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A COMPACTIFