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HARDY-SOBOLEV SPACES OF COMPLEX TANGENTIAL DERIVATIVES OF HOLOMORPHIC FUNCTIONS IN DOMAINS OF FINITE TYPE

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Abstract

In this paper, we prove Fefferman-Stein like characterizations of Hardy-Sobolev spaces of complex tangential derivatives of holomorphic functions in domains of finite type in $\mathbb{C}^n$. We also study the relationship between these complex tangential Hardy-Sobolev spaces and the usual ones. We also obtain partial results on domains not necessarily of finite type.

0. Introduction

In this paper, we consider Hardy-Sobolev spaces of complex tangential derivatives of holomorphic functions in some domain Ω in $\mathbb{C}^n$. Let us precise the definition when $n = 2$. For L a complex tangential derivative in Ω, $k \in \mathbb{N}$ and u a holomorphic function in Ω, we denote by $\nabla_T^k u$ the $(k+1)$-tuple of functions given by $(u, Lu, \dots, L^k u)$. Then, we consider, for $0 < p < \infty$, the space $\mathscr{H}_{k,T}^p(\Omega)$ of holomorphic functions u in Ω for which the normal maximal function of $|\nabla_T^k u|$ belongs to $L^p(\partial\Omega)$. We call the complex tangential Hardy-Sobolev space of order k $\mathscr{H}_{k,T}^p(\Omega)$. One has to put in parallel the usual Hardy-Sobolev space $\mathscr{H}_k^p(\Omega)$ which is defined in terms of the total gradient. For this last one, Fefferman-Stein like characterizations hold in terms of the Littlewood-Paley function, the area integral or the maximal admissible function. These characterizations are proved when Ω is strictly pseudoconvex or of finite type in $\mathbb{C}^2$ where one can define geometrically adapted admissible approach regions. Since derivation preserves holomorphy, this follows from the corresponding characterizations of the Hardy space of holomorphic functions. We prove here analogous characterizations of $\mathscr{H}_{k,T}^p(\Omega)$ when Ω is of finite type in $\mathbb{C}^n$ with the main difficulty that complex tangential derivation does not preserve holomorphy. Here, we say that Ω *is of finite type m when the Lie*

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brackets up to order m of the complex tangential vector fields generate all the tangential space. Let us point out that part of the characterizations of $\mathscr{H}^p_{k,T}(\Omega)$ (as well as all the characterizations of $\mathscr{H}^p_k(\Omega)$) hold without any assumptions of finite type on Ω. In this case, we use a family of admissible approach regions $\mathscr{A}^{(m)}_\alpha(\zeta)$, $\zeta \in \partial\Omega$, which are arbitrarily large, as m increases, around Levi flat points ζ but which coïncide with the hyperbolic approach regions around stricly pseudoconvex points and which fit the domain around points of finite type m in $\mathbb{C}^2$.

Moreover, we study the relationship between $\mathscr{H}^p_{k,T}(\Omega)$ and $\mathscr{H}^p_k(\Omega)$. Note that in [11] and in [6], results were given in strictly pseudoconvex domains (or more generally in domains of finite type 2, the case of the unit ball in $\mathbb{C}^n$ have been done previously in [1]). In this case, $\mathscr{H}^p_k(\Omega)$ identifies with $\mathscr{H}^p_{2k,T}(\Omega)$. The situation cannot be as simple in the general case, since the inclusion $\mathscr{H}^p_k(\Omega) \subset \mathscr{H}^p_{2k,T}(\Omega)$ cannot be improved because of the strictly pseudoconvex points. To obtain converse inclusions, some finite type hypothesis is necessary. One needs to recover all complex derivatives from complex tangential ones. When Ω is of finite type m, we prove that a holomorphic function in $\mathscr{H}^p_{k,T}(\Omega)$ is in the usual Hardy-Sobolev space of order k/m.

Let us now describe precisely the setting.

Let $\Omega \subset \mathbb{C}^n$ be a bounded, smooth domain, given by

$$\Omega = \{z \in \mathbb{C}^n; r(z) < 0\}$$

with r a $\mathscr{C}^\infty$ function such that $|\nabla r| = 1$ on $\partial\Omega = \{r = 0\}$. For $\delta > 0$ and $z \in \Omega$, denote by $\tau(z, \delta)$ the function (eventually infinite) constructed by Catlin which gives, when Ω is of finite type m, the size in the complex tangential directions of the polydiscs that fits the domain around z (we will recall the precise definition of $\tau(z, \delta)$ in §1.1). For $m \geq 2$ an integer, denote by $\tau_m(z, \delta) := \min\{\tau(z, \delta), \delta^{1/m}\}$ and by $Q_m(z, \delta)$ the corresponding polydiscs. It gives a non-isotropic pseudo-distance d_m on $\partial\Omega$. This is equivalent to Catlin's pseudo-distance when Ω is of finite type μ, for any $m \geq \mu$ and gives arbitrarily large balls in complex tangential directions around flat points as m grows.

We identify a small neighborhood of $\partial\Omega$ in $\overline{\Omega}$, denoted by $\overline{\Omega} \cap U$, with $\partial\Omega \times [0, s_0[$ via a diffeomorphism Φ:

$$\Phi : \partial\Omega \times [0, s_0[\to \overline{\Omega} \cap U \quad \Phi(\zeta, 0) = \zeta,\ \zeta \in \partial\Omega.$$

For $z \in \overline{\Omega} \cap U$, let $\pi(z) \in \partial\Omega$ and $\delta(z) \geq 0$ be such that $\Phi(\pi(z), \delta(z)) = z$; $\delta(z)$ is equivalent to the distance to $\partial\Omega$. In the following, we will write $\tau_m(z)$ for $\tau_m(z, \delta(z))$ and we will forget the subscript m when there is no ambiguity.

We define the following quantities for any smooth function u and any aperture $\alpha > 0$:

- The normal maximal function:

$$\text{for any } \zeta \in \partial\Omega, \quad \mathscr{N} u(\zeta) = \sup\{|u(\Phi(\zeta, t))|;\ 0 < t < s_0\}.$$

- The maximal admissible function:

$$\text{for any } \zeta \in \partial\Omega, \quad \mathscr{M}_\alpha^{(m)} u(\zeta) = \sup\left\{|u(z)|;\ z \in \mathscr{A}_\alpha^{(m)}(\zeta)\right\}$$

where $\mathscr{A}_\alpha^{(m)}(\zeta)$ denotes the admissible approach region:

$$\mathscr{A}_\alpha^{(m)}(\zeta) = \{\Phi(z, t);\ \ z \in \partial\Omega,\ 0 < t < s_0,\ \ d_m(z, \zeta) < \alpha t\}.$$

- The Littlewood-Paley function:

$$\text{for any } \zeta \in \partial\Omega, \quad g(u)(\zeta) = \left(\int_0^{s_0} |u \circ \Phi(\zeta, t)|^2 \frac{dt}{t}\right)^{1/2}.$$

- The admissible area function:

$$\text{for any } \zeta \in \partial\Omega, \quad S_\alpha^{(m)} u(\zeta) = \left(\int_{\mathscr{A}_\alpha^{(m)}(\zeta)} |u(z)|^2 \frac{dV(z)}{\delta(z)^2 \tau(z, \delta(z))^{2n-2}}\right)^{1/2}.$$

Define the complex Hardy space $\mathscr{H}^p$ as the space of holomorphic functions u whose normal maximal functions are in $L^p(\partial\Omega)$. It follows from standard method (see [7] and [4] for harmonic functions and [3] and [11] in this context) that $\mathscr{H}^p$, $0 < p < \infty$, is characterized in terms of any of the preceding functionals. Namely, it is equivalent for a holomorphic function u to be in $\mathscr{H}^p(\Omega)$, to have $\mathscr{M}_\alpha^{(m)} u \in L^p(\partial\Omega)$, or $g(\delta\nabla u) \in L^p(\partial\Omega)$ or $S_\alpha^{(m)}(\delta\nabla u) \in L^p(\partial\Omega)$, independently on the aperture α and on the choice of m.

One can then consider Hardy-Sobolev spaces $\mathscr{H}_k^p(\Omega)$ of holomorphic functions, that is the spaces of holomorphic functions which have derivatives up to order k in $\mathscr{H}^p(\Omega)$. Since derivatives of holomorphic functions are still holomorphic, it is a corollary of the previous characterizations of $\mathscr{H}^p(\Omega)$ that similar characterizations hold for $\mathscr{H}_k^p(\Omega)$.

For $k \in \mathbb{N}$, $r \in \mathbb{N}^*$, $m \in \mathbb{N} \setminus \{0, 1\}$, $0 < p < \infty$, for a holomorphic function u in Ω, the following are equivalent

$$\begin{aligned}
u &\in \mathscr{H}_k^p(\Omega)\\
\mathscr{N}(|\nabla^k u|) &\in L^p(\partial\Omega)\\
\mathscr{M}_\alpha^{(m)}(|\nabla^k u|) &\in L^p(\partial\Omega) \ \textit{for some}\ \alpha \in]0, 1[\\
g(\delta^r |\nabla^{r+k} u|) &\in L^p(\partial\Omega)\\
S_\alpha^{(m)}(\delta^r |\nabla^{r+k} u|) &\in L^p(\partial\Omega) \ \textit{for some}\ \alpha \in]0, 1[.
\end{aligned}$$

(the symbol ∇^k denotes the collection of all the derivatives of order less than k).

We want here to prove the analogs for spaces involving only complex tangential derivatives. We want also to link these spaces to the usual Hardy-Sobolev spaces. Namely, denote by $\nabla_T^k u$ the collection of all possible composition of order less than k of the L_{ij}'s, $i < j$, given by

$$L_{ij} = \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_i} - \frac{\partial r}{\partial z_i}\frac{\partial}{\partial z_j}.$$

As before, denote by $\mathscr{H}^p_{k,T}(\Omega)$ the set of holomorphic functions u in Ω such that $\mathscr{N}(|\nabla_T^k u|) \in L^p(\partial\Omega)$.

Our first result holds without any assumption on the type of Ω.

THEOREM 0.1. *For $k \in \mathbb{N}$, $m \geq 2$ an integer and $0 < p < \infty$, the following are equivalent for a holomorphic function u in Ω.*

(i) $u \in \mathscr{H}^p_{k,T}(\Omega)$,

(ii) $\mathscr{M}^{(m)}_\alpha(|\nabla_T^k u|) \in L^p(\partial\Omega)$ *for some* $\alpha \in]0, 1[$.

Furthermore, if $S^{(m)}_\alpha(\tau_m^{-k}\delta^r|\nabla^r u|) \in L^p(\partial\Omega)$ for some $r \in \mathbb{N}$ so that $2r-k \geq 1$ then $u \in \mathscr{H}^p_{k,T}(\Omega)$.

REMARK 0.2. The last statement implies in particular that $\mathscr{H}^p_{k/2}(\Omega) \subset \mathscr{H}^p_{k,T}(\Omega)$ (since $c\delta(z)^{1/2} \leq \tau_m(z)$).

REMARK 0.3. When Ω is Levi flat around some point, part (ii) states that the supremum can be taken over arbitrarily large admissible regions around this point.

THEOREM 0.4. *Let Ω be a bounded smooth domain of finite type m in $\mathbb{C}^n$. For $k \in \mathbb{N}$, and $1 - \frac{1}{mn+1} < p < \infty$, the following are equivalent for a holomorphic function u in Ω.*

(i) $u \in \mathscr{H}^p_{k,T}(\Omega)$,

(ii) $\mathscr{M}^{(m)}_\alpha(|\nabla_T^k u|) \in L^p(\partial\Omega)$ *for some* $\alpha \in]0, 1[$,

(iii) $g(\delta|\nabla\nabla_T^k u|) \in L^p(\partial\Omega)$,

(iv) $S^{(m)}_\alpha(\delta|\nabla\nabla_T^k u|) \in L^p(\partial\Omega)$ *for some* $\alpha \in]0, 1[$,

(v) $S^{(m)}_\alpha(\tau_m^{-k}\delta^r|\nabla^r u|) \in L^p(\partial\Omega)$ *for some $r \in \mathbb{N}$ so that* $2r - k \geq 1$.

REMARK 0.5. The last statement implies that, when Ω is of finite type m, a function in $\mathscr{H}^p_{k,T}(\Omega)$ is also in the ordinary Hardy-Sobolev space $\mathscr{H}^p_{k/m}(\Omega)$ (since $\tau_m(z) \leq C\delta(z)^{1/m}$). We recover in this context the well known phenomenon of finite type domains: complex tangential derivatives of holomorphic functions behave at least as well as global derivatives of order $1/m$ in domains of finite type m. It actually says something more subtle. A complex tangential gradient of order k behaves like $\tau_m^{-k}\delta^r|\nabla^r u|$ and conversely in domains of finite type m. In particular, this means that $\nabla_T^k u$ behaves as *an ordinary gradient whose order changes from point to point.*

REMARK 0.6. In this paper, we only give the proof of Theorem 0.4 when $0 < p < 2$. When $p \geq 2$, the result follows from singular integrals machinery and some commutation properties (see [12]).

The key point in the proofs of Theorem 0.1 and 0.4 is the use of mean-value properties for complex tangential derivatives. For $z \in \Omega$, denote by $Q_m(z)$ the set

$$Q_m(z) := \{w \in \Omega;\ \delta(z)/2 \leq \delta(w) \leq 2\delta(z);\ d_m(\pi(z), \pi(w)) \leq \delta(z)/2\}.$$

Denote by $\mathrm{Mean}^{Q_m(z)}(|F|)$ the mean-value of $|F|$ over $Q_m(z)$. We prove the following.

THEOREM 0.7 (Mean-value inequality). *For $k, l \in \mathbb{N}$, $0 < p < \infty$ and $m \geq 2$ an integer, there exists a constant $C > 0$ such that, for u holomorphic function in Ω and z in $\Omega \cap U$,*

$$\delta(z)^{lp}|\nabla^l\nabla_T^k u(z)|^p \leq C\,\mathrm{Mean}^{Q_m(z)}(|\nabla_T^k u|^p).$$

To get these mean-value properties, we improve the usual freezing coefficient method which consists in taking the coefficients of L to be constant up to a remaining term so that it preserves holomorphy. As this is not sufficient here, we "freeze" the coefficients to a higher order by using a Taylor expansion of the coefficients of L up to a sufficiently large order.

