MATH. SCAND. 91 (2002), 55–66

A REDUCTION OF THE PROBLEM OF CHARACTERIZING PERFECT SEMIGROUPS

TORBEN MAACK BISGAARD and NOBUHISA SAKAKIBARA

1. Introduction

Suppose $(S, +, *)$ is an abelian semigroup equipped with an *involution*, that is, an involutory automorphism, written $s \mapsto s^*$. Such a structure will be called a *$*$-semigroup*, abbreviated "semigroup" when confusion is unlikely, such as when applying an adjective which is defined only for $*$-semigroups (e.g., "perfect semigroup"). Define $S + S = \{ s + t \mid s, t \in S \}$ and define $\overbrace{S + \cdots + S}^{N}$ similarly for arbitrary $N \in \mathsf{N}$. A *positive definite function on S* is a function $\varphi: S + S \to \mathsf{C}$ such that

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^*) \geq 0$$

for every choice of $n \in \mathsf{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathsf{C}$. Denote by $\mathscr{P}(S)$ the set of all positive definite functions on S. A *character* on S is a function $\sigma: S \to \mathsf{C}$, not identically zero, such that $\sigma(s^*) = \overline{\sigma(s)}$ and $\sigma(s + t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S. Denote by $\mathscr{A}(S^*)$ the least σ-ring of subsets of S^* rendering the mapping $\sigma \mapsto \sigma(s): S^* \to \mathsf{C}$ measurable for each $s \in S$. A function $\varphi: S + S \to \mathsf{C}$ is a *moment function* if there is a measure μ defined on $\mathscr{A}(S^*)$ such that

$$\varphi(s) = \int_{S^*} \sigma(s)\, d\mu(\sigma)$$

for all $s \in S + S$, and a moment function φ is *determinate* if there is only one such μ. (In writing an equation such as the preceding, it is understood that μ should integrate the integrands.) The semigroup S is *perfect* if every positive definite function on S is a determinate moment function. For positive definite functions and (Radon) moment functions on semigroups, we refer to

Received October 31, 1999.

[1], especially Section 6.5 on semigroups called “perfect” in that book but now called “Radon perfect”.

A $*$-semigroup H is *$*$-archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathsf{N}$ such that $n(x + x^*) = y + z$. A *$*$-archimedean component* of a $*$-semigroup S is a $*$-archimedean $*$-subsemigroup of S which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components. It was shown in [2], Theorem 3.1, that a $*$-semigroup S with zero is perfect if and only if $H \cup \{0\}$ is perfect for each $*$-archimedean component H of S. We shall be concerned with extending this result to semigroups without zero. It is not true that a $*$-semigroup S, even with zero, is perfect if and only if each $*$-archimedean component of S is perfect. For example, if $H = \mathsf{Q} \cap [1, \infty[$ and $S = H \cup \{0\}$ then S is perfect ([4], Corollary 2) and H is a $*$-archimedean component of S, yet H is not perfect ([6], Remark 3.6). We shall define “quasi-perfect” semigroups in such a way that a $*$-semigroup S is quasi-perfect if and only if each $*$-archimedean component of S is quasi-perfect.

2. Reduction to the $*$-archimedean case

Say that a $*$-semigroup H is *determinate of order* $N \geq 1$ if whenever μ and ν are measures on $\mathscr{A}(H^*)$ such that

$$\int_{H^*} \eta(x)\, d\mu(\eta) = \int_{H^*} \eta(x)\, d\nu(\eta), \qquad x \in \overbrace{H + \cdots + H}^{N}$$

then $\mu = \nu$. An *ideal* of a $*$-semigroup X is a nonempty $*$-stable subset H of X such that $X + H \subset H$. In particular, $H + H \subset H$, so H is again a $*$-semigroup.

