Gruson and Peskine for curves on rational quartic surfaces, can be found on smooth quartic surfaces as well.

K. Oguiso [13] showed in 1994 that for all $n \geq 2$ and $d > 0$ there exists a $K3$ surface of degree $2n$ containing a smooth rational curve of degree d.

The main aim of this paper is to prove the following general result:

THEOREM 1.1. *Let $n \geq 2$, $d > 0$, $g \geq 0$ be integers. Then there exists a $K3$ surface[1] X of degree $2n$ in P^{n+1} containing a smooth curve C of degree d and genus g if and only if*

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[1] By a $K3$ surface is meant a smooth $K3$ surface.

(i) $g = d^2/4n + 1$ *and there exist integers* $k, m \geq 1$ *and* $(k, m) \neq (2, 1)$ *such that* $n = k^2m$ *and* $2n$ *divides* kd,

(ii) $d^2/4n < g < d^2/4n + 1$ *except in the following cases*
  (a) $d \equiv \pm 1, \pm 2 \pmod{2n}$,
  (b) $d^2 - 4n(g-1) = 1$ *and* $d \equiv n \pm 1 \pmod{2n}$,
  (c) $d^2 - 4n(g-1) = n$ *and* $d \equiv n \pmod{2n}$,
  (d) $d^2 - 4n(g-1) = 1$ *and* $d - 1$ *or* $d + 1$ *divides* $2n$,

(iii) $g = d^2/4n$ *and* d *is not divisible by* $2n$,

(iv) $g < d^2/4n$ *and* $(d, g) \neq (2n+1, n+1)$.

Furthermore, in case (i) X *can be chosen such that* $\operatorname{Pic} X = \mathbb{Z}\frac{2n}{dk}C = \mathbb{Z}\frac{1}{k}H$ *and in cases* (ii)–(iv) *such that* $\operatorname{Pic} X = \mathbb{Z}H \oplus \mathbb{Z}C$, *where* H *is the hyperplane section of* X.

If $n \geq 4$, X *can be chosen to be scheme-theoretically an intersection of quadrics in cases* (i), (iii) *and* (iv), *and also in case* (ii), *except when* $d^2 - 4n(g-1) = 1$ *and* $3d \equiv \pm 3 \pmod{2n}$ *or* $d^2 - 4n(g-1) = 9$ *and* $d \equiv \pm 3 \pmod{2n}$, *in which case* X *has to be an intersection of both quadrics and cubics.*

REMARK 1.2. If one allows X to be a birational projective model of a $K3$ surface (which automatically yields with at worst rational double points as singularities), then the result above remains the same, except that the case (ii)-(c) occurs.

The most general results concerning construction of $K3$ fibered Calabi-Yau threefolds are due to H. P. Kley [6], who constructs rigid curves of bounded genera on complete intersection Calabi-Yau threefolds ($CICY$s). The approach of Kley requires that the smooth curve C on the $K3$ surface X used to construct the $CICY$ is linearly independent of the hyperplane section H of X and also that $h^1(C', \mathcal{O}_{C'}(k)) = 0$ for all $C' \in |C|$ for $k = 1$ or 2 (depending on the different types of $CICY$s). Motivated by this, we also prove the following result, which is an improvement of the results in [6] and gives the corresponding existence of more rigid curves in $CICY$s than is shown in [6].

PROPOSITION 1.3. *Let* $k \geq 1$ *be an integer. We can find* X *and* C *as in Theorem 1.1 such that* $h^1(C', \mathcal{O}_{C'}(k)) = 0$ *for all* $C' \in |C|$ *if and only if*

$$d \leq 2nk \qquad \text{or} \qquad dk > nk^2 + g.$$

So far one has only used the $K3$ surfaces that are complete intersections (more specifically the smooth complete intersections of type (4) in $\mathbb{P}^3$, (2, 3) in $\mathbb{P}^4$ and (2, 2, 2) in $\mathbb{P}^5$, see Section 6) to construct $CICY$s containing rigid curves. S. Mukai showed in [11] that general $K3$ surfaces of degrees 10, 12, 14, 16 and 18 are complete intersections in homogeneous spaces. For the triples (n, d, g) in Theorem 1.1 corresponding to such general surfaces, one can then construct $K3$ fibered Calabi-Yau threefolds that are complete intersections in homogeneous spaces containing rigid curves. This is the topic in [8].

We work over the field of complex numbers, although the results will probably hold for any algebraically closed field of characteristic zero.

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2. Preliminaries

A *curve* will always be reduced and irreducible in this paper.

We now quote some results which will be needed in the rest of the paper. Most of these results are due to Saint-Donat [14].

PROPOSITION 2.1 ([14, Cor. 3.2]). *Let Σ be a complete linear system on a $K3$ surface. Then Σ has no base points outside its fixed components.*

PROPOSITION 2.2 ([14, Prop. 2.6(i)]). *Let $|C| \neq \emptyset$ be a complete linear system without fixed components on a $K3$ surface X such that $C^2 > 0$. Then the generic member of $|C|$ is smooth and irreducible and $h^1(\mathcal{O}_X(C)) = 0$*

PROPOSITION 2.3. *Let $|C| \neq \emptyset$ be a complete linear system without fixed components on a $K3$ surface such that $C^2 = 0$. Then every member of $|C|$ can be written as a sum $E_1 + E_2 + \cdots + E_k$, where $E_i \in |E|$ for $i = 1, \ldots, k$ and E is a smooth curve of genus* 1.

In other words, $|C|$ is a multiple k of an elliptic pencil.

In particular, if C is part of a basis of Pic X*, then the generic member of $|C|$ is smooth and irreducible.*

PROOF. This is [14, Prop. 2.6(ii)]. For the last statement, since C is part of a basis of Pic X, $k = 1$ and $|C| = |E|$.

We will also need the following criteria for base point freeness and very ampleness of a line bundle on a $K3$ surface.

LEMMA 2.4 ([14], see also [7, Thm. 1.1]). *Let L be a nef line bundle on a $K3$ surface. Then*

(a) *$|L|$ is not base point free if and only if there exist curves E, Γ and an integer $k \geq 2$ such that*

$$L \sim kE + \Gamma, \qquad E^2 = 0, \qquad \Gamma^2 = -2, \qquad E.\Gamma = 1.$$

In this case, every member of $|L|$ is of the form $E_1 + \cdots + E_k + \Gamma$, where $E_i \in |E|$ for all i.

Equivalently, L is not base point free if and only if there is a divisor E satisfying $E^2 = 0$ and $E.L = 1$.

(b) *L is very ample if and only if $L^2 \geq 4$ and*

(I) *there is no divisor E such that $E^2 = 0$, $E.L = 1, 2$,*

(II) *there is no divisor E such that $E^2 = 2$, $L \sim 2E$, and*

(III) *there is no divisor E such that $E^2 = -2$, $E.L = 0$,*

Note that (II) in (b) is automatically fulfilled if L is a part of a basis of Pic X, which it often will be in our cases.

Let L be a base point free line bundle on a $K3$ surface with dim $|L| = r \geq 2$. Then $|L|$ defines a morphism

$$\phi_L : X \longrightarrow \mathbf{P}^r,$$

whose image $\phi_L(X)$ is called a projective model of X.

We have the following result:

Proposition 2.5 ([14]). (i) *If there is a divisor E such that $E^2 = 0$ and $E.L = 2$, or $E^2 = 2$ and $L \sim 2E$, then ϕ_L is $2:1$ onto a surface of degree $\frac{1}{2}L^2$.*

(ii) *If there is no such divisor, then ϕ_L is birational onto a surface of degree L^2 (in fact it is an isomorphism outside of finitely many contracted smooth rational curves), and $\phi_L(X)$ is normal with only rational double points.*

In [14] an L which is base point free and as in (i) is called *hyperelliptic*, as in this case all smooth curves in $|L|$ are hyperelliptic. We will call an L which is base point free and as in (ii) *birationally very ample* (see [7] for a generalization).

We will need the criteria for very ampleness to prove Theorem 1.1 and for birational very ampleness to prove the statement in Remark 1.2.

We also have the following result about the ideal of $\phi_L(X)$, when L is very ample.

Proposition 2.6 ([14, Thm. 7.2]). *Let L with $H^2 \geq 8$ be a very ample divisor on a $K3$ surface X. Then the ideal of $\phi_L(X)$ is generated by its elements of degree 2, except if there exists a curve E such that $E^2 = 0$ and $E.L = 3$, in which case the ideal of X is generated by its elements of degree 2 and 3.*

We will concentrate on the proof of Theorem 1.1, and give the main ideas of the proof of the statement in Remark 1.2 in Remark 4.7 below.

The immediate restrictions on the degree and genus of a divisor come from the Hodge index theorem. Indeed, if H is any divisor on a $K3$ surface satisfying

$H^2 = 2n > 0$ and C is any divisor satisfying $C.H = d$ and $C^2 = 2(g-1)$, we get from the Hodge index theorem:

$$(C.H)^2 - C^2H^2 = d^2 - 4n(g-1) \geq 0,$$

with equality if and only if

$$dH \sim 2nC.$$

3. The case $d^2 - 4n(g-1) = 0$

We have

PROPOSITION 3.1. *Let $n \geq 2$, $d > 0$, $g \geq 0$ be integers satisfying $d^2 - 4n(g-1) = 0$. Then there is a $K3$ surface X of degree $2n$ in P^{n+1} containing a smooth curve C of degree d and genus g if and only if there exist integers $k, m \geq 1$ and $(k, m) \neq (2, 1)$ such that*

$$n = k^2m \qquad \text{and} \qquad 2n \qquad \text{divides} \qquad kd.$$

Furthermore, X can be chosen such that $\operatorname{Pic} X = \mathsf{Z}\frac{2n}{dk}C = \mathsf{Z}\frac{1}{k}H$, *where H is the hyperplane section of X, and if $n \geq 4$, such that X is scheme-theoretically an intersection of quadrics.*

PROOF. First we show that these conditions are necessary. Let X be a projective $K3$ surface containing a smooth curve of type (d, g) such that $d^2 - 4n(g-1) = 0$. Let H be a hyperplane section of X. Since $C \sim \frac{d}{2n}H$, there has to exist a divisor D and an integer $k \geq 1$ such that $H \sim kD$ and $2n$ divides kd. Furthermore, letting $D^2 = 2m$, $m \geq 1$, one gets

$$H^2 = 2n = 2mk^2,$$

so $n = k^2m$.

If $(k, m) = (2, 1)$, then $n = 4$ and $H \sim 2D$ for a divisor D such that $D^2 = 2$, but this is impossible by Lemma 2.4.

Now we show that these conditions are sufficient by explicitly constructing a projective $K3$ surface of degree $2n$ containing a smooth curve of type (d, g) under the above hypotheses.

Consider the rank 1 lattice $L = \mathsf{Z}D$ with intersection form $D^2 = 2m$. This lattice is integral and even, and it has signature $(1, 0)$.

Now [10, Thm. 2.9(i)] (see also [12]) states that:

If $\rho \leq 10$, then every even lattice of signature $(1, \rho - 1)$ occurs as the Néron-Severi group of some algebraic $K3$ surface.

Therefore there exists an algebraic $K3$ surface X such that $\operatorname{Pic} X = \mathsf{Z}D$. Since $D^2 > 0$, either $|D|$ or $|-D|$ contains an effective member, so we can assume D is effective (possibly after having changed D with $-D$). Since D generates $\operatorname{Pic} X$, D is ample by Nakai's criterion (in particular, it is nef). If $m \geq 2$, then $D^2 \geq 4$, and by Lemma 2.4 D is very ample. Indeed, since $\operatorname{Pic} X = \mathbb{Z}D$ and $D^2 > 0$, there can exist no divisor E such that $E^2 = 0, -2$ or $D \sim 2E$.

If $m = 1$, then by our assumptions $k \geq 3$, so by [14, Thm. 8.3] kD is very ample.

So in all cases, $H := kD$ is very ample under the above assumptions, and by $H^2 = k^2 D^2 = 2mk^2 = 2n$, we can embed X as a $K3$ surface of degree $2n$ in $\mathbb{P}^{n+1}$. Define

$$C := \frac{dk}{2n} D.$$

then C is nef, since it is a non-negative multiple of a nef divisor, and by Lemma 2.4 it is base point free (no curves with self-intersection -2 or 0 can occur on X since D generates the Picard group).

So by Propositions 2.1 and 2.2 the generic member of $|C|$ is smooth and irreducible, and one easily checks that $C.H = d$ and $C^2 = 2g - 2$.

The two last assertions follow from the construction (the last one follows from Proposition 2.6).

4. The case $d^2 - 4n(g-1) > 0$

We have the following result of Oguiso:

THEOREM 4.1. *Let $n \geq 2$ and $d \geq 1$ be positive integers. Then there exist a $K3$ surface X of degree $2n$ in $\mathbb{P}^{n+1}$ and a smooth rational curve C of degree d on X.*

Furthermore, X can be chosen such that $\operatorname{Pic} X = \mathbb{Z}H \oplus \mathbb{Z}C$, *where H is the hyperplane section of X, and if $n \geq 4$, then X can be chosen to be scheme-theoretically an intersection of quadrics.*

PROOF. This is [13, Thm 3]. The last statement follows again by Proposition 2.6 since $|\operatorname{disc}(H, C)| = d^2 + 4n > 16$ and a divisor E such as the one in the proposition would give $|\operatorname{disc}(E, H)| = 9$.

We can make a more general construction:

PROPOSITION 4.2. *Let $n \geq 1$, $d \geq 1$, $g \geq 0$ be positive integers satisfying $d^2 - 4n(g-1) > 0$. Then there exist an algebraic $K3$ surface X and two divisors H and C on X such that* $\operatorname{Pic} X = \mathbb{Z}H \oplus \mathbb{Z}C$, *$H^2 = 2n$, $C.H = d$, $C^2 = 2(g-1)$ and H is nef.*

PROOF. Consider the lattice $L = \mathbb{Z}H \oplus \mathbb{Z}C$ with intersection matrix

$$\begin{bmatrix} H^2 & H.C \\ C.H & C^2 \end{bmatrix} = \begin{bmatrix} 2n & d \\ d & 2(g-1) \end{bmatrix}.$$

This lattice is integral and even, and it has signature $(1, 1)$ if and only if $d^2 - 4n(g-1) > 0$.

By [10, Thm. 2.9(i)] again, we conclude that the lattice L occurs as the Picard group of some algebraic $K3$ surface X. It remains to show that H can be chosen nef.

