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SMALL EIGENVALUES OF LARGE HANKEL MATRICES: THE INDETERMINATE CASE

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Abstract

In this paper we characterize the indeterminate case by the eigenvalues of the Hankel matrices being bounded below by a strictly positive constant. An explicit lower bound is given in terms of the orthonormal polynomials and we find expressions for this lower bound in a number of indeterminate moment problems.

1. Introduction

Let α be a positive measure on R with infinite support and finite moments of all orders

$$s_n = s_n(\alpha) = \int_{\mathsf{R}} x^n d\alpha(x). \tag{1.1}$$

With α we associate the infinite Hankel matrix $\mathscr{H}_\infty = \{H_{jk}\}$,

$$H_{jk} = s_{j+k}. \tag{1.2}$$

Let $\mathscr{H}_N$ be the $(N+1)\times(N+1)$ matrix whose entries are H_{jk}, $0 \le j, k \le N$. Since $\mathscr{H}_N$ is positive definite, then all its eigenvalues are positive. The large N asymptotics of the smallest eigenvalue, denoted as λ_N, of the Hankel matrix $\mathscr{H}_N$ has been studied in papers by Szegő [11], Widom and Wilf [13], Chen and Lawrence [6]. See also the monograph by Wilf [14]. All the cases considered by these authors are determinate moment problems, and it was shown in each case that $\lambda_N \to 0$, and asymptotic results were obtained about how fast λ_N tends to zero.

The smallest eigenvalue can be obtained from the classical Rayleigh quotient:

$$\lambda_N = \min\left\{\sum_{j=0}^{N}\sum_{k=0}^{N} s_{j+k} v_j v_k : \sum_{k=0}^{N} v_j^2 = 1,\ v_j \in \mathsf{R},\ 0 \le j \le N\right\}. \tag{1.3}$$

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It follows that λ_N is a decreasing function of N.

The main result of this paper is Theorem 1.1, which we state next.

THEOREM 1.1. *The moment problem associated with the moments* (1.1) *is determinate if and only if* $\lim_{N\to\infty} \lambda_N = 0$.

We shall compare this result with a theorem of Hamburger [8, Satz XXXI], cf. [1, p. 83] or [10, p. 70].

Let μ_N be the minimum of the Hankel form $\mathscr{H}_N$ on the hyperplane $v_0 = 1$, i.e.

$$\mu_N = \min\left\{\sum_{j=0}^{N}\sum_{k=0}^{N} s_{j+k} v_j v_k : v_0 = 1,\ v_j \in \mathsf{R},\ 0 \le j \le N\right\}, \tag{1.4}$$

and let μ'_N be the corresponding minimum for the moment sequence $s'_n = s_{n+2}$, $n \ge 0$, i.e.

$$\begin{aligned}\mu'_N &= \min\left\{\sum_{j=0}^{N}\sum_{k=0}^{N} s_{j+k+2} v'_j v'_k : v'_0 = 1,\ v'_j \in \mathsf{R},\ 0 \le j \le N\right\}\\ &= \min\left\{\sum_{j=0}^{N+1}\sum_{k=0}^{N+1} s_{j+k} v_j v_k : v_0 = 0,\ v_1 = 1,\ v_j \in \mathsf{R},\ 0 \le j \le N+1\right\}.\end{aligned}$$

The theorem of Hamburger can be stated that the moment problem is determinate if and only if at least one of the limits $\lim_{N\to\infty} \mu_N$, $\lim_{N\to\infty} \mu'_N$ are zero.

It is clear from (1.3), (1.4) that $\mu_N \ge \lambda_N$ and similarly $\mu'_N \ge \lambda_{N+1}$. From these inequalities and Hamburger's theorem, we obtain the "only if" statement in Theorem 1.1. The "if" statement will be proved by finding a positive lower bound for the eigenvalues λ_N in the indeterminate case, cf. Theorem 1.2 below.

We think that Theorem 1.1 has the advantage over the theorem of Hamburger that it involves only the moment sequence (s_n) and not the shifted sequence (s_{n+2}). In section 2 we give another proof of the "only if" statement to make the proof of Theorem 1.1 independent of Hamburger's theorem.