LEMMA 2.1. *Every ideal of a $*$-semigroup which is determinate of order* 1 *has the same property.*

PROOF. Suppose X is a $*$-semigroup which is determinate of order 1 and H is an ideal of X; we have to show that H is determinate of order 1. The following observation, which does not use determinacy, will be of use later on. For each $\eta \in H^*$ we can define $\tilde{\eta} \in X^*$ by choosing $y \in H$ such that $\eta(y) \neq 0$ (which is possible by the definition of a character) and setting

$$\tilde{\eta}(x) = \frac{\eta(x + y)}{\eta(y)}, \qquad x \in X.$$

We leave it as an exercise to verify that the definition is independent of the choice of y and that the function $\tilde{\eta}$ so defined is indeed a character on X. Note that $\tilde{\eta}$ extends η. Now the mapping $\eta \mapsto \tilde{\eta}$ is a one-to-one correspondence between H^* and the set of those $\xi \in X^*$ such that $\xi|H \neq 0$, the inverse of this mapping being the mapping taking $\xi \in X^*$ (with $\xi|H \neq 0$) to $\xi|H$. To see this, first note that $\eta = \tilde{\eta}|H$ for $\eta \in H^*$. Secondly, suppose $\xi \in X^*$ is such that $\eta := \xi|H \neq 0$; we have to show $\xi = \tilde{\eta}$. Choose $y \in H$ such that $\eta(y) \neq 0$. For arbitrary $x \in X$ we have $\xi(x) = \xi(x+y)/\xi(y) = \eta(x+y)/\eta(y) = \tilde{\eta}(x)$.

Now to see that H is determinate of order 1, suppose μ and ν are measures on $\mathscr{A}(H^*)$ such that

$$\int_{H^*} \eta(x)\, d\mu(\eta) = \int_{H^*} \eta(x)\, d\nu(\eta), \qquad x \in H;$$

we have to show $\mu = \nu$. Let $\tilde{\mu}$ and $\tilde{\nu}$ be the images of μ and ν, respectively, under the mapping $\eta \mapsto \tilde{\eta}: H^* \to X^*$. Since $\eta = \tilde{\eta}|H$ for $\eta \in H^*$ then

$$\int_{X^*} \xi(x)\, d\tilde{\mu}(\xi) = \int_{X^*} \xi(x)\, d\tilde{\nu}(\xi), \quad x \in H.$$

If $x \in X$ and $y \in H$ then $x + y \in H$ and $\xi(x + y) = \xi(x)\xi(y)$ for $\xi \in X^*$, so

$$\int_{X^*} \xi(x)\xi(y)\, d\tilde{\mu}(\xi) = \int_{X^*} \xi(x)\xi(y)\, d\tilde{\nu}(\xi).$$

This being so for all $x \in X$, since X is determinate of order 1 it follows that $\xi(y)\, d\tilde{\mu}(\xi) = \xi(y)\, d\tilde{\nu}(\xi)$. (The property in the definition of determinacy extends to complex measures, cf. [1], 6.5.2.) Hence, defining $D_y = \{\, \xi \in X^* \mid \xi(y) \neq 0 \,\}$, we have $\tilde{\mu}|D_y = \tilde{\nu}|D_y$. Equivalently, defining $G_y = \{\, \eta \in H^* \mid \eta(y) \neq 0 \,\}$, we have $\mu|G_y = \nu|G_y$. Since every set in $\mathscr{A}(H^*)$ is contained in the union of countably many G_y, it follows that $\mu = \nu$, as desired.

LEMMA 2.2. *For a $*$-semigroup H, the following two conditions are equivalent*:

(i) *H is determinate of every order $N \geq 1$*

(ii) *H is determinate of some order $N \geq 1$.*

PROOF. We have to show that if M and N are positive integers such that H is determinate of order M then H is determinate of order N. This is trivial if $M \geq N$, so suppose $M < N$. Since H is determinate of order M then the semigroup $X = \overbrace{H + \cdots + H}^{M}$ is determinate of order 1. By Lemma 2.1 it follows that the semigroup $Y = \overbrace{H + \cdots + H}^{N}$ (which is an ideal of X) is determinate of order 1, that is, H is determinate of order N, as desired.