Consider the group generated by the Picard-Lefschetz reflections

$$\begin{aligned} \operatorname{Pic} X &\xrightarrow{\pi_\Gamma} \operatorname{Pic} X \\ D &\longmapsto D + (D.\Gamma)\Gamma, \end{aligned}$$

where $\Gamma \in \operatorname{Pic} X$ satisfies $\Gamma^2 = -2$ and $D \in \operatorname{Pic} X$ satisfies $D^2 > 0$. Now [1, VIII, Prop. 3.9] states that a fundamental domain for this action is the *big-and-nef cone* of X. Since $H^2 > 0$, we can assume that H is nef.

We would like to investigate under which conditions H is very ample and $|C|$ contains a smooth irreducible member. To show the latter for $g > 0$, it will be enough to show that $|C|$ is base point free, by Propositions 2.2 and 2.3.

We first need a basic lemma.

LEMMA 4.3. *Let H, C, X, n, d and g be as in Proposition 4.2 and $k \geq 1$ an integer. If $(d, g) = (2nk, nk^2)$ (resp. $(d, g) = (nk, \frac{nk^2+3}{4})$), we can assume (after a change of basis of* $\operatorname{Pic} X$*) that $kH - C > 0$ (resp. $kH - 2C > 0$).*

PROOF. We calculate $(kH - C)^2 = -2$ (resp. $(kH - 2C)^2 = -2$), so by Riemann-Roch either $kH - C > 0$ or $C - kH > 0$ (resp. either $kH - 2C > 0$ or $2C - kH > 0$). If the latter is the case, define $C' := 2kH - C$ (resp. $C' := kH - C$). Then one calculates $C'.H = d$ and $C'^2 = 2(g-1)$, and since clearly $\operatorname{Pic} X \simeq \mathbb{Z}H \oplus \mathbb{Z}C'$, we can substitute C with C'.

PROPOSITION 4.4. *Let H, C, X, n, d and g be as in Proposition 4.2 with $g \geq 1$, and with the additional assumptions that $kH - C > 0$ (resp. $kH - 2C > 0$) if $(d, g) = (2nk, nk^2)$ (resp. if $(d, g) = (nk, \frac{nk^2+3}{4})$). Assume H is base point free. Then $|C|$ contains a smooth irreducible member if and only if we are not in one of the following cases:*

(i) $(d, g) = (2n + 1, n + 1)$,

(ii) $d^2 - 4n(g - 1) = 1$ *and* $d - 1$ *or* $d + 1$ *divides* $2n$.

PROOF. We first show that C is nef except for the case (i).

Assume that C is not nef. Then there is a curve Γ (necessarily contained in the fixed component of $|C|$) such that $C.\Gamma < 0$ and $\Gamma^2 = -2$.

We now consider the two cases $\Gamma.H > 0$ and $\Gamma.H = 0$ [2].

[2] This latter case occurs only if H is not ample, so it is only interesting in order to prove the statement in Remark 1.2. To prove Theorem 1.1 we could assume that H is ample and thus get an easier proof of Proposition 4.4.

If $\Gamma.H > 0$, define $a := -C.\Gamma \geq 1$ and

$$C' := C - a\Gamma,$$

then

$$C'^2 = 2(g-1) \geq 0,$$

so by Riemann-Roch either $|C'|$ or $|-C'|$ contains an effective member, and since $h^0(\mathscr{O}_X(a\Gamma)) = 1$, it must be $|C'|$. Hence

$$0 < d' := C'.H = C.H - a(\Gamma.H) < d,$$

where we have used that $\Gamma.H > 0$ to get the strict inequality on the right and $C'^2 \geq 0$ to get the strict inequality on the left, by the Hodge index theorem. Since $H^2 > 0$, one must have $d'^2 - 4n(g-1) \geq 0$ by the Hodge index theorem, and equality occurs if and only if $d'H \sim 2nC'$.

We now show that $d'^2 - 4n(g-1) = 0$ only if $(d, g) = (2n+1, n+1)$ and that C is not nef in this case.

Write $\Gamma \sim xH + yC$, for two integers x and y. We have

$$C' \sim C - a\Gamma \sim C - a(xH + yC) \sim -axH + (1-ay)C \sim \frac{d'}{2n}H,$$

which implies $y = a = 1$. We then have

$$-1 = \Gamma.C = dx + 2(g-1)y = dx + 2(g-1),$$

which yields $x = -\frac{2g-1}{d}$, whence

$$\Gamma \sim -\frac{2g-1}{d}H + C.$$

Note that this implies

$$d \mid 2g-1. \tag{1}$$

We now use

$$-2 = \Gamma^2 = \frac{(2g-1)^2}{d^2}2n - 2(2g-1) + 2(g-1) = 2\left(\frac{(2g-1)^2 n}{d^2} - g\right)$$

to conclude

$$n = \frac{(g-1)d^2}{(2g-1)^2}. \tag{2}$$

Using this, we calculate

$$\Gamma.H = -2n\frac{2g-1}{d} + d = -2\frac{(g-1)d^2}{(2g-1)^2}\frac{2g-1}{d} + d = \frac{d}{2g-1},$$

which yields

$$2g - 1 \mid d. \tag{3}$$

Comparing (1) and (3), we get $d = 2g - 1$, which gives by (2) that

$$(d, g) = (2n + 1, n + 1).$$

So we have shown that $d'^2 - 4n(g - 1) = 0$ occurs only when $(d, g) = (2n + 1, n + 1)$ and that C is not nef in this case.