To prove the link between complex tangential derivatives and ordinary derivatives, we use the pointwise estimates between complex tangential gradients and ordinary gradients proved in [10]. Namely, one has the following:

POINTWISE ESTIMATES ([10]). *For $k \in \mathbb{N}$, u a holomorphic function in Ω, and $z \in \Omega$,*

$$\tau(z)^{2k}|\nabla_T^k u(z)|^2 \leq C\,\mathrm{Mean}^{Q(z)}(|u|^2). \tag{1}$$

Moreover if Ω is of finite type m in $\mathbb{C}^n$ then for $\epsilon > 0$ there exists $C(\epsilon)$ so that

$$\delta(z)^{2k}|\nabla^k u(z)|^2 \leq \mathrm{Mean}^{Q(z)}(C(\epsilon)\tau^{2k}|\nabla_T^k u|^2 + \epsilon^2|u|^2). \tag{2}$$

The paper is organized as follows. In section 1, we recall some basic definitions and properties of the geometry and prove Theorem 0.7. Theorem 0.1

follows at once. In section 2, we establish the relations between usual area integrals and area integrals of complex tangential derivatives. In section 3, we conclude by showing the links between area integrals and maximal functions of complex tangential derivatives.

As said before, we proved these results in the context of domains of finite type 2 in [11]. The main innovation in this paper is to develop a new technic which allows to overcome the technical difficulties which appear for $m > 2$.

In the following, we will use the symbol $A \lesssim B$ if there exists a universal constant C so that $A \leq CB$. Similarly, we will write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

1. Geometry and mean-value properties

In this paragraph, we will assume for simplicity that $n = 2$.

1.1. Geometry

Assume Ω is a domain in $\mathbb{C}^2$. Let us recall the following facts from [5] (see also [8]). Let $z_0 \in \partial\Omega$, as $|\nabla r|(z_0) = 1$, we may assume that $\frac{\partial r}{\partial z_1} \neq 0$ in a neighborhood $V(z_0)$ of z_0. Then

LEMMA 1.1. *Let $M \in \mathbb{N}$, $M \geq 2$. For any $z \in V(z_0) \cap \Omega$, there exists a biholomorphic mapping $\Phi_z : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\varrho := r \circ \Phi_z$ satisfies:*

$$\varrho(\zeta) = r(z) + Re(\zeta_1) + \sum_{\substack{j,k\in\mathbb{N} \\ j,k\geq 1; j+k\leq M}} a_{j,k}(z)\zeta_2{}^j\overline{\zeta_2}^k + \mathcal{O}\left(|\zeta_2|^{M+1} + |\zeta_1|\,|\zeta|\right).$$

Moreover

$$\Phi_z(\zeta) = \left(z_1 + d_0(z)\zeta_1 + \sum_{k=1}^{M} d_k(z)\zeta_2{}^k, z_2 + \zeta_2\right)$$

where $d_0(.)$, $d_k(.)$; $k = 1, \ldots, M$ depend smoothly on z and $d_0(.) \neq 0$ in $V(z_0)$.

It is easy to extend this result to arbitrary dimension (this is done for instance in [10]). It is important to note that this change of variables is independent on any assumption on the type of Ω. Now fix $m \geq 2$ an integer and take $M \geq m$ in the preceding lemma. Define $A_l(z) := \max\left\{|a_{j,k}(z)|;\ j + k = l\right\}$. For $\delta > 0$, denote by $\tau(z, \delta) = \min\left\{\left(\frac{\delta}{A_l(z)}\right)^{1/l},\ l = 2, \ldots, m\right\}$. This defines a function on $V(z_0) \cap \Omega$ with values in $\overline{\mathbb{R}_+}$. When Ω is of finite type m, there exists $l \in \{2, \ldots, m\}$ such that $A_l(z) \neq 0$ for $z \in \partial\Omega$ and by continuity for $z \in V(z_0)$ sufficiently small so that $\tau(z, \delta)$ takes finite values. Now define

$\tau_m(z,\delta) := \min\{\tau(z,\delta), \delta^{1/m}\}$. Remark that if Ω is of finite type m, then, for any $\mu \geq m$, $\tau_m \simeq \tau_\mu \simeq \tau$. Define the polydisc around z by

$$Q_m(z,\delta) = \Phi_z\left(R_m(z,\delta)\right) = \Phi_z\big(\{\zeta \in \mathbb{C}^2;\ \ |\zeta_1| < \delta,\ \ |\zeta_2| < \tau_m(z,\delta)\}\big).$$

The following properties hold:

(1) there exists a constant $C > 0$ such that, for any $z \in V(z_0)$ and $0 < \delta < 1$,

$$\frac{1}{C}\delta^{1/2} \leq \tau_m(z,\delta) \leq C\delta^{1/m}.$$

(2) if $\delta' < \delta$ then $\left(\frac{\delta'}{\delta}\right)^{1/2}\tau_m(z,\delta) \leq \tau_m(z,\delta') \leq \left(\frac{\delta'}{\delta}\right)^{1/m}\tau_m(z,\delta)$.

(3) for any $0 < \delta < 1$ and $z \in Q_m(z',\delta)$, $\tau_m(z,\delta) \simeq \tau_m(z',\delta)$.

(4) there exists a constant $C > 0$ such that, if $z \in Q_m(z',\delta)$, then $Q_m(z,\delta) \subset Q_m(z',\delta)$ and $Q_m(z',\delta) \subset Q_m(z, C\delta)$.

By definition, there exists a constant c such that, for any $z \in V(z_0)$,

$$Q_m(z, c\delta(z)) \subset \Omega.$$

We will note $Q_m(z) = Q_m(z, c\delta(z)) = \Phi_z(R_m(z))$ and $\tau_m(z) = \tau_m(z, c\delta(z))$.

(5) In addition, for any $\zeta \in Q_m(z)$, $\tau_m(\zeta) \simeq \tau_m(z)$.

It follows from these properties that

$$d_m(z,\zeta) = \inf\{\delta > 0, z \in Q_m(\zeta,\delta) \cap \partial\Omega\}$$

defines a pseudo-distance on $\partial\Omega$.

1.2. Mean-value property for complex tangential derivatives and applications

Let E be a measurable subset of Ω. Denote by $\mathrm{Mean}^E(F)$ the mean-value of $|F|$ over E with respect to the Lebesgue measure.

We prove the following proposition.

PROPOSITION 1.2. *For $k, l, r, m \in \mathbb{N}$, $m \geq 2$, $0 < p < \infty$, there exists a constant $C > 0$ such that, for any holomorphic function u in Ω and any z in $\Omega \cap U$,*

$$\delta(z)^{lp}|\nabla^{l+r}\nabla_T^k u(z)|^p \leq C\,\mathrm{Mean}^{Q_m(z)}(|\nabla^r\nabla_T^k u|^p).$$

Once this proposition is proved it follows by standard methods (see [7] or [14] for instance) that

COROLLARY 1.3. *For $k, m \in \mathbb{N}$, $m \geq 2$, $\alpha > 0$, $0 < p < \infty$ and a holomorphic function u,*

$$\left\| \mathscr{M}_\alpha^{(m)}(|\nabla_T^k u|) \right\|_{L^p(\partial\Omega)} \lesssim \left\| \mathscr{N}(|\nabla_T^k u|) \right\|_{L^p(\partial\Omega)},$$

$$\text{and for any } \zeta \in \partial\Omega, \quad g(\delta |\nabla\nabla_T^k u|)(\zeta) \lesssim S_\alpha^{(m)}(\delta|\nabla\nabla_T^k u|)(\zeta).$$

This gives the equivalence between (i) and (ii) of Theorem 0.1 and that (iv) implies (iii) in Theorem 0.4. Note that, in fact, the implication (iv) $\Rightarrow$ (iii) does not need any finite type hypothesis.

Let us now prove Proposition 1.2. First, remark that $|\nabla^r \nabla_T^k u| \simeq |\nabla_T^k \nabla^r u|$: for $k = r = 1$, the commutator of any first order derivative and ∇_T is a differential operator of order 1 with smooth coefficients. As ∇_T contains the identity by definition, we can write $|\nabla\nabla_T u| \simeq |\nabla_T \nabla u|$. For larger r and k, the result follows from induction.