(We used twice the fact that for X as above, determinacy of H of order M is equivalent to determinacy of X of order 1. This is because the correspondence between X^* and a subset of H^* from the proof of Lemma 2.1 (with the roles of X and H interchanged) is in this case a one-to-one correspondence between X^* and all of H^*.)

Say that H is *determinate* if the equivalent conditions of Lemma 2.2 are satisfied. Say that H is *quasi-perfect of order* $N \geq 2$ if H is determinate and for each $\varphi \in \mathscr{P}(H)$ there is some (hence, a unique) measure μ on $\mathscr{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta), \qquad x \in \overbrace{H + \cdots + H}^{N}.$$

Then H is perfect if and only if H is quasi-perfect of order 2.

THEOREM 2.3. *For a $*$-semigroup H, possibly without zero, the following three conditions are equivalent* :

(i) *H is quasi-perfect of every order greater than or equal to* 3

(ii) *H is quasi-perfect of some order greater than or equal to* 3

(iii) *H is quasi-perfect of order* 3.

PROOF. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Suppose H is quasi-perfect of order $N \geq 3$. We shall show that H is quasi-perfect of order M for $3 \leq M \leq N$ by backwards induction on M. Suppose $3 \leq M < N$ and that we have shown that H is quasi-perfect of order $M + 1$. Suppose $\varphi \in \mathscr{P}(H)$. Since H is quasi-perfect of order $M + 1$, there is a unique measure μ on $\mathscr{A}(H^*)$ such that

$$\varphi(x) = \int_{H^*} \eta(x)\, d\mu(\eta) \tag{1}$$

for $x \in \overbrace{H + \cdots + H}^{M+1}$. Clearly there is at most one measure with the corresponding property for x in the larger set $\overbrace{H + \cdots + H}^{M}$. Thus we only have to show that (1) extends to all $x \in \overbrace{H + \cdots + H}^{M}$. Suppose $h \in H$. For $x_1, \ldots, x_n \in \overbrace{H + \cdots + H}^{M-1}$ and $c_1, \ldots, c_n \in \mathsf{C}$ we have by the Cauchy-

Schwarz inequality

$$\left|\sum_{j=1}^{n} c_j\varphi(h+x_j)\right|^2 \le \varphi(h+h^*)\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(x_j+x_k^*)$$
$$= \varphi(h+h^*)\int_{H^*}\left|\sum_{j=1}^{n} c_j\eta(x_j)\right|^2 d\mu(\eta).$$

We used the fact that for $j, k = 1, \ldots, n$ we have

$$x_j + x_k^* \in \overbrace{H+\cdots+H}^{M-1} + \overbrace{H+\cdots+H}^{M-1} = \overbrace{H+\cdots+H}^{2M-2} \subset \overbrace{H+\cdots+H}^{M+1}$$

since $2M-2 \ge M+1$ because of $M \ge 3$. The above inequality shows that the mapping

$$\left(\eta \mapsto \sum_{j=1}^{n} c_j\eta(x_j)\right) \mapsto \sum_{j=1}^{n} c_j\varphi(h+x_j)$$

is a well-defined bounded linear form on a linear subspace of $L^2(\mu)$. Extend this linear form to a bounded linear form L_h on all of $L^2(\mu)$. Then there is a unique $\psi_h \in L^2(\mu)$ such that $L_h(f) = \int f\psi_h\, d\mu$ for all $f \in L^2(\mu)$. In particular,

$$\varphi(h+x) = \int_{H^*} \eta(x)\psi_h(\eta)\, d\mu(\eta) \tag{2}$$

for $x \in \overbrace{H+\cdots+H}^{M-1}$. For $x \in \overbrace{H+\cdots+H}^{M}$ we have by (1) and (2),

$$\int_{H^*} \eta(h)\eta(x)\, d\mu(\eta) = \int_{H^*} \eta(x)\psi_h(\eta)\, d\mu(\eta).$$

Since H is determinate (by definition) it follows that $\psi_h(\eta)d\mu(\eta) = \eta(h)d\mu(\eta)$, so (2) reduces to

$$\varphi(h+x) = \int_{H^*} \eta(h)\eta(x)\, d\mu(\eta).$$

This is the desired representation since each $y \in \overbrace{H+\cdots+H}^{M}$ can be written as $y = h + x$ with $h \in H$ and $x \in \overbrace{H+\cdots+H}^{M-1}$.