We now consider the case when $d'^2 - 4n(g - 1) > 0$. Since

$$0 \neq d'^2 - 4n(g-1) = |\operatorname{disc}(H, C')| < d^2 - 4n(g-1) = |\operatorname{disc}(H, C)|,$$

then $\operatorname{disc}(H, C)$ cannot divide $\operatorname{disc}(H, C')$ and we have a contradiction, so C is nef.

If $\Gamma.H = 0$, write $\Gamma \sim xH + yC$. We have

$$-2 = \Gamma^2 = \Gamma.(xH + yC) = yC.\Gamma,$$

which gives the two possibilities

(a) $y = 1$, $C.\Gamma = -2$, and

(b) $y = 2$, $C.\Gamma = -1$.

In case (a), we get from $\Gamma.H = 2nx + dy = 2nx + d = 0$ that $x = -d/2n$, which means that

$$d = 2nk \qquad \text{and} \qquad x = -k,$$

for some integer $k \geq 1$. From $\Gamma.C = dx + 2(g-1)y = -2nk^2 + 2(g-1) = -2$, we get $g = nk^2$. So $(d, g) = (2nk, nk^2)$ and $\Gamma \sim -kH + C$, which by assumption is not effective, a contradiction.

In case (b), we get from $\Gamma.H = 2nx + dy = 2nx + 2d = 0$ that $x = -d/n$, which means that

$$d = nk \qquad \text{and} \qquad x = -k,$$

for some integer $k \geq 1$. We get from $\Gamma.C = dx + 2(g-1)y = -nk^2 + 4(g-1) = -1$, that $g = (nk^2 + 3)/4$. So $(d, g) = (nk, \frac{nk^2+3}{4})$ and $\Gamma \sim -kH + 2C$, which by assumption is not effective, again a contradiction.

So we have proved that C is nef except for the case (i).

If $|C|$ is not base point free, then using Lemma 2.4(a), X must contain two divisors E and Γ such that $E^2 = 0$ and

$$|\operatorname{disc}(E, \Gamma)| = 1,$$

and this must be divisible by $|\operatorname{disc}(H, C)| = d^2 - 4n(g-1)$, which then must be equal to 1.

Setting $E \sim xH + yC$ one finds from $E.C = dx + 2(g-1)y = 1$ and $E^2 = 2nx^2 + 2dxy + 2(g-1)y^2 = 0$ that

$$x = \pm 1 \qquad \text{and} \qquad y = \frac{1 \mp d}{2(g-1)}.$$

Using the fact that $d^2 - 4n(g-1) = 1$, we get $2(g-1) = \frac{(d+1)(d-1)}{2n}$, which gives

$$(x, y) = \left(1, -\frac{2n}{d+1}\right) \qquad \text{or} \qquad \left(-1, \frac{2n}{d-1}\right).$$

So if $d^2 - 4n(g-1) = 1$ and $d+1$ or $d-1$ divides $2n$, then the divisor E will satisfy $E^2 = 0$ and $E.C = 1$, whence C is not base point free by Lemma 2.4(a).

This concludes the proof of the proposition.

Note that we have also proved

COROLLARY 4.5. *Let H and C be divisors on a $K3$ surface X such that H is nef, $H^2 = 2n$, $C.H = d$ and $C^2 = 2(g-1)$ for some integers $n \geq 1$, $d > 0$ and $g \geq 1$. If either*

(a) $(d, g) = (2n+1, n+1)$, *or*

(b) $d^2 - 4n(g-1) = 1$ *and* $d+1$ *or* $d-1$ *divides* $2n$,

then C is not base point free.

In particular, a projective $K3$ surface of degree $2n$, for an integer $n \geq 2$ (or even a birational projective model of a $K3$ surface) cannot contain an effective, irreducible divisor of degree d and arithmetic genus g for any values of d and g as in (a) *or* (b).

PROOF. In case (a) the divisor $\Gamma := C - H$ is effective and satisfies $\Gamma.C = -1$, so C is not even nef.

In case (b) the divisor $E := H - \frac{2n}{d+1}C$ or $E := -H + \frac{2n}{d-1}C$ is effective and satisfies $E^2 = 0$ and $E.C = 1$. Thus C is not base point free by Lemma 2.4(a).