So, as ordinary derivatives preserve holomorphy, it is enough to consider the case $r = 0$. We are going to write $L^k u$ as a sum of a function satisfying mean-value properties and of a remaining term. For this we introduce the following class of functions.

DEFINITION 1.4. Let $K = (k_1, k_2)$ be a multi-index of positive integers. A function $F \in \mathscr{C}^\infty(\Omega)$ is called $(AB)_K$ if $\frac{\partial^{k_j} F}{\partial \overline{\zeta_j}^{k_j}} = 0$ for $j = 1, 2$ in Ω.

To simplify notation, we will assume that K is fixed in the following and we will write (AB) instead of $(AB)_K$.

For any $\zeta \in \mathbb{C}$ and $r > 0$, we denote by $\mathsf{D}(\zeta, r)$ the disc $\{z \in \mathbb{C};\ |z - \zeta| \leq r\}$. The terminology (AB) comes from Ahern and Bruna who proved the following lemma (cf [1]):

LEMMA 1.5. *For (l_1, l_2) and $(m_1, m_2) \in \mathbb{N}^2$, $0 < p < \infty$, there exists a constant C such that, for any (AB)-function F in Ω, any $\zeta = (\zeta_1, \zeta_2) \in \Omega$ and any $r = (r_1, r_2) \in (]0, +\infty[)^2$ such that $\mathsf{D}(\zeta_1, r_1) \times \mathsf{D}(\zeta_2, r_2) \subset \Omega$,*

$$r_1^{p(l_1+m_1)} r_2^{p(l_2+m_2)} \left| \frac{\partial^{l_1+l_2+m_1+m_2} F}{\partial \overline{\zeta_1}^{l_1} \partial \overline{\zeta_2}^{l_2} \partial \zeta_1^{m_1} \partial \zeta_2^{m_2}}(\zeta) \right|^p \leq C\, \mathrm{Mean}^{\mathsf{D}(\zeta_1, r_1) \times \mathsf{D}(\zeta_2, r_2)}(|F|^p).$$

Given $z \in \Omega \cap U$, let $w = \Phi_z(\zeta)$ and $\varrho = r \circ \Phi_z(\zeta)$. Denote by $L' = \frac{\partial \varrho}{\partial \zeta_2} \frac{\partial}{\partial \zeta_1} - \frac{\partial \varrho}{\partial \zeta_1} \frac{\partial}{\partial \zeta_2}$ a holomorphic complex tangential vector field. Recall that $Q_m(z) = \Phi_z(R_m(z))$ where $R_m(z) = R_m(z, c\delta(z))$ and $R_m(z, \delta) = \{\zeta \in \mathbb{C}^2;\ |\zeta_1| < \delta,\ |\zeta_2| < \tau_m(z, \delta)\}$. We have the following lemma:

LEMMA 1.6. *For $\zeta \in R_m(z)$, $k, l \in \mathbb{N}$ and a holomorphic function f in $\Phi_z(\Omega)$,*

$$L'^k f(\zeta) = F_{kl} f(\zeta) + R_{kl} f(\zeta)$$

where $F_{kl}f$ is an (AB)-function and for $0 < p < \infty$, $\zeta \in R_m(z)$, $0 \leq j \leq l$,

$$\delta(z)^{jp}|\nabla^j R_{kl} f(\zeta)|^p \leq C\,\mathrm{Mean}^{R_m(z)}\left(|f|^p\right).$$

Proof. It follows easily by induction on $k \in \mathbb{N}$ that there exist some constants $c_{r,s}$, $1 \leq r+s \leq k$, such that

$$L'^k = \sum_{1\leq r+s\leq k}\sum_{E_{k,r,s}} c_{r,s}\left(\prod_{j=1}^{k}\frac{\partial^{m_j+n_j}\varrho}{\partial\zeta_1^{m_j}\partial\zeta_2^{n_j}}\right)\frac{\partial^{r+s}}{\partial\zeta_1^r\partial\zeta_2^s},$$

where $E_{k,r,s}$ denotes the set of couples (m_j, n_j), $j = 1, dots, k$, in lexicographical order, which satisfy $\sum_{j=1}^k m_j = k-r$ and $\sum_{j=1}^k n_j = k-s$ with $m_j + n_j \geq 1$.

For any $N \in \mathbb{N}$, we can write $\varrho = T_0^N\varrho + R_0^N\varrho$ where $T_0^N\varrho$ stands for the Taylor expansion of ϱ up to order N.

Assume for simplicity that $l = 0$. Choose $N = 2k-1$. Since ϱ is $\mathscr{C}^\infty$, one has

$$\frac{\partial^{m_j+n_j}\varrho}{\partial\zeta_1^{m_j}\partial\zeta_2^{n_j}} = \frac{\partial^{m_j+n_j}T_0^N\varrho}{\partial\zeta_1^{m_j}\partial\zeta_2^{n_j}} + r_{N,m_j,m_j} \quad \text{where } r_{N,m_j,m_j} = \mathscr{O}\big(|\zeta|^{N+1-m_j-n_j}\big).$$

Now, for $\zeta \in R_m(z)$, $|\zeta| \leq \tau_m(z)$ so one obtains, since $m_j+n_j \leq 2k-(r+s)$,

$$\begin{aligned} L'^k f(\zeta) &= \sum_{1\leq r+s\leq k}\left(\prod_{j=1}^{k}\left(\frac{\partial^{m_j+n_j}T_0^N\varrho}{\partial\zeta_1^{m_j}\partial\zeta_2^{n_j}}\right) + \mathscr{O}(\tau_m(z)^{N+1-2k+(r+s)})\right)\frac{\partial^{l+s} f}{\partial\zeta_1^r\partial\zeta_2^s}(\zeta) \\ &= F_{k0}f(\zeta) + R_{k0}f(\zeta). \end{aligned}$$

By definition, F_{k0} is an $(AB)_K$-function for $K = K(N)$ large enough.

But, by the mean-value properties satisfied by f, for any $\zeta \in R_m(z)$

$$\begin{aligned} \sum_{1\leq r+s\leq k}\mathscr{O}\big(\tau_m(z)^{N+1-2k+(r+s)}\big)^p\left|\frac{\partial^{r+s}f}{\partial\zeta_1^r\partial\zeta_2^s}\right|^p(\zeta) &\leq C\,\mathrm{Mean}^{R_m(z)}\big(\tau_m^{p(N+1-2k)}|f|^p\big) \\ &\leq C'\,\mathrm{Mean}^{R_m(z)}(|f|^p). \end{aligned}$$

This gives the lemma.

Proof of the proposition. Denote by f the holomorphic function in $\Phi_z(\Omega)$ given by $f = u\circ\Phi_z$. Write $\delta^{lp}(z)|\nabla^l L'^k f|^p \lesssim \delta^{lp}(z)|\nabla^l F_{kl} f|^p + \delta(z)^{lp}|\nabla^l\nabla^l R_{kl} f|^p$. The second term is bounded by the mean- value of $|f|^p$

on $R_m(z)$. For the first term, we use Lemma . to get a bound as $\mathrm{Mean}^{R_m(z)}$ $(|F_{kl} f|^p)$. This in turn is bounded by

$$\mathrm{Mean}^{R_m(z)}(|L'^k f|^p + |R_{kl} f|^p) \le C\,\mathrm{Mean}^{R_m(z)}(|L'^k f|^p + |f|^p).$$

Going back to Ω, it gives the result for L since L' corresponds to a smooth non-vanishing function times L.

REMARK 1.7. It is by an analogous method that the pointwise estimates quoted in the introduction are proved in [10].

In the following, we will forget the subscript m to simplify the notations.

2. Area Integrals

2.1. Area integrals and area integrals of complex tangential derivatives

First, recall that usual methods, involving Hardy inequality and mean-value properties, allow to prove that, for $0 < p \le 2$ and u holomorphic in Ω,

$$(*) \qquad \|S_\alpha(\delta^{r+\eta}\tau^{-k}|\nabla^r u|)\|_{L^p(\partial\Omega)} \lesssim \|S_\alpha(\delta^{l+\eta}\tau^{-k}|\nabla^l u|)\|_{L^p(\partial\Omega)} + \sup_K |u|$$

as long as $r+\eta-k/2$ and $l+\eta-k/2$ are positive, where K denotes a compact subset of Ω (see [4] and [3]). The same kind of method using part (1) of the pointwise estimates of the introduction gives that

$$\|S_\alpha(\delta^{\eta+1}|\nabla\nabla_T^k u|)\|_{L^p(\partial\Omega} \lesssim \|S_\alpha(\delta^{r+\eta}\tau^{-k}|\nabla^r u|)\|_{L^p(\partial\Omega)} + \sup_K |u|$$

for $r+\eta-k/2 > 0$ and $\eta > -1$ (see [11] in the context of domains of type 2 and [9]).

We prove now a converse inequality. For $0 < p \le 2$, $\|S_\alpha(\delta^{r+\eta}\tau^{-k}|\nabla^r u|)\|_p$ can be estimated by the L^p-norm of $S_\alpha(\delta^{j+\eta}|\nabla^j\nabla_T^k u|)$ when $r+\eta-k/2 > 0$ and $j+\eta > 0$. This estimate is proved in [9]. We give here a simplified proof.

By $(*)$, it is sufficient to prove the required estimate for some r big enough. Apply the converse pointwise estimates (2) to the component of $\nabla^l u$, l will be chosen large enough, and integrate over $\mathscr{A}_\alpha(\zeta)$ to get

$$\begin{aligned} &S_\alpha(\delta^{k+l+\eta}\tau^{-k}|\nabla^{k+l}u|)(\zeta) \\ &\qquad\lesssim C(\epsilon)S_\beta(\delta^{l+\eta}|\nabla_T^k\nabla^l u|)(\zeta) + \epsilon S_\beta(\delta^{l+\eta}\tau^{-k}|\nabla^l u|)(\zeta)\end{aligned}$$

for some $\beta > \alpha$. Now, by the mean-value properties, the first term is majorized by $S_\gamma(\delta^{j+\eta}|\nabla^j\nabla_T^k u|)(\zeta)$ for any $j \in \mathbb{N}$ so that $j+\eta > 0$. And for l large enough, the L^p-norms of $S_\beta(\delta^{l+\eta}\tau^{-k}|\nabla^l u|)$ and of $S_\alpha(\delta^{k+l+\eta}\tau^{-k}|\nabla^{k+l}u|)$ are equivalent to $\|S_\alpha(\delta^{r+\eta}\tau^{-k}|\nabla^r u|)\|_p$ for any $r+\eta-k/2>0$. So, as the L^p-norms of the area integrals S_α are independent on the aperture α, it gives an a priori estimate for ϵ small enough. We get rid of the a priori assumption as in [11] by applying this inequality in $\Omega_\epsilon=\{z\in\Omega;\ \delta(z)>\epsilon\}$ and letting ϵ goes to 0. Eventually we get the following result.

PROPOSITION 2.1. *Assume Ω is of finite type in $\mathbb{C}^n$. For $k, r, j \in \mathbb{N}$, $0<p\le 2$, $\alpha, \eta \in \mathbb{R}$ so that $r+\eta-k/2>0$, $j+\eta>0$, for u holomorphic in Ω*

$$\|S_\alpha(\delta^{r+\eta}\tau^{-k}|\nabla^r u|)\|_p \lesssim \|S_\alpha(\delta^{\eta+j}|\nabla^j\nabla_T^k u|)\|_p+\sup_K |u|.$$

2.2. *An embedding result*

In this section, we prove a key estimate to deal with the remaining terms.

PROPOSITION 2.2. *Assume Ω is of finite type m in $\mathbb{C}^n$.*
Let $\mu\in]0,1/m[$. For $1-\frac{(1/m-\mu)}{n+(1/m-\mu)}<p\le 2$, there exists $q\ge p$, $q>1$ so that

$$\|S_\alpha(\delta^{1-\mu}|\nabla\nabla_T^{k-1}u|)\|_q \le \|\mathscr{M}_\alpha(|\nabla_T^k u|)\|_p.$$

PROOF. By the preceding paragraph, $\|\mathscr{S}_\alpha(\delta^{1-\mu}|\nabla\nabla_T^{k-1}u|)\|_q$ is successively bounded by

$$\|\mathscr{S}_\alpha(\delta^{k-\mu}\tau^{-k+1}|\nabla^k u|)\|_q,\quad \|\mathscr{S}_\alpha(\delta^{-\mu}\tau|\nabla_T^k u|)\|_q,\quad \|\mathscr{S}_\alpha(\delta^{1/m-\mu}|\nabla_T^k u|)\|_q$$

up to $\sup_K|u|$. This in turn is bounded by

$$C\|\mathscr{M}_\alpha(\delta^{1/m-\mu-\epsilon}|\nabla_T^k u|)\|_{L^q(\partial\Omega)}$$

for any $\epsilon>0$ since

$$\mathscr{S}_\alpha(\delta^{1/m-\mu}|\nabla_T^k u|)(\zeta) \le \mathscr{M}_\alpha(\delta^{1/m-\mu-\epsilon}|\nabla_T^k u|)(\zeta)\times\left(\int_{\mathscr{A}_\alpha(\zeta)}\frac{\delta(z)^\epsilon\,dV(z)}{\delta(z)^2\tau^2(z,\delta(z))}\right)^{1/2}.$$

Now, using the atomic decomposition of spaces of homogeneous type (see [2]), one can show (see [11]) that

$$\|\mathscr{M}_\alpha(\delta^{1/m-\mu-\epsilon}|\nabla_T^k u|)\|_{L^q(\partial\Omega)}\le\|\mathscr{M}_\alpha(|\nabla_T^k u|)\|_{L^p(\partial\Omega)}$$

if $1/m-\mu-\epsilon\ge n/p-n/q$.

It is possible to find such a q by assumption on the range of p.

3. Characterizations of complex tangential Hardy-Sobolev spaces

3.1. Estimate on the normal maximal function by the Littlewood-Paley function

In this paragraph, we prove that (iii) implies (i) of Theorem 0.4. More precisely, we prove, without finite type hypothesis the following result.

PROPOSITION 3.1. *For $k \in \mathbb{N}$, $0 < p < \infty$ and u holomorphic in Ω,*

$$\|\mathcal{N}(|\nabla_T^k u|)\|_{L^p(\partial\Omega)} \lesssim \|g(\delta|\nabla\nabla_T^k u|)\|_{L^p(\partial\Omega)} + s_0^\theta \|S_\alpha(\delta^{1-\theta}|\nabla\nabla_T^{k-1} u|)\|_{L^q(\partial\Omega)}$$

for any $q > 1$, $q \geq p$, any $\theta \in]0, 1[$.

REMARK 3.2. When Ω is of finite type, it gives an a-priori estimate when $1 - \frac{1}{mn+1} < p \leq 2$, since by Proposition 2.2, for θ sufficiently close to 0, one can choose $q > 1, q \geq p$, so that

$$\|S_\alpha(\delta^{1-\theta}|\nabla\nabla_T^{k-1} u|)\|_{L^q(\partial\Omega)} \lesssim \|\mathcal{M}_\alpha(\nabla_T^k u)\|_{L^p(\partial\Omega)} \lesssim \|\mathcal{N}(|\nabla_T^k u|)\|_{L^p(\partial\Omega)}.$$

So, if $u \in \mathscr{C}^\infty(\overline{\Omega}) \cap \mathscr{H}(\Omega)$, for s_0 small enough, we have

$$\|\mathcal{N}(|\nabla_T^k u|)\|_{L^p(\partial\Omega)} \leq C\|g(\delta\nabla\nabla_T^k u)\|_{L^p(\partial\Omega)}.$$

To obtain the general result, one has to apply this estimate in $\Omega_\epsilon = \{\Phi(z, t), t > \epsilon\}$ (since a holomorphic function in Ω is in particular $\mathscr{C}^\infty(\overline{\Omega}_\epsilon)$) and to let ϵ goes to zero. On one hand

$$\int_\epsilon^{\epsilon_0} t^2|f|^2(\Phi(\zeta, t))\frac{dt}{t} \leq g(\delta|f|)(\zeta),$$

on the other hand, the monotone convergence theorem proves that

$$\lim_{\epsilon\to 0} \|\sup_{\epsilon<t<\epsilon_0} |\nabla_T^k u|\|_{L^p(\partial\Omega)} = \|\mathcal{N}(|\nabla_T^k u|)\|_{L^p(\partial\Omega)}.$$

PROOF. The method is analogous to the one used in [11]. The trick is to write $\nabla_T^k u$ as the sum of a harmonic function and of a remaining term.

Write $\nabla_T^k u = (\nabla_T^k u)_0 + (\nabla_T^k u)_h$ where $(\nabla_T^k u)_0$ is the (vector)-solution to the Dirichlet problem

$$\begin{cases} \Delta v = \Delta(\nabla_T^k u) \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega. \end{cases}$$

Then,

$$\begin{aligned}\left|(\nabla_T^k u)_0 \circ \Phi(\zeta,t)\right| &\leq \int_0^{s_0} |\nabla(\nabla_T^k u)_0 \circ \Phi(\zeta,s)|ds + \sup_K |u| \\ &\leq s_0^\theta \left(\int_0^{s_0} \left|\nabla(\nabla_T^k u)_0 \circ \Phi(\zeta,s)\right|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2} + \sup_K |u|\end{aligned}$$

where K is a compact subset of Ω, $0 < \theta < 1$.

So, it gives

$$\begin{aligned}&\|\mathcal{N}(|\nabla_T^k u|)\|_{L^p(\partial\Omega)} \leq \|\mathcal{N}(|(\nabla_T^k u)_h|)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(|(\nabla_T^k u)_0|)\|_{L^p(\partial\Omega)} \\ &\quad \lesssim \|g(\delta\nabla(\nabla_T^k u)_h)\|_{L^p(\partial\Omega)} \\ &\qquad + s_0^\theta \left\|\left(\int_0^{s_0} |\nabla(\nabla_T^k u)_0 \circ \Phi(.,s)|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2}\right\|_{L^p(\partial\Omega)} + \sup_K |u| \\ &\quad \lesssim \|g(\delta\nabla\nabla_T^k u)\|_{L^p(\partial\Omega)} + \|g(\delta\nabla\nabla_T^k u)_0\|_{L^p(\partial\Omega)} \\ &\qquad + s_0^\theta \left\|\left(\int_0^{s_0} |\nabla(\nabla_T^k u)_0 \circ \Phi(.,s)|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2}\right\|_{L^p(\partial\Omega)} + \sup_K |u| \\ &\quad \lesssim \|g(\delta\nabla\nabla_T^k u)\|_{L^p(\partial\Omega)} \\ &\qquad + 2s_0^\theta \left\|\left(\int_0^{s_0} |\nabla(\nabla_T^k u)_0 \circ \Phi(.,s)|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2}\right\|_{L^p(\partial\Omega)} + \sup_K |u|.\end{aligned}$$

Now, by estimates on the Dirichlet problem (see [11] appendix for study in this context or [13]) we obtain that, for some $q > 1$, $q \geq p$

$$\begin{aligned}(*) &= \left\|\left(\int_0^{s_0} \left|\nabla(\nabla_T^k u)_0 \circ \Phi(.,s)\right|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2}\right\|_{L^p(\partial\Omega)} \\ &\leq \left\|\left(\int_0^{s_0} \left|\nabla(\nabla_T^k u)_0 \circ \Phi(.,s)\right|^2 s^{2-2\theta}\frac{ds}{s}\right)^{1/2}\right\|_{L^q(\partial\Omega)} \\ &\leq \left\|\Delta\big(\nabla_T^k u\big)\right\|_{W_\theta^{-1,(q,2)}(\Omega)}\end{aligned}$$

where $W_\theta^{-1,(q,2)}$ denotes the usual Sobolev space. Now, since u is holomorphic, $|\Delta\nabla_T^k u| = |[\Delta, \nabla_T^k]u|$. Note that $|[\Delta, \nabla_T^k]u| \simeq |\nabla^2\nabla_T^{k-1}u|$: First, recall that $|\nabla_T^k\nabla^r u| \simeq |\nabla^r\nabla_T^k u|$. The commutator $[\Delta, \nabla_T^k]$ is obtained by derivating at least one and at most two complex tangential vector fields of the ∇_T^k. If only one is derivated, one gets a term $\simeq |\nabla^2\nabla_T^{k-1}u|$, if two are derivated, one obtains a term $\simeq |\nabla^2 \nabla_T^{k-2} u|$. So,

$$\left\| \Delta\left(\nabla_T^k u\right) \right\|_{W_\theta^{-1,(q,2)}(\Omega)} \simeq \left\| \left(\int_0^{s_0} |\nabla \nabla_T^{k-1} u|^2 s^{2-2\theta} \frac{ds}{s} \right)^{1/2} \right\|_{L^q(\partial\Omega)}.$$

Now, by the mean-value properties, this is bounded by

$$\left\| \mathscr{S}_\alpha\left(\delta^{1-\theta} \left|\nabla \nabla_T^{k-1} u\right|\right) \right\|_{L^q(\partial\Omega)}.$$

This ends the proof of the proposition.

3.2. Estimate of the area integral by the admissible maximal function

In this paragraph, we adapt the method of [7] to our setting. We are going to prove the following result.

PROPOSITION 3.3. *Let* $\epsilon > 0$. *For* $0 < \mu < 1$, $0 < p < 2$, $\alpha > 0$ *and* u *holomorphic in* Ω,

$$\begin{aligned} \left\| S_\alpha\left(\delta |\nabla \nabla_T^k u|\right) \right\|_p \lesssim & \left(\frac{1}{\epsilon^2} + 1 \right) \left\| \mathscr{M}_\alpha\left(\nabla_T^k u\right) \right\|_p \\ & + (\epsilon + s_0^\mu) \left\| S_\alpha\left(\delta |\nabla \nabla_T^k u|\right) \right\|_p + \left\| S_\alpha\left(\delta^{1-\mu} |\nabla \nabla_T^{k-1} u|\right) \right\|_p + \sup_K |u|. \end{aligned}$$

REMARK 3.4. Implication (ii) $\Rightarrow$ (iv) of Theorem 0.7 follows: By §2.1, this gives an a-priori estimate when Ω is of finite type m and $0 < p < 2$. Indeed, $\|S_\alpha(\delta^{1-\mu} |\nabla \nabla_T^{k-1} u|)\|_p$ is estimated by $\|S_\alpha(\delta^{1-\mu} \tau |\nabla \nabla_T^k u|)\|_p \lesssim s_0^{1/m-\mu} \|S_\alpha(\delta |\nabla \nabla_T^k u|)\|_p$ if $1/m - \mu > 0$.