(iii) $\Rightarrow$ (i): Trivial.

We call a $*$-semigroup H *quasi-perfect* if the equivalent conditions of Theorem 2.3 are satisfied. In [4], Theorem, it was shown that if S is a (Radon) perfect semigroup with zero and if T is a $*$-subsemigroup of S containing 0 and such that $T \setminus \{0\}$ is an ideal of S then T is likewise (Radon) perfect. The following result seems to be the closest analogue of this for semigroups without zero. The proof, like the proof of Theorem 2.3, is strongly inspired by the argument in [4].

THEOREM 2.4. *If X is a quasi-perfect semigroup and H is an ideal of X then H is quasi-perfect.*

PROOF. Suppose $\varphi \in \mathscr{P}(H)$. For $h, k \in H$ and $n = 0, 1, 2, 3$ define $\varphi_{h,k,n} \in \mathscr{P}(X)$ by

$$\varphi_{h,k,n}(x) = \varphi(h+h^*+x)+i^n\varphi(h+k^*+x)+i^{-n}\varphi(k+h^*+x)+\varphi(k+k^*+x)$$

for $x \in X + X$. Since X is quasi-perfect, there is a unique measure $\lambda_{h,k,n}$ on $\mathscr{A}(X^*)$ such that

$$\varphi_{h,k,n}(x) = \int_{X^*} \xi(x)\, d\lambda_{h,k,n}(\xi)$$

for $x \in X + X + X$. We now introduce a subring $\mathscr{A}_0(X^*)$ of $\mathscr{A}(X^*)$ which generates the latter as a σ-ring, as follows. For $x \in X$ and $n \in \mathsf{N}$ define $G_{x,n} = \left\{\xi \in X^* \,\middle|\, |\xi(x)| > 1/n\right\}$. Then let $\mathscr{A}_0(X^*)$ be the set of those elements of $\mathscr{A}(X^*)$ which are contained in the union of finitely many $G_{x,n}$. Clearly $\mathscr{A}_0(X^*)$ is a subring of $\mathscr{A}(X^*)$. This subring generates $\mathscr{A}(X^*)$ as a σ-ring since for each $A \in \mathscr{A}(X^*)$ there is a countable subset Y of X such that for each $\xi \in A$ there is some $y \in Y$ such that $\xi(y) \neq 0$. This is because the set of all subsets A of X^* with the property just described is a σ-ring of subsets of X^* which renders the mapping $\xi \mapsto \xi(x)$ measurable for each $x \in X$, hence contains $\mathscr{A}(X^*)$ by the definition of $\mathscr{A}(X^*)$. Since $\mathscr{A}_0(X^*)$ generates $\mathscr{A}(X^*)$, every measure μ on $\mathscr{A}_0(X^*)$ which is *finite* in the sense that $\mu(A) < \infty$ for all $A \in \mathscr{A}_0(X^*)$ extends to a unique measure on $\mathscr{A}(X^*)$. Thus measures on $\mathscr{A}(X^*)$ can be identified with their restrictions to $\mathscr{A}_0(X^*)$. If μ is a measure on $\mathscr{A}(X^*)$ which integrates the function $\xi \mapsto \xi(x)$ for all $x \in \overbrace{X + \cdots + X}^{N}$ for some $N \in \mathsf{N}$ then $\mu|\mathscr{A}_0(X^*)$ is finite since for $x \in X$ and $n \in \mathsf{N}$ we have

$$\frac{\mu(G_{x,n})}{n^N} \leq \int_{G_{x,n}} |\xi(x)|^N \, d\mu(\xi) \leq \int_{X^*} |\xi(x)|^N \, d\mu(\xi)$$
$$= \int_{X^*} |\xi(Nx)| \, d\mu(\xi) < \infty.$$

If we now define

$$\lambda_{h,k} = \frac{1}{4}\sum_{n=0}^{3} i^{-n}\lambda_{h,k,n}$$

(as a set function on $\mathscr{A}_0(X^*)$) then $\lambda_{h,k}$ is the unique complex measure on $\mathscr{A}(X^*)$ such that

$$\varphi(h + k^* + x) = \int_{X^*} \xi(x)\, d\lambda_{h,k}(\xi)$$

for $x \in X + X + X$. We see that for each $y \in H + H$ there is a unique complex measure λ_y on $\mathscr{A}(X^*)$ such that

$$\varphi(x + y) = \int_{X^*} \xi(x)\, d\lambda_y(\xi) \tag{3}$$

for $x \in X + X + X$. Indeed, any $y \in H + H$ can be written as $y = h + k^*$ with $h, k \in H$, and then we have to define $\lambda_y = \lambda_{h,k}$. This definition of λ_y is independent of the choice of h and k since if $h_1, h_2, k_1, k_2 \in H$ and $h_1 + k_1^* = h_2 + k_2^*$ then $\lambda_{h_1,k_1} = \lambda_{h_2,k_2}$ since these two measures represent the same function on $X + X + X$. If $y, z \in H + H$ then

$$\int_{X^*} \xi(x)\xi(y)\, d\lambda_z(\xi) = \varphi(x + y + z) = \int_{X^*} \xi(x)\xi(z)\, d\lambda_y(\xi)$$

for $x \in X + X + X$, and by the quasi-perfectness of X it follows that

$$\xi(y)\, d\lambda_z(\xi) = \xi(z)\, d\lambda_y(\xi). \tag{4}$$

For $h \in H$ write $G_h = \{\xi \in X^* \mid \xi(h) \neq 0\}$. If for $y \in H + H$ we define a measure κ_y on G_y by

$$d\kappa_y(\xi) = \xi(y)^{-1}\, d\lambda_y(\xi)|G_y$$

then (4) shows that for $y, z \in H + H$ we have

$$\kappa_y|(G_y \cap G_z) = \kappa_z|(G_y \cap G_z).$$

Hence there is a unique measure κ on the set

$$G = \bigcup_{h\in H} G_h = \left\{\xi \in X^* \;\middle|\; \xi|H \neq 0\right\}$$

such that

$$\kappa_y = \kappa|G_y$$

for all $y \in H+H$. More precisely, κ is defined on the σ-ring $\mathscr{A}_*(G)$ consisting of those elements of $\mathscr{A}(X^*)$ which are contained in the union of countably many G_h. We claim that

$$d\lambda_y(\xi)|G = \xi(y)\,d\kappa(\xi) \tag{5}$$

for $y \in H+H$ where $|G$ denotes the operation of restriction of a measure to the σ-ring $\mathscr{A}_*(G)$. Since G is the union of the sets G_z, $z \in H+H$, it suffices to verify $d\lambda_y(\xi)|G_z = \xi(y)\,d\kappa(\xi)|G_z$ for $z \in H+H$. But the right-hand side is equal to $\xi(y)\xi(z)^{-1}\,d\lambda_z(\xi)|G_z$, so the desired equality follows from (4). This proves (5). For $x \in H+H+H$ and $y \in H+H$, since characters outside G vanish on H, by (3) and (5) we have

$$\varphi(x+y) = \int_{X^*} \xi(x)\,d\lambda_y(\xi) = \int_G \xi(x)\,d\lambda_y(\xi) = \int_G \xi(x)\xi(y)\,d\kappa(\xi).$$

Thus, for $x \in H+H+H+H+H$ we have

$$\varphi(x) = \int_G \xi(x)\,d\kappa(\xi).$$

Now κ is uniquely determined by this property. To see this, note that if κ_1 and κ_2 are two measures with this property then $\varphi(3h+3h^*+x) = \int_G |\xi(h)|^6 \xi(x)\,d\kappa_i(\xi)$ for $h \in H$, $x \in X$, and $i = 1, 2$, so by the quasi-perfectness of X it follows that $|\xi(h)|^6\,d\kappa_1(\xi) = |\xi(h)|^6 d\kappa_2(\xi)$, hence $\kappa_1|G_h = \kappa_2|G_h$. Since every measurable subset of G is contained in the union of countably many G_h, it follows that $\kappa_1 = \kappa_2$.

Now the mapping $\xi \mapsto \xi|H\colon G \to H^*$ is a bijection, cf. the proof of Lemma 2.1. We leave it as an exercise to verify that the mapping $\xi \mapsto \xi|H$ is an isomorphism between the measurable spaces $\big(G, \mathscr{A}_*(G)\big)$ and $\big(H^*, \mathscr{A}(H^*)\big)$. Now if μ is the image measure of κ under the mapping $\xi \mapsto \xi|H$ then $\varphi(x) = \int_{H^*} \eta(x)\,d\mu(\eta)$ for $x \in H+H+H+H+H$, and μ is uniquely determined by this property (since κ is unique). Thus H is quasi-perfect of order 5, that is, H is quasi-perfect.

It is not true that every ideal of a perfect semigroup is perfect. For example, if $H = \mathbb{Q} \cap [1, \infty[$ and $X = H \cup \{0\}$ then X is perfect and H is an ideal of X, yet H is not perfect.

COROLLARY 2.5. *A $*$-semigroup H is quasi-perfect if and only if the $*$-semigroup $X = H \cup \{0\}$ obtained by adjoining to H a zero external to H is perfect.*

PROOF. If X is perfect then H is quasi-perfect by Theorem 2.4 since H is an ideal of X. Conversely, if H is quasi-perfect then the perfectness of

X follows just as in the proof of [6], Theorem 3.2, that perfectness of H implies perfectness of $H \cup \{0\}$, only the argument with the Cauchy-Schwarz inequality has to be applied twice, first to get from the integral representation on $H + H + H + H$ ($\subset H + H + H$) to the integral representation on $H + H$ and a second time to get it on all of H.

Corollary 2.6. *Every $*$-homomorphic image of a quasi-perfect $*$-semigroup is quasi-perfect.*

Proof. Suppose h is a $*$-homomorphism of a quasi-perfect $*$-semigroup H_1 onto some $*$-semigroup H_2; we have to show that H_2 is quasi-perfect. For $i = 1, 2$ let $S_i = H_i \cup \{0\}$ be the $*$-semigroup with zero obtained by adjoining to H_i a zero external to H_i. By Corollary 2.5, S_1 is perfect. Extending h to a $*$-homomorphism of S_1 onto S_2 by defining $h(0) = 0$, S_2 is a $*$-homomorphic image of the perfect $*$-semigroup S_1, hence perfect by [3], Theorem 1. By Corollary 2.5 it follows that H_2 is quasi-perfect.

Suppose $(S_i)_{i\in I}$ is a family of $*$-semigroups with zero. The *direct sum* $S = \bigoplus_{i\in I} S_i$ is the set of all families $(s_i) \in \prod_{i\in I} S_i$ such that $s_i \neq 0$ for only finitely many $i \in I$. Addition and involution in S are defined componentwise. It is known that if each S_i is perfect, so is S ([3], Theorem 3). Now suppose $(H_i)_{i\in I}$ is a family of $*$-semigroups not necessarily having zeros. The *free sum* H of the family (H_i) is defined as follows. For $i \in I$ let $S_i = H_i \cup \{0\}$ be the $*$-semigroup with zero obtained by adjoining to H_i a zero external to H_i. Define $S = \bigoplus_{i\in I} S_i$. Then $H = S \setminus \{0\}$.

Corollary 2.7. *The free sum of an arbitrary family of quasi-perfect $*$-semigroups is quasi-perfect.*

Proof. With notation as above, for $i \in I$, from the quasi-perfectness of H_i it follows that S_i is perfect (Corollary 2.5). By the result from [3] cited above it follows that S is perfect. By Corollary 2.5 it follows that H is quasi-perfect.