One gets the following

THEOREM 4.6. *Let $n \geq 2$, $d \geq 1$, $g \geq 0$ be positive integers satisfying $d^2 - 4ng > 0$ and $(d, g) \neq (2n+1, n+1)$. Then there exists a projective $K3$*

surface X of degree $2n$ in P^{n+1} containing a smooth curve C of degree d and genus g. Furthermore, we can find an X such that Pic $X = \mathsf{Z}H \oplus \mathsf{Z}C$, *where H is the hyperplane section of X, and X is scheme-theoretically an intersection of quadrics if $n \geq 4$.*

PROOF. The case $g = 0$ is Theorem 4.1, so we can assume $g > 0$. Since $|\operatorname{disc}(H, C)| = d^2 - 4n(g-1) > 4n$, the H constructed as in Proposition 4.2 is very ample, since the existence of such divisors as in (I) and (III) in Lemma 2.4(b) implies that $|\operatorname{disc}(H, C)|$ must divide $|\operatorname{disc}(H, E)| = 1, 4, 4n$, respectively. Now Proposition 4.4 gives the rest.

To prove that X is an intersection of quadrics when $n \geq 4$, by Prop 2.6 it is sufficient to show that there cannot exist any divisor E such that $E^2 = 0$ and $E.H = 3$.

Such an E would give

$$|\operatorname{disc}(E, H)| = 9$$

but we have $|\operatorname{disc}(H, C)| > 4n \geq 16$, when $n \geq 4$.

Now we only have to investigate the pairs (d, g) for which

$$0 < d^2 - 4n(g-1) \leq 4n.$$

We proceed as follows. For given n, d, g we use the construction of Proposition 4.2 and then investigate whether H is very ample by using Lemma 2.4. Then two cases may occur:

(1) Using the fact that Pic $X = \mathsf{Z}H + \mathsf{Z}C$ and $H^2 = 2n$, $C.H = d$, $C^2 = 2(g-1)$, we find that there cannot exist any divisor $E \sim xH + yC$ as in cases (I) and (III) of Lemma 2.4(b), so H is very ample and by Proposition 4.4, $|C|$ contains a smooth irreducible member and there exists a projective $K3$ surface X of degree $2n$ in P^{n+1} containing a smooth curve C of degree d and genus g.

(2) Using the numerical properties $H^2 = 2n$, $C.H = d$ and $C^2 = 2(g-1)$, we find a divisor $E \sim aH + bC$ for $a, b \in \mathsf{Z}$ satisfying case (I) or (III) of Lemma 2.4(b), thus contradicting the very ampleness of H. This then implies that there cannot exist any projective $K3$ surface of degree $2n$ in P^{n+1} containing a divisor of degree d and genus g.

(To prove the statement in Remark 1.2, we proceed in an analogous way, but check the conditions for H to be birationally very ample instead. We then have to investigate whether any of the smooth curves in $|C|$ are mapped isomorphically to a smooth curve. See Remark 4.7 below.)

For any triple of integers (n, d, g) such that $n \geq 2$, $d > 0$, $g \geq 1$ and $0 < d^2 - 4n(g-1) \leq 4n$, define

$$\Lambda(n, d, g) := d^2 - 4n(g-1) = |\operatorname{disc}(H, C)|.$$

We check conditions (I) and (III) in Lemma 2.4. We let

$$E \sim xH + yC,$$

and use the values of $E^2 = 2nx^2 + 2dxy + 2(g-1)y^2$ and $E.H = 2nx + dy$ to find the integers x and y (if any).

We get the two equations

$$n\Lambda(n, d, g)x^2 - (E.H)\Lambda(n, d, g)x - (g-1)(E.H)^2 + \frac{d^2}{2}E^2 = 0$$

and

$$y = \frac{(E.H) - 2nx}{d}.$$

(a) If $E.H = 1$ and $E^2 = 0$, then

$$x = \frac{\Lambda(n, d, g) \pm d\sqrt{\Lambda(n, d, g)}}{2n\Lambda(n, d, g)},$$

$$y = \mp\frac{1}{\sqrt{\Lambda(n, d, g)}},$$

and the only possibility is $\Lambda(n, d, g) = 1$, so

$$x = \frac{1 \pm d}{2n}, \qquad y = \mp 1,$$

and we must have $d \equiv \pm 1 \pmod{2n}$.

(b) If $E.H = 2$ and $E^2 = 0$, then

$$x = \frac{\Lambda(n, d, g) \pm d\sqrt{\Lambda(n, d, g)}}{n\Lambda(n, d, g)},$$

$$y = \mp\frac{2}{\sqrt{\Lambda(n, d, g)}},$$

and the only possibilities are $\Lambda(n, d, g) = 1$ or 4.

If $\Lambda(n, d, g) = 1$, then

$$x = \frac{1 \pm d}{n}, \qquad y = \mp 2,$$

and we must have $d \equiv \pm 1 \pmod{2n}$ or $d \equiv n \pm 1 \pmod{2n}$.