If

$$\pi_N(x) := \sum_{j=0}^{N} v_j x^j, \qquad v_j \in \mathsf{R} \tag{1.5}$$

then a simple calculation shows that

$$\sum_{0\le j,\,k\le N} s_{j+k} v_j v_k = \int_E \pi_N^2(x)\, d\alpha(x), \tag{1.6}$$

and

$$\sum_{k=0}^{N} v_k^2 = \int_0^{2\pi} \left|\pi_N\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi}. \tag{1.7}$$

We could also study the reciprocal of λ_N given by

$$\frac{1}{\lambda_N} = \max\left\{\int_0^{2\pi} \left|\pi_N\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} : \pi_N,\ \int_E \pi_N^2(x)d\alpha(x) = 1\right\}. \tag{1.8}$$

Let $\{p_k\}$ denote the orthonormal polynomials with respect to α, normalized so that p_k has positive leading coefficient.

We recall that the moment problem is indeterminate, cf. [1], [10], if and only if there exists a non-real number z_0 such that

$$\sum_{k=0}^{\infty} |p_k(z_0)|^2 < \infty. \tag{1.9}$$

In the indeterminate case the series in (1.9) actually converges for all z_0 in $\mathbb{C}$, uniformly on compact sets. In the determinate case the series in (1.9) diverges for all non-real z_0 and also for all real numbers except the at most countably many points, where α has a positive mass.

If we expand the polynomial (1.5) as a linear combination of the orthonormal system

$$\pi_N(x) = \sum_{j=0}^{N} c_j p_j(x),$$

then

$$\int_0^{2\pi} \left|\pi_N\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \sum_{0\le j,\,k\le N} c_j c_k \int_0^{2\pi} p_j\left(e^{i\theta}\right) p_k\left(e^{-i\theta}\right) \frac{d\theta}{2\pi}$$
$$= \sum_{0\le j,\,k\le N} \mathcal{K}_{jk} c_j c_k,$$

where we have defined

$$\mathcal{K}_{jk} = \int_0^{2\pi} p_j\left(e^{i\theta}\right) p_k\left(e^{-i\theta}\right) \frac{d\theta}{2\pi}. \tag{1.10}$$

Thus

$$\frac{1}{\lambda_N} = \max\left\{\sum_{0\le j,\,k\le N} \mathcal{K}_{jk} c_j c_k : c_j,\ \sum_{j=0}^{N} c_j^2 = 1\right\}. \tag{1.11}$$

Since the eigenvalues of the matrix $(\mathscr{K}_{jk})_{0\le j,k\le N}$ are positive, and their sum is its trace, then

$$\frac{1}{\lambda_N} \le \sum_{k=0}^{N} \mathscr{K}_{kk} = \int_0^{2\pi} \sum_{k=0}^{N} \left|p_k\big(e^{i\theta}\big)\right|^2 \frac{d\theta}{2\pi}. \tag{1.12}$$

Thus in the case of indeterminacy,

$$\frac{1}{\lambda_N} \le \int_0^{2\pi} \sum_{k=0}^{\infty} \left|p_k(e^{i\theta})\right|^2 \frac{d\theta}{2\pi} < \infty, \tag{1.13}$$

which shows that

$$\lim_{N\to\infty} \lambda_N \ge \left(\int_0^{2\pi} \frac{1}{\rho(e^{i\theta})} \frac{d\theta}{2\pi}\right)^{-1}, \tag{1.14}$$

where

$$\rho(z) = \left(\sum_{k=0}^{\infty} |p_k(z)|^2\right)^{-1}. \tag{1.15}$$

We recall that for $z \in \mathsf{C} \setminus \mathsf{R}$ the number $\rho(z)/|z-\overline{z}|$ is the radius of the Weyl circle at z.