A *face* of a $*$-semigroup X is a $*$-subsemigroup H of X such that if $x, y \in X$ and $x + y \in H$ then $x, y \in H$.

Corollary 2.8. *Suppose H is a face of a $*$-semigroup X. Then X is quasi-perfect if and only if both H and $X \setminus H$ are quasi-perfect.*

Proof. This is clear from Theorem 2.4, Corollary 2.6, Corollary 2.7, and [5], Theorem 2.1.

Corollary 2.9. *Suppose H is a $*$-semigroup and $S = H \cup \{0\}$. Under the assumption that H is quasi-perfect, the following two conditions are equivalent:*

(i) *H is perfect*

(ii) *if $\varphi \in \mathscr{P}(H)$ and $h, k \in H$ then the measure $\mu_{h,k}$ on S^* defined as in* [6], (3.1)*, satisfies $\mu_{h,k}(\{1_{\{0\}}\}) = 0$.*

PROOF. This is clear from [6], Proposition 3.5.

A $*$-semigroup H is *$*$-divisible* if for each $x \in H$ there exist $y \in H$ and $m, n \in \mathsf{N}_0$ with $m + n \geq 2$ such that $x = my + ny^*$. It is known that every $*$-divisible $*$-semigroup with zero is perfect ([3], Theorem 4).

COROLLARY 2.10. *Every $*$-divisible $*$-semigroup is perfect.*

PROOF. Suppose H is a $*$-divisible $*$-semigroup. The $*$-semigroup $S = H \cup \{0\}$ is a $*$-divisible $*$-semigroup with zero, hence perfect. By Corollary 2.5 it follows that H is quasi-perfect. But the $*$-divisibility of H implies that $H = H + H$, so H is perfect.

For a $*$-semigroup S we denote by $\mathscr{J}(S)$ the set of all $*$-archimedean components of S. For every nonempty subset H of S there is a least face of S containing H, viz., the intersection of all faces of S containing H, the set of such faces being nonempty since S itself is such a face. If H is a $*$-subsemigroup of S then the least face X of S containing H is the set of those $x \in S$ such that $x+y \in H$ for some $y \in S$. If H is a $*$-archimedean component of S then X is the set of those $x \in S$ such that $x + H \subset H$. In particular, $X + H \subset H$.

THEOREM 2.11. *A $*$-semigroup S is quasi-perfect if and only if each $*$-archimedean component of S is quasi-perfect.*

PROOF. First suppose S is quasi-perfect and $H \in \mathscr{J}(S)$. Let X be the least face of S containing H. By Corollary 2.8, X is quasi-perfect. Moreover, $X + H \subset H$. By Theorem 2.4 it follows that H is quasi-perfect. Conversely, suppose each $H \in \mathscr{J}(S)$ is quasi-perfect. Then for $H \in \mathscr{J}(S)$ the semigroup $H \cup \{0\}$ is perfect (Corollary 2.5). By [3], Theorem 3, it follows that the direct sum $R = \bigoplus_{H \in \mathscr{J}(S)}(H \cup \{0\})$ is perfect. Now $S \cup \{0\}$ is perfect, being the image of R under the $*$-homomorphism $(s_H)_{H \in \mathscr{J}(S)} \mapsto \sum_{H \in \mathscr{J}(S)} s_H$ ([3], Theorem 1). By Corollary 2.5, it follows that S is quasi-perfect.