If $\Lambda(n, d, g) = 4$, then

$$x = \frac{2 \pm d}{2n}, \qquad y = \mp 1,$$

and we must have $d \equiv \pm 2 \pmod{2n}$.

(c) Finally, if $E.H = 0$ and $E^2 = -2$, we get

$$-2 = E^2 = E.(xH + yC) = yE.C,$$

so $y = -1$ or -2 (since C is nef), and by $E.H = 2nx + dy = 0$, we get

$$d \equiv 0 \pmod{2n} \text{ and } x = \frac{d}{2n}, \quad \text{or} \quad d \equiv 0 \pmod{n} \text{ and } x = \frac{d}{n}$$

respectively.

One now easily calculates (using $E.C = 2$ (resp. 1))

$$\Lambda(n, d, g) = 4n \qquad \text{and} \qquad n$$

respectively. Furthermore, in the latter case, if $d \equiv 0 \pmod{2n}$, writing $d = 2nk$, for some integer $k \geq 1$, we get from $E.C = 1$ the absurdity $g = nk^2 + \frac{3}{4}$. So we actually have $d \equiv n \pmod{2n}$ in this case.

What we have left to prove in Theorem 1.1 is that X can be chosen as an intersection of quadrics in case (iii) and under the given assumptions in case (ii).

We have to show, by Proposition 2.6, that there cannot exist any divisor E such that $E.H = 3$ and $E^2 = 0$. Since such an E would give $|\operatorname{disc}(E, H)| = 9$, and we have $|\operatorname{disc}(H, C)| = d^2 - 4n(g-1) = 4n \geq 16$ if $n \geq 4$ in case (iii), this case is proved.

For case (ii), we proceed as above and set $E \sim xH + yC$, and try to find the integers x and y. We find

$$x = \frac{3(\Lambda(n, d, g) \pm d\sqrt{\Lambda(n, d, g)})}{2n\Lambda(n, d, g)},$$

$$y = \mp \frac{3}{\sqrt{\Lambda(n, d, g)}},$$

so we must have $\Lambda(n, d, g) = 1$ or 9.

If $\Lambda(n, d, g) = 1$, then

$$x = \frac{3(1 \pm d)}{2n}, \qquad y = \mp 3,$$

and E exists if and only if $3d \equiv \pm 3 \pmod{2n}$.

If $\Lambda(n, d, g) = 9$, then

$$x = \frac{3 \pm d}{2n}, \qquad y = \mp 1,$$

and E exists if and only if $d \equiv \pm 3 \pmod{2n}$.

So, under the given assumptions, the constructed X is an intersection of quadrics. Conversely, given a $K3$ surface X of degree $2n$ containing a smooth curve C of degree d and genus g such that $d^2 - 4(n-1) = 1$ and $3d \equiv \pm 3 \pmod{2n}$ or $d^2 - 4(n-1) = 9$ and $d \equiv \pm 3 \pmod{2n}$, then the divisor E above, which is a linear combination of C and the hyperplane section H, will be a divisor such that $E.H = 3$ and $E^2 = 0$. Hence X must be an intersection of both quadrics and cubics.

This concludes the proof of Theorem 1.1.

REMARK 4.7. If we allow $\phi_H(X)$ to be a birational projective model of X (i.e. we require H to be birationally very ample only), we can allow the existence of a divisor E such that $E^2 = -2$ and $E.H = 0$. That is, we can allow the cases

$$d^2 - 4n(g-1) = n \qquad \text{and} \qquad d \equiv n \pmod{2n},$$

and

$$d^2 - 4n(g-1) = 4n \qquad \text{and} \qquad d \equiv 0 \pmod{2n}.$$

In the first case, with a lattice as in Proposition 4.2, the only contracted curve Γ satisfies $\Gamma.C = 1$, whence every smooth curve in $|C|$ is mapped isomorphically by ϕ_H to a smooth curve of degree d and genus g.

In the second case, define

$$\Gamma := \frac{d}{2n} H - C.$$

Then $\Gamma^2 = -2$, $\Gamma.H = 0$ and $\Gamma.C = 2$, so any irreducible member of $|C|$ contains a length two scheme where H fails to be very ample. Thus $\phi_H(C)$ is singular for all irreducible $C' \in |C|$.

This shows the statement in Remark 1.2.

5. Proof of Proposition 1.3

This section is devoted to the proof of Proposition 1.3.

Let $C' \in |C|$. By the exact sequence

$$0 \to \mathcal{O}_X(kH - C) \to \mathcal{O}_X(kH) \to \mathcal{O}_{C'}(kH) \to 0,$$

and using that $H^1(\mathcal{O}_X(kH)) = H^2(\mathcal{O}_X(kH)) = 0$ by the Kodaira vanishing theorem, we see that (using Serre duality)

$$h^1(\mathcal{O}_{C'}(k)) = h^0(C - kH).$$

If $d \leq 2nk$, then $(C-kH).H = d-2nk \leq 0$, which implies $h^0(C-kH) = 0$, since H is ample.