We conclude that, when Ω is of finite type m, for $u \in \mathscr{C}^\infty(\overline{\Omega}) \cap \mathscr{H}(\Omega)$, we have

$$\left\| S_\alpha\left(\delta \nabla \nabla_T^k u\right) \right\|_{L^p(\partial\Omega)} \le C\left(\left\| \mathscr{M}_\alpha\left(\nabla_T^k u\right) \right\|_{L^p(\partial\Omega)} + \sup_K |u| \right).$$

It remains to show that this inequality is still valid for general u. We apply this inequality in $\Omega_\epsilon = \{z \in \mathbb{C}^n;\ \delta(z) > \epsilon\}$. One can verify that the constant involved is independent of $\epsilon > 0$. We want to let $\epsilon \to 0$ in the inequality. Let us observe that, for $\zeta_\epsilon = \Phi(\zeta, c\epsilon) \in \partial\Omega_\epsilon$, $\mathscr{R}_\alpha(\zeta_\epsilon) \subset \mathscr{R}_\beta(\zeta)$, for some $\beta > \alpha$. This allows to show that

$$\left\| \mathscr{M}_\alpha\left(\nabla_T^k u\right) \right\|_{L^p(\partial\Omega_\epsilon)} \le \left\| \mathscr{M}_\beta\left(\nabla_T^k u\right) \right\|_{L^p(\partial\Omega)}.$$

Then, we conclude by Fatou's Lemma that

$$\left\| S_\alpha\left(\delta \nabla \nabla_T^k u\right) \right\|_{L^p(\partial\Omega)} \le C\left(\left\| \mathscr{M}_\beta\left(\nabla_T^k u\right) \right\|_{L^p(\partial\Omega)} + \sup_K |u| \right).$$

In the following, it will be convenient to have a defining function for Ω which is harmonic near $\partial\Omega$. We choose a point $x_0 \in K$ and denote by δ the Green's function for Ω with singularity x_0. Thus, δ is harmonic in $\Omega \setminus \{x_0\}$ and $\delta(z)$ is comparable with the distance to the boundary, for $z \in \Omega \cap U$. Let λ and ϵ be any real positive numbers and E be the set

$$E_{\epsilon,\lambda} = E := \Big\{ z \in \partial\Omega;\ \mathscr{M}_\alpha\big(\nabla_T^k u\big)(z) \leq \lambda,\ S_\gamma\big(\delta^{1-\mu}\big|\nabla\nabla_T^{k-1}u\big|\big)(z) \leq \lambda,\ S_\gamma\big(\delta\big|\nabla\nabla_T^k u\big|\big)(z) \leq \frac{\lambda}{\epsilon + s_0^\mu} \Big\};$$

for some $\gamma > \alpha$.

Let E_0 be those points of E of relative density $\frac{1}{2}$, D_0, D their complements. By the maximal Theorem, $\sigma(D_0) \leq C\sigma(D)$. Proposition 3.3. follows from the following lemma.

LEMMA 3.5. *There exists a constant C and $\gamma > \alpha$ such that, for every $\epsilon > 0$*

$$\begin{aligned} \int_{E_0} S_\alpha\big(\delta\big|\nabla\nabla_T^k u\big|\big)^2(z)\, d\sigma(z) &\leq C\Bigg(\left(\frac{1}{\epsilon^2}+1\right)\lambda^2\sigma(D_0) + \sup_K |u|^2 \\ &\quad + \int_0^\lambda t\sigma\left(\left\{\mathscr{M}_\alpha(\nabla_T^k u) \geq t\right\}\right) dt \\ &\quad + \big(\epsilon + s_0^\mu\big)^2 \int_E S_\gamma\big(\delta\big|\nabla\nabla_T^k u\big|\big)^2(z)d\sigma(z) \\ &\quad + \int_E S_\gamma\big(\delta^{1-\mu}\big|\nabla\nabla_T^{k-1}u\big|\big)^2(z)\, d\sigma(z)\Bigg). \end{aligned}$$

Assume this lemma proved and let us prove Proposition 3.1. Write

$$\begin{aligned} (*) &= \big\|S_\alpha\big(\delta\big|\nabla\nabla_T^k u\big|\big)\big\|^p_{L^p(\partial\Omega)} \\ &= p\int_0^\infty \lambda^{p-1}\sigma\big(\big\{S_\alpha\big(\delta\big|\nabla\nabla_T^k u\big|\big) \geq \lambda\big\}\big)d\lambda \\ &\leq p\int_0^\infty \lambda^{p-1}\sigma(D_0)\, d\lambda \\ &\quad + p\int_M^\infty \lambda^{p-3}\int_{E_0} S_\alpha\big(\delta\big|\nabla\nabla_T^k u\big|\big)^2(z)\, d\sigma(z)d\lambda + M^p\sigma(\partial\Omega). \end{aligned}$$

Use Lemma 3.5 to get

$$
\begin{aligned}
(*) \lesssim \left(\frac{1}{\epsilon^2}+1\right) \int_0^\infty & \lambda^{p-1}\sigma(D)\,d\lambda \\
& + \int_M^\infty \lambda^{p-3} \int_0^\lambda t\sigma\left(\left\{\mathscr{M}_\alpha\left(\nabla_T^k u\right) \geq t\right\}\right) dt\, d\lambda \\
& + \int_M^\infty \lambda^{p-3} \int_E S_\gamma\left(\delta^{1-\mu}\left|\nabla\nabla_T^{k-1}u\right|\right)^2(z)\,d\sigma(z)\,d\lambda \\
& + \left(\epsilon + s_0^\mu\right)^2 \int_M^\infty \lambda^{p-3} \int_E S_\gamma\left(\delta\left|\nabla\nabla_T^k u\right|\right)^2(z)\,d\sigma(z)\,d\lambda \\
& + M^p\sigma(\partial\Omega) + \sup_K |u|^2 M^{p-2} \\
\lesssim \left(\frac{1}{\epsilon^2}+1\right) & \left\|\mathscr{M}_\alpha\left(\nabla_T^k u\right)\right\|_{L^p(\partial\Omega)}^p + \left\|S_\gamma\left(\delta^{1-\mu}\left|\nabla\nabla_T^{k-1}u\right|\right)\right\|_{L^p(\partial\Omega)}^p \\
& + \left(\epsilon + s_0^\mu\right)^p \left\|S_\gamma\left(\delta\left|\nabla\nabla_T^k u\right|\right)\right\|_{L^p(\partial\Omega)}^p + M^p\sigma(\partial\Omega) + \sup_K |u|^2 M^{p-2}.
\end{aligned}
$$

It gives proposition 3.3.

PROOF OF LEMMA 3.5. We note $\mathscr{R}_\alpha = \cup_{z\in E_0}\mathscr{A}_\alpha(z)$ and

$$I_{E_0} = \int_{E_0} S_\alpha\left(\delta\nabla\nabla_T^k u\right)^2(z)\,d\sigma(z).$$

Then

$$
\begin{aligned}
I_{E_0} &= \iint_{\mathscr{R}_\alpha} \delta^2\left|\nabla\nabla_T^k u\right|^2 \sigma\left(\{\zeta \in E_0;\ z\in\mathscr{R}_\alpha(\zeta)\}\right) \frac{dV(z)}{\delta^2\tau^{2n-2}} \\
&\leq C\iint_{\mathscr{R}_\alpha} \delta\left|\nabla\nabla_T^k u\right|^2 dV.
\end{aligned}
$$

We write that

$$2\left|\nabla\nabla_T^k u\right|^2 \leq 2\left|\Delta(\nabla_T^k u).\nabla_T^k u\right| + \Delta\left|\nabla_T^k u\right|^2$$

and, following the method of Fefferman and Stein, we will estimate

$$\iint_{\mathscr{R}_\alpha} \delta\Delta\left|\nabla_T^k u\right|^2 dV$$

by applying Green's Theorem. Let us denote by $d\hat{\sigma}$ the surface measure on $\partial\mathscr{R}_\alpha$. So, we obtain

$$\begin{aligned}
I_{E_0} &\leq C \left(\iint_{\mathscr{R}_\alpha} \delta \left|\nabla_T^k u . \Delta\left(\nabla_T^k u\right)\right| dV \right. \\
&\qquad \left. + \left(\int_{\partial\mathscr{R}_\alpha} \delta \frac{\partial |\nabla_T^k u|^2}{\partial \nu} \, d\hat{\sigma} - \int_{\partial\mathscr{R}_\alpha} \frac{\partial \delta}{\partial \nu} \left|\nabla_T^k u\right|^2 d\hat{\sigma} \right) \right) \\
&= (1) + (2) + (3),
\end{aligned}$$

where $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative on $\partial\mathscr{R}_\alpha$.

Estimate of the first term (1): As u is holomorphic in Ω, we have $\Delta\nabla_T^k u = [\Delta, \nabla_T^k]u$ and so, as in the preceding paragraph $|\Delta\nabla_T^k u| \simeq |\nabla^2\nabla_T^{k-1} u|$. So

$$\begin{aligned}
(1) &\leq \iint_{\mathscr{R}_\alpha} \delta \left|\nabla_T^k u\right| . \left|\nabla^2 \nabla_T^{k-1} u\right| dV \\
&\leq \left(\iint_{\mathscr{R}_\alpha} \delta^{-1+\mu} \left|\nabla_T^k u\right|^2 dV \right)^{1/2} \times \left(\iint_{\mathscr{R}_\alpha} \delta^{3-\mu} \left|\nabla^2 \nabla_T^{k-1} u\right|^2 dV \right)^{1/2} \\
&\lesssim \iint_{\mathscr{R}_\alpha} \delta^{-1+\mu} \left|\nabla_T^k u\right|^2 dV + \iint_{\mathscr{R}_\alpha} \delta^{3-\mu} \left|\nabla^2 \nabla_T^{k-1} u\right|^2 dV \\
&\lesssim s_0^\mu \iint_{\mathscr{R}_\alpha} \delta \left|\nabla \nabla_T^k u\right|^2 dV + \iint_{\mathscr{R}_\alpha} \delta^{1-\mu} \left|\nabla \nabla_T^{k-1} u\right|^2 dV + \sup_K |u|
\end{aligned}$$

for every $0 < \mu < 1$, some $\beta > \alpha$, by Hardy inequality and mean-value property.

Estimate of the second term (2): (2) $\leq \int_{\partial\mathscr{R}_\alpha} \delta |\nabla\nabla_T^k u| . |\nabla_T^k u| \, d\hat{\sigma}$.

We split $\partial\mathscr{R}_\alpha$ into three pieces $\partial\mathscr{R}_\alpha = F \cup F^{E_0} \cup F^{D_0}$ where

$$\Phi^{-1}(F) \subset \partial\Omega \times \{s_0\}, \quad \Phi^{-1}\left(F^{E_0}\right) \subset E_0 \text{ and } \Phi^{-1}\left(F^{D_0}\right) \subset D_0 \times (0, s_0).$$

So, we write

$$(2) \leq \left(\int_F + \int_{F^{E_0}} + \int_{F^{D_0}} \right).$$

First, we have

$$\int_F \delta \left|\nabla \nabla_T^k u\right| . \left|\nabla_T^k u\right| d\hat{\sigma} \leq C \sup_K |u|^2$$

and

$$\int_{F^{E_0}} \delta \left|\nabla \nabla_T^k u\right| . \left|\nabla_T^k u\right| d\hat{\sigma} = 0 \quad \text{since} \quad F^{E_0} \subset \partial\Omega.$$

For every $\epsilon > 0$, the last part is majorized by

$$\leq C\left(\frac{1}{\epsilon^2}\int_{F^{D_0}}\left|\nabla_T^k u\right|^2 d\hat{\sigma} + \epsilon^2\int_{F^{D_0}}\delta^2\left|\nabla\nabla_T^k u\right|^2 d\hat{\sigma}\right).$$

As $\mathscr{M}_\alpha(\nabla_T^k u) \leq \lambda$ on E, we deduce that

$$\frac{1}{\epsilon^2}\int_{F^{D_0}}\left|\nabla_T^k u\right|^2 d\hat{\sigma} \leq \frac{1}{\epsilon^2}\lambda^2\int_{F^{D_0}} d\hat{\sigma} \leq \frac{C}{\epsilon^2}\lambda^2\sigma(D_0).$$

Now, by the mean-value property,

$$\epsilon^2\int_{F^{D_0}}\delta^2\left|\nabla\nabla_T^k u\right|^2 d\hat{\sigma} \leq C\bigg(\epsilon^2\iint_{\mathscr{R}_\beta}\delta\left|\nabla\nabla_T^k u\right|^2 dV + \sup_K |u|^2\bigg)$$

(this follows from the fact that

$$\int_{\partial\mathscr{R}_\alpha}\delta^{l+1}\tau^r\,\mathrm{Mean}^Q(|f|^2)\,d\hat{\sigma} \leq C\iint_{\mathscr{R}_\beta}\delta^l\tau^r|f|^2\,dV,$$

for β sufficiently large).

So

$$(2) \leq \frac{C}{\epsilon^2}\lambda^2\sigma(D_0) + C\bigg(\epsilon^2\iint_{\mathscr{R}_\beta}\delta\left|\nabla\nabla_T^k u\right|^2 dV + \sup_K |u|^2\bigg).$$

Estimate of the third term: The third term is majorized by

$$\begin{aligned}(3) &\leq C\int_{\partial\mathscr{R}_\alpha}\left|\nabla_T^k u\right|^2 d\hat{\sigma} \leq C\bigg(\int_F + \int_{F^{E_0}} + \int_{F^{D_0}}\bigg)\\ &\leq C\bigg(\sup_K |u|^2 + \int_0^\lambda t\sigma\big(\big\{\mathscr{M}_\alpha(\nabla_T^k u) \geq t\big\}\big)\,dt + \lambda^2\sigma(D_0)\bigg),\end{aligned}$$

since $\mathscr{M}_\alpha(\nabla_T^k u) \leq \lambda$ on E.

To conclude for Lemma 3.5, it suffices to remark that

$$\iint_{\mathscr{R}_\beta}|f|^2\frac{dV}{\delta} \leq \int_E S_\gamma(f)^2\,d\sigma,$$

for some $\gamma > \beta$.

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