3. Reduction to the rational case

A $*$-semigroup S is *rational* if S is isomorphic to a subsemigroup of a rational vector space carrying the identical involution. The condition is equivalent to saying that S carries the identical involution, is cancellative, and the group $S - S$ is torsion-free. For an arbitrary $*$-semigroup S, denote by $\overline{S}$ the greatest rational $*$-homomorphic image of S, that is, the pair $(\overline{S}, s \mapsto \overline{s})$ consisting of a rational semigroup $\overline{S}$ and a $*$-homomorphism $s \mapsto \overline{s}$ of S onto $\overline{S}$ such that for every rational semigroup T and every $*$-homomorphism $f\colon S \to T$ there is a unique homomorphism $h\colon \overline{S} \to T$ such that $f(s) = h(\overline{s})$ for all $s \in S$. (This property is what we mean by "greatest".) To see that such a pair $(\overline{S}, s \mapsto \overline{s})$ exists, let $\sim$ be the least congruence relation in S such that $s^* \sim s$ for all $s \in S$, define $T = S/\!\sim$, and let $f\colon S \to T$ be the quotient mapping. Then T is the greatest identical-involution $*$-homomorphic image of S. Now let (G, g) be the pair—unique up to isomorphism—consisting of an abelian group G and a homomorphism $g\colon T \to G$ such that for every abelian group H and every homomorphism $h\colon T \to H$ there is a unique homomorphism $k\colon G \to H$ such that $h = k \circ g$. This construction is well-known from algebra. The semigroup $g(T)$ generates G as a group, and for $x, y \in T$ we have $g(x) = g(y)$ if and only if $a + x = a + y$ for some $a \in T$. Finally, let H be G modulo the torsion of G and denote by $\phi\colon G \to H$ the quotient mapping. Then $\overline{S}$ can be identified with $\phi \circ g \circ f(S)$, at the same time identifying the mapping $s \mapsto \overline{s}$ with the mapping $\phi \circ g \circ f$.

THEOREM 3.1. *A $*$-semigroup S is quasi-perfect if and only if $\overline{H \cup \{0\}}$ is perfect for each $*$-archimedean component H of S.*

PROOF. See [2] for the definition of the concept "Stieltjes perfect". We have the chain of bi-implications:

$$\begin{aligned} S \text{ is quasi-perfect} &\Leftrightarrow H \text{ is quasi-perfect for all } H \in \mathscr{J}(S) \\ &\Leftrightarrow H \cup \{0\} \text{ is perfect for all } H \in \mathscr{J}(S) \\ &\Leftrightarrow \overline{H \cup \{0\}} \text{ is Stieltjes perfect for all } H \in \mathscr{J}(S) \\ &\Leftrightarrow \overline{H \cup \{0\}} \text{ is perfect for all } H \in \mathscr{J}(S). \end{aligned}$$

The first bi-implication is by Theorem 2.11, the second by Corollary 2.5, and the last two by [2].

ACKNOWLEDGEMENT. This work of the second author was partly supported by Grant-in-Aid for Encouragement of Young Scientists, Japan Society for the Promotion of Science, the Saneyoshi Scholarship Foundation, and the Kayamori Foundation of Informational Science Advancement.

REFERENCES

1. Berg, C., Christensen, J. P. R. and Ressel, P., *Harmonic Analysis on Semigroups*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1984.
2. Bisgaard, T. M., *On perfect semigroups*, Acta Math. Hungar. 79 (1998), 269–294.
3. Bisgaard, T. M. and Ressel, P., *Unique disintegration of arbitrary positive definite functions on $*$-divisible semigroups*, Math. Z. 200 (1989), 511–525.

4. Nakamura, Y. and Sakakibara, N., *Perfectness of certain subsemigroups of a perfect semigroup*, Math. Ann. 287 (1990), 213–220.
5. Sakakibara, N., *Moment problems on subsemigroups of* N_0^k *and* $\mathbf{Z}^k$, Semigroup Forum 45 (1992), 241–248.
6. Sakakibara, N., *Perfectness and semiperfectness of abelian $*$-semigroups without zero*, Hokkaido Math. J. 24 (1995), 113–125.

NANDRUPSVEJ 7 ST. TH.
DK-2000 FREDERIKSBERG C
DENMARK
E-mail: torben.bisgaard@get2net.dk

FACULTY OF ENGINEERING
IBARAKI UNIVERSITY
NAKANARUSAWA
HITACHI 316-8511
JAPAN
E-mail: sakaki@base.ibaraki.ac.jp