If $d > 2nk$ and $dk \leq nk^2+g$, we have $(C-kH)^2 \geq -2$ and $(C-kH).H > 0$. So by Riemann-Roch, $C - kH > 0$.

To finish the proof, let $d > 2nk$ and $dk > nk^2 + g$ and assume that there is an element $D \in |C - kH|$. Then $D^2 < -2$, so D has to contain an irreducible curve Γ such that $D.\Gamma < 0$ and $\Gamma^2 = -2$. As seen above, we can assume that Pic X is either generated by some rational multiple of the hyperplane section or generated by H and C. Clearly, in the first case, all divisors have non-negative self-intersection, so we can assume Pic $X = \mathsf{Z}H \oplus \mathsf{Z}C$.

We can write

$$D = m\Gamma + E,$$

with Γ not appearing as a component of E and $E \geq 0$.

If $E = 0$, then $m = 1$ since D is a part of a basis of Pic X, but then $D^2 = -2$, which is a contradiction, so $E > 0$. We then see that $D.\Gamma \geq -2m$ and $\Gamma.H < D.H/m$.

Now we define the divisor

$$D' := D + (D.\Gamma)\Gamma.$$

Then $D'^2 = D^2$ and

$$-D.H < D'.H = D.H + (D.\Gamma)(\Gamma.H) < D.H.$$

By the Hodge index theorem and the fact that $D'^2 < 0$, we get $D'^2 H^2 < (D'.H)^2$, so we have

$$\begin{aligned} 0 \neq (D'.H)^2 - D'^2 H^2 &= |\operatorname{disc}(H, D')| \\ &< (D.H)^2 - D^2 H^2 = |\operatorname{disc}(H, D)|, \end{aligned}$$

a contradiction, since clearly Pic $X = \mathsf{Z}H \oplus \mathsf{Z}D$.

This concludes the proof of Proposition 1.3.

6. Application to complete intersection $K3$ surfaces

$K3$ surfaces of degree 4 in P^3 or of degree 6 in P^4 have to be smooth quartics and smooth complete intersections of type (2, 3) respectively. Furthermore, by Proposition 2.6 a complete intersection $K3$ surface of degree 8 in P^5 has to be either an intersection of quadrics, in which case it is easily seen to be a complete intersection of type (2, 2, 2), or an intersection of both quadrics and cubics, in which case it cannot be a complete intersection.

Applying Theorem 1.1 to the three kinds of complete intersection $K3$ surfaces, one gets:

THEOREM 6.1. *Let $d > 0$ and $g \geq 0$ be integers. Then:*

(1) (Mori [9]) *There exists a smooth quartic surface X in $\mathbb{P}^3$ containing a smooth curve C of degree d and genus g if and only if a) $g = d^2/8 + 1$, or b) $g < d^2/8$ and $(d, g) \neq (5, 3)$. Furthermore, a) holds if and only if $\mathcal{O}_X(1)$ and $\mathcal{O}_X(C)$ are dependent in* Pic *X, in which case C is a complete intersection of X and a hypersurface of degree $d/4$.*

(2) *There exists a $K3$ surface X of type $(2, 3)$ in $\mathbb{P}^4$ containing a smooth curve C of degree d and genus g if and only if a) $g = d^2/12 + 1$, b) $g = d^2/12 + 1/4$ or c) $g < d^2/12$ and $(d, g) \neq (7, 4)$. Furthermore, a) holds if and only if $\mathcal{O}_X(1)$ and $\mathcal{O}_X(C)$ are dependent in* Pic *X, in which case C is a complete intersection of X and a hypersurface of degree $d/6$.*

(3) *There exists a $K3$ surface X of type $(2, 2, 2)$ in $\mathbb{P}^5$ containing a smooth curve C of degree d and genus g if and only if a) $g = d^2/16 + 1$ and d is divisible by 8, b) $g = d^2/16$ and $d \equiv 4 \pmod 8$, or c) $g < d^2/16$ and $(d, g) \neq (9, 5)$. Furthermore, a) holds if and only if $\mathcal{O}_X(1)$ and $\mathcal{O}_X(C)$ are dependent in* Pic *X, in which case C is a complete intersection of X and a hypersurface of degree $d/8$.*

NOTE ADDED IN PROOF. In remark 1.2, in case (ii)-(c) X can be chosen to be scheme-theoretically an intersection of quadrics unless $n = 5$, in which case X has to be an intersection of both quadrics and cubics. The reason for this is that only for $n = 5$ we have $H \sim 2B + \Gamma$, for two divisors B and Γ satisfying $B^2 = 2$, $\Gamma^2 = -2$ and $B.\Gamma = 1$ (see [14, Thm. 7.2]).

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DEPT. OF MATHEMATICS
UNIVERSITY OF BERGEN
JOHS. BRUNSGT 12
5008 BERGEN
NORWAY
E-mail: andreask@mi.uib.no