The above argument establishes the following result:

THEOREM 1.2. *In the indeterminate case the smallest eigenvalue λ_N of the Hankel matrix $\mathscr{H}_N$ is bounded below by the harmonic mean of the function ρ along the unit circle.*

We shall conclude this paper with examples, where we have calculated or estimated the quantity

$$\rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} \left|p_k\big(e^{i\theta}\big)\right|^2 \frac{d\theta}{2\pi}. \tag{1.16}$$

This will be done for the moment problems associated with the Stieltjes-Wigert polynomials, cf. [4], [12], the Al-Salam-Carlitz polynomials [2], the symmetrized version of polynomials of Berg-Valent ([3]) leading to a Freud-like weight [5], and the q^{-1}-Hermite polynomials of Ismail and Masson [9].

If we introduce the coefficients of the orthonormal polynomials as

$$p_k(x) = \sum_{j=0}^{k} \beta_{k,j} x^j \tag{1.17}$$

then

$$\int_0^{2\pi} \left|p_k(\mathrm{e}^{i\theta})\right|^2 \frac{d\theta}{2\pi} = \sum_{j=0}^{k} \beta_{k,j}^2,$$

and therefore

$$\rho_0 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \beta_{k,j}^2. \tag{1.18}$$

Another possibility for calculating ρ_0 is to use the entire functions B, D from the Nevanlinna matrix since it is well known that [1, p. 123]

$$\sum_{k=0}^{\infty} |p_k(z)|^2 = \frac{B(z)D(\overline{z}) - D(z)B(\overline{z})}{z - \overline{z}}. \tag{1.19}$$

It follows that

$$\sum_{k=0}^{\infty} \left|p_k(\mathrm{e}^{i\theta})\right|^2 = \mathrm{Im}\left\{B(\mathrm{e}^{i\theta})D(\mathrm{e}^{-i\theta})\right\} / \sin\theta. \tag{1.20}$$

2. Indeterminate Moment Problems

In this section we shall give a proof of Theorem 1.1 which is independent of Hamburger's result. We have already established that if $\lim_{N\to\infty} \lambda_N = 0$, then the problem is determinate. We shall next prove that if $\lambda_N \geq \gamma$ for all N, where $\gamma > 0$, then the problem is indeterminate. Since $1/\lambda_N \leq 1/\gamma$ for all N, and $1/\lambda_N$ is the biggest eigenvalue of the positive definite matrix $(\mathcal{K}_{jk})_{0\leq j,k\leq N}$, we get

$$\sum_{0\leq j,k\leq N} \mathcal{K}_{jk} c_j \overline{c_k} \leq \frac{1}{\gamma} \sum_{j=0}^{N} |c_j|^2, \tag{2.1}$$

for all vectors $(c_0, \ldots, c_N) \in \mathbb{C}^{N+1}$. If we consider an arbitrary complex polynomial p of degree $\leq N$ written as $p(x) = \sum_{k=0}^{N} c_k p_k(x)$, the inequality (2.1) can be formulated

$$\int_0^{2\pi} \left|p(\mathrm{e}^{i\theta})\right|^2 \frac{d\theta}{2\pi} \leq \frac{1}{\gamma} \int |p(x)|^2 \, d\alpha(x). \tag{2.2}$$

Let now z_0 be an arbitrary non-real number in the open unit disc. By the Cauchy integral formula

$$p(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(\mathrm{e}^{i\theta})}{\mathrm{e}^{i\theta} - z_0} \mathrm{e}^{i\theta} d\theta,$$

and therefore

$$|p(z_0)|^2 \le \int_0^{2\pi} \left|p\left(\mathrm{e}^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{1}{|\mathrm{e}^{i\theta} - z_0|^2} \frac{d\theta}{2\pi}. \tag{2.3}$$

Combined with (2.2) we see that there is a constant K such that for all complex polynomials p

$$|p(z_0)|^2 \le K \int |p(x)|^2 \, d\alpha(x), \tag{2.4}$$

where $K = 1/(\gamma(1 - |z_0|^2))$.

This inequality implies indeterminacy in the following way. Applying it to the polynomial

$$p(x) = \sum_{k=0}^{N} p_k(\overline{z_0}) p_k(x),$$

we get

$$\sum_{k=0}^{N} |p_k(z_0)|^2 \le K, \tag{2.5}$$

and since N is arbitrary, indeterminacy follows.

REMARK. We see that the infinite positive definite matrix $\mathscr{K}_\infty = \{\mathscr{K}_{j,k}\}$ is bounded on ℓ^2 if and only if $\lambda_N \ge \gamma$ for all N for some $\gamma > 0$. Furthermore $\mathscr{K}_\infty$ is of trace class if and only if $\rho_0 < \infty$. The result of Theorem 1.1 can be reformulated to say that boundedness implies trace class for this family of operators.

3. Examples

We shall follow the notation and terminology for q-special functions as those in Gasper and Rahman [7].

EXAMPLE 3.1 (The Stieltjes-Wigert Polynomials). These polynomials are orthonormal with respect to the weight function

$$\omega(x) = \frac{k}{\sqrt{\pi}} \exp\left(-k^2 (\log x)^2\right), \qquad x > 0, \tag{3.1}$$

where $k > 0$ is a positive parameter, cf. [4], [12]. They are given by

$$p_n(x) = (-1)^n q^{\frac{n}{2}+\frac{1}{4}} (q;q)_n^{-\frac{1}{2}} \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2} \left(-q^{\frac{1}{2}} x\right)^k, \tag{3.2}$$

where we have defined $q = \exp\{-(2k^2)^{-1}\}$.

It follows by (1.18) that

$$\rho_0 = \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{(q;q)_n} \sum_{k=0}^{n} q^{k(2k+1)} \binom{n}{k}_q^2 = \sum_{k=0}^{\infty} q^{2k^2+k+\frac{1}{2}} \sum_{n=k}^{\infty} \frac{q^n}{(q;q)_n} \binom{n}{k}_q^2. \tag{3.3}$$

Putting $n = k + j$, the inner sum is

$$\sum_{j=0}^{\infty} \frac{q^{k+j}}{(q;q)_k^2} \frac{(q;q)_{k+j}}{(q;q)_j^2} = \frac{q^k}{(q;q)_k} {}_2\phi_1(q^{k+1}, 0; q; q, q)$$

and hence

$$\rho_0 = \sum_{k=0}^{\infty} \frac{q^{2(k+\frac{1}{2})^2}}{(q;q)_k} {}_2\phi_1(0, q^{k+1}; q; q, q). \tag{3.4}$$

We can obtain another expression for ρ_0. We apply the transformation [7, (III.5)]

$${}_2\phi_1(a, b; c; q, z) = \frac{(abz/c;q)_\infty}{(bz/c;q)_\infty} {}_3\phi_2(a, c/b, 0; c, cq/bz; q, q) \tag{3.5}$$

to see that

$$\sum_{n=k}^{\infty} \frac{q^n}{(q;q)_n} \binom{n}{k}_q^2 = \frac{1}{(q;q)_\infty} \sum_{j=0}^{k} \frac{q^{k+j}}{(q;q)_j^2}. \tag{3.6}$$

We then find

$$\rho_0 = \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} q^{2(k+\frac{1}{2})^2} \sum_{j=0}^{k} \frac{q^j}{(q;q)_j^2}. \tag{3.7}$$

A formula more general than (3.6) is

$$\sum_{n=k}^{\infty} \frac{\omega^n}{(q;q)_n} \binom{n}{k}_q^2 = \frac{1}{(\omega;q)_\infty} \sum_{j=0}^{k} \frac{(\omega;q)_j \omega^{2k-j}}{(q;q)_j (q;q)_{k-j}^2}$$

and is stated in [2]. This more general identity also follows from (3.5) and the simple observation

$$\frac{(q^{-k};q)_j}{(q^{1-k}/\omega;q)_j} = \frac{(q;q)_k\,(\omega;q)_{k-j}}{(\omega;q)_k\,(q;q)_{k-j}}(\omega/q)^j.$$

We have numerically computed the smallest eigenvalue of the Hankel matrix of various dimensions with the Stieltjes-Wigert weight from which we extrapolate to determine the smallest eigenvalue $s = \lim_{N\to\infty}\lambda_N$ of the infinite Hankel matrix for different values of q. This is then compared with the numerically computed lower bound $l = 1/\rho_0$. For $q = \frac{1}{2}$ we have $s = 0.3605\ldots, l = 0.3435\ldots$. The percentage error $100(s-l)/s$ is plotted for various values of q and is shown in figure 1.

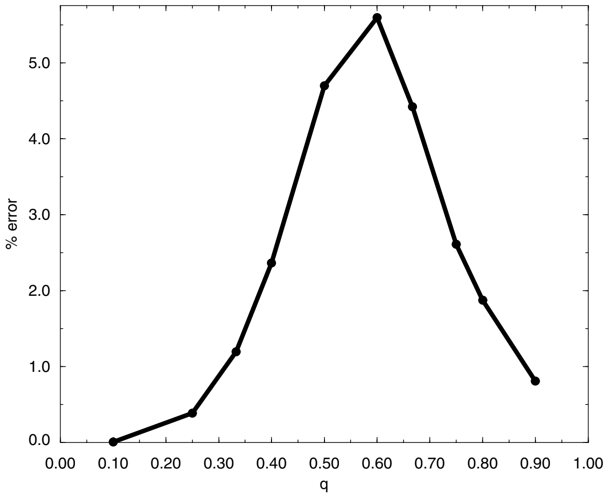

FIGURE 1. Percentage error plotted for various values of q.

EXAMPLE 3.2 (Al-Salam-Carlitz polynomials). The Al-Salam-Carlitz polynomials were introduced in [2]. We consider the indeterminate polynomials $V_n^{(a)}(x;q)$, where $0 < q < 1$ and $q < a < 1/q$, cf. [3]. For the corresponding orthonormal polynomials $\{p_k\}$ we have by [3, (4.24)]

$$(3.8)\quad \sum_{k=0}^{\infty}\left|p_k(\mathrm{e}^{i\theta})\right|^2 = \frac{(q\mathrm{e}^{i\theta}, q\mathrm{e}^{-i\theta};q)_\infty}{(aq,q,q;q)_\infty}\,{}_3\phi_2(\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}, aq; q\mathrm{e}^{i\theta}, q\mathrm{e}^{-i\theta}; q, q/a).$$

Therefore

(3.9)
$$\rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} \left| p_k\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \frac{1}{(aq, q, q; q)_\infty} \sum_{n=0}^{\infty} I_n \frac{(aq;q)_n}{(q;q)_n} \left(\frac{q}{a}\right)^n,$$

where

$$\begin{aligned} I_n &= \int_0^{2\pi} \frac{(e^{i\theta}, e^{-i\theta}; q)_\infty}{(1 - q^n e^{i\theta})(1 - q^n e^{-i\theta})} \frac{d\theta}{2\pi} \\ &= \int_{|z|=1} \frac{(z, 1/z; q)_\infty}{(1 - q^n z)(1 - q^n/z)} \frac{dz}{2\pi i z}. \end{aligned} \tag{3.10}$$

Recall the Jacobi triple product identity [7],

$$j(z) := (q, z, 1/z; q)_\infty = \sum_{k=-\infty}^{\infty} c_k z^k, \tag{3.11}$$

with

$$c_k = (-1)^k \left[q^{k(k+1)/2} + q^{k(k-1)/2}\right]. \tag{3.12}$$

Note that $c_k = c_{-k}$.

Using the partial fraction decomposition

$$\frac{q^n}{1 - q^n z} - \frac{q^{-n}}{1 - q^{-n} z} = \frac{1 - q^{2n}}{(1 - q^n z)(z - q^n)}$$

we find by the residue theorem and the Jacobi triple product identity (3.11) that for $n \geq 1$, I_n is given by

$$\begin{aligned} &(1 - q^{2n})(q;q)_\infty I_n \\ &\quad = q^n \operatorname{Res}\left(\frac{j(z)}{1 - q^n z}, z = 0\right) - q^{-n} \operatorname{Res}\left(\frac{j(z)}{1 - q^{-n} z}, z = 0\right) \\ &\quad = q^n \sum_{k=0}^{\infty} q^{nk} c_{-k-1} - q^{-n} \sum_{k=0}^{\infty} q^{-nk} c_{-k-1} \\ &\quad = \sum_{k=1}^{\infty} \left(q^{nk} - q^{-nk}\right) c_k, \end{aligned}$$

while for $n = 0$, I_0 is

$$(q;q)_\infty I_0 = \int_{|z|=1} \frac{j(z)}{(1-z)(z-1)} \frac{dz}{2\pi i} = -\mathrm{Res}\left(\frac{j(z)}{(1-z)^2}, z=0\right)$$
$$= -\sum_{k=0}^{\infty}(k+1)c_{-k-1} = \sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}.$$

The conclusion is

$$I_0 = \frac{1}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)/2}, \tag{3.13}$$
$$I_n = \frac{1}{(1-q^{2n})(q;q)_\infty}\sum_{k=1}^{\infty} c_k\left(q^{nk}-q^{-nk}\right), \qquad n \geq 1$$

The above formulas can be further simplified. Using the Jacobi triple product identity (3.11) we find for integer values of n

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{nk} q^{\binom{k}{2}} = 0,$$

hence

$$\sum_{k=0}^{\infty}(-1)^k q^{nk}\, q^{\binom{k}{2}} = -\sum_{k=1}^{\infty}(-1)^k q^{-nk}\, q^{\binom{k+1}{2}}, \qquad n = 0, \pm 1, \ldots. \tag{3.14}$$

This analysis implies

$$(q;q)_\infty(1-q^{2n})\, I_n = 2\sum_{k=1}^{\infty}(-1)^k\, q^{\binom{k}{2}}\left[q^{nk}-q^{-nk}\right]. \tag{3.15}$$

Thus we have established the representation for $n \geq 1$

$$I_n = \frac{2q^{-n}}{(q;q)_\infty}\sum_{k=1}^{\infty}(-1)^{k-1}\, q^{\binom{k}{2}}\, \frac{\sin(nk\tau)}{\sin(n\tau)}, \qquad q = e^{-i\tau}. \tag{3.16}$$

It is clear that I_0 is the limiting case of I_n as $n \to 0$. The representation (3.16) indicates that I_n is a theta function evaluated at the special point $n\tau$, hence we do not expect to find a closed form expression for I_n.

EXAMPLE 3.3 (Freud-like weight). In [3] Berg-Valent found the Nevanlinna matrix in the case of the indeterminate moment problem corresponding to a birth and death process with quartic rates. Later Chen and Ismail, cf. [5], considered the corresponding symmetrized moment problem, found the Nevanlinna matrix and observed that there are solutions which behave as the Freud weight $\exp(-\sqrt{|x|})$. In particular they found the entire functions

$$B(z) = -\delta_0\big(K_0\sqrt{z/2}\big), \qquad D(z) = \frac{4}{\pi}\,\delta_2\big(K_0\sqrt{z/2}\big), \tag{3.17}$$

where

$$\delta_l(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+l)!} z^{4n+l}, \qquad l = 0, 1, 2, 3, \tag{3.18}$$

$$K_0 = \frac{\Gamma(1/4)\Gamma(5/4)}{\sqrt{\pi}}. \tag{3.19}$$

Note that

$$\delta_0(z) = \frac{1}{2}\Big[\cosh\big(z\sqrt{i}\,\big) + \cos\big(z\sqrt{i}\,\big)\Big], \tag{3.20}$$

$$\delta_2(z) = \frac{1}{2i}\Big[\cosh\big(z\sqrt{i}\,\big) - \cos\big(z\sqrt{i}\,\big)\Big]. \tag{3.21}$$

If $\omega := \exp(i\pi/4) = (1+i)/\sqrt{2}$, then a simple calculation shows that

$$\begin{aligned} B(x)D(y) - D(x)B(y) = \frac{-2i}{\pi}\Big[&\cos\big(\omega^3 K_0\sqrt{x/2}\,\big)\cos\big(\omega K_0\sqrt{y/2}\,\big) \\ &- \cos\big(\omega^3 K_0\sqrt{y/2}\,\big)\cos\big(\omega K_0\sqrt{x/2}\,\big)\Big]. \end{aligned} \tag{3.22}$$

If $x = e^{i\theta}$, and $y = e^{-i\theta}$, then we linearize the products of cosines and find that the right-hand side of (3.22) is

$$\begin{aligned} \frac{-i}{\pi}\Big\{&\cos\big[K_0\big(\omega^3 e^{i\theta/2} + \omega e^{-i\theta/2}\big)/\sqrt{2}\,\big] + \cos\big[K_0\big(\omega^3 e^{i\theta/2} - \omega e^{-i\theta/2}\big)/\sqrt{2}\,\big] \\ &- \cos\big[K_0\big(\omega^3 e^{-i\theta/2} + \omega e^{i\theta/2}\big)/\sqrt{2}\,\big] - \cos\big[K_0\big(\omega^3 e^{-i\theta/2} - \omega e^{i\theta/2}\big)/\sqrt{2}\,\big]\Big\}. \end{aligned}$$

We now combine the first and third terms, then combine the second and fourth terms and apply the addition theorem for trigonometric functions. We then see that the above is

$$\begin{aligned} \frac{2i}{\pi}\big\{&\sinh[K_0\cos(\theta/2)]\sinh[K_0\sin(\theta/2)] \\ &+ \sin[K_0\cos(\theta/2)]\sin[K_0\sin(\theta/2)]\big\}. \end{aligned}$$

Thus we have proved that

$$\begin{aligned}(3.23)\quad &\frac{B\left(e^{i\theta}\right)D\left(e^{-i\theta}\right)-B\left(e^{-i\theta}\right)D\left(e^{i\theta}\right)}{e^{i\theta}-e^{-i\theta}}\\ &\quad=\frac{1}{\pi\sin\theta}\big\{\sinh[K_0\cos(\theta/2)]\sinh[K_0\sin(\theta/2)]\\ &\qquad\qquad+\sin[K_0\cos(\theta/2)]\sin[K_0\sin(\theta/2)]\big\}.\end{aligned}$$

Thus in the case under consideration, after some straightforward calculations and the evaluation of a beta integral, we obtain

$$\begin{aligned}\rho_0&=\int_0^{2\pi}\sum_{n=0}^{\infty}\left|p_n(e^{i\theta})\right|^2\frac{d\theta}{2\pi}\\ &=\frac{K_0^2}{\pi}\sum_{m,\,n\geq 0,\,m+n\ \text{even}}\frac{(K_0/2)^{2m+2n}}{(2m+1)(2n+1)\,m!\,n!\,(m+n)!}.\end{aligned}\tag{3.24}$$

EXAMPLE 3.4 (q^{-1}-Hermite polynomials). Ismail and Masson [9] proved that for this moment problem the functions B and D are given by

$$B(\sinh\xi)=-\frac{(qe^{2\xi},qe^{-2\xi};q^2)_\infty}{(q,q;q^2)_\infty},\tag{3.25}$$

$$D(\sinh\xi)=\frac{\sinh\xi}{(q;q)_\infty}(q^2e^{2\xi},q^2e^{-2\xi};q^2)_\infty,\tag{3.26}$$

[9, (5.32)], [9, (5.36)]; respectively. Ismail and Masson also showed that [9, (6.25)]

$$\begin{aligned}(3.27)\quad &B(\sinh\xi)D(\sinh\eta)-B(\sinh\eta)D(\sinh\xi)\\ &\quad=\frac{-e^{\eta}}{2(q;q)_\infty}\prod_{n=0}^{\infty}\left[1-2e^{-\eta}q^n\sinh\xi-e^{-2\eta}q^{2n}\right]\\ &\qquad\qquad\cdot\left[1+2e^{\eta}q^{n+1}\sinh\xi-e^{2\eta}q^{2n+2}\right].\end{aligned}$$

We rewrite the infinite product as

$$\prod_{n=0}^{\infty}a_nb_n=a_0\prod_{n=1}^{\infty}a_nb_{n-1},$$

and with $\sinh\xi = e^{i\theta}$ and $\sinh\eta = e^{-i\theta}$ we get the following representation

(3.28)
$$\begin{aligned}
&\frac{B(e^{i\theta})D(e^{-i\theta}) - B(e^{-i\theta})D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} \\
&\qquad = \frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}\left[1 + 4q^n - 2q^{2n} + 4q^{3n} + q^{4n} - 8q^{2n}\cos(2\theta)\right] \\
&\qquad = \frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}\left[(1+q^n)^4 - 16q^{2n}\cos^2\theta\right].
\end{aligned}$$

Writing the infinite product as a power series in $\cos^2\theta$ and using

$$\int_{-\pi}^{\pi}\cos^{2k}\theta\,\frac{d\theta}{2\pi} = 2^{-2k}\binom{2k}{k},$$

we evaluate the integral of (3.28) with respect to $d\theta/2\pi$ as

$$\rho_0 = \frac{(-q;q)_\infty^4}{(q;q)_\infty}\sum_{k=0}^{\infty}\binom{2k}{k}\sum_{1\le n_1<\cdots<n_k}\frac{(-2)^{2k}q^{2(n_1+\cdots+n_k)}}{[(1+q^{n_1})\ldots(1+q^{n_k})]^4}. \tag{3.29}$$

The formula (3.28) can be transformed further by putting $\cos^2\psi = -\cos\theta$ and $p^2 = q$, because then

$$\prod_{n=1}^{\infty}\left[(1+q^n)^2 + 4q^n\cos\theta\right] = \prod_{n=1}^{\infty}\left[1 + p^{4n} - 2p^{2n}\cos(2\psi)\right]$$

can be expressed by means of the theta function $\vartheta_1(p;\psi)$. We find

$$\prod_{n=1}^{\infty}\left[(1+q^n)^2 + 4q^n\cos\theta\right] = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}U_{2n}(\cos\psi), \tag{3.30}$$

where

$$U_{2n}(\cos\psi) = \frac{\sin(2n+1)\psi}{\sin\psi}$$

is the Chebyshev polynomium of the second kind given by

$$U_{2n}(x) = \sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k x^{2(n-k)}(1-x^2)^k. \tag{3.31}$$

Similarly putting $\cos^2\varphi = \cos\theta$ we find

$$\prod_{n=1}^{\infty}\left[(1+q^n)^2 - 4q^n\cos\theta\right] = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}U_{2n}(\cos\varphi). \tag{3.32}$$

If we let U_n^* be the polynomial of degree n such that $U_{2n}(x) = U_n^*(x^2)$, we get

$$\frac{B(e^{i\theta})D(e^{-i\theta}) - B(e^{-i\theta})D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{1}{(q;q)_\infty^2} \sum_{n,m=0}^{\infty} (-1)^m q^{\binom{n+1}{2}+\binom{m+1}{2}} U_n^*(-\cos\theta) U_m^*(\cos\theta). \tag{3.33}$$

For non-negative integers k, l, r we have

$$C(k,l,r) := \frac{1}{2\pi}\int_0^{2\pi} (1+\cos\theta)^k (1-\cos\theta)^l \cos^r\theta \, d\theta = \frac{2^{k+l}}{\pi}(-1)^r B\left(k+\frac{1}{2}, l+\frac{1}{2}\right) {}_2F_1\left(k+\frac{1}{2}, -r; k+l+1; 2\right), \tag{3.34}$$

which gives

$$\frac{1}{2\pi}\int_0^{2\pi} U_n^*(-\cos\theta) U_m^*(\cos\theta)\, d\theta = \sum_{k=0}^{n}\sum_{l=0}^{m} \binom{2n+1}{2k+1}\binom{2m+1}{2l+1} (-1)^{n+l} C(k,l,n+m-k-l). \tag{3.35}$$

Putting these formulas together we get a 5-fold sum for ρ_0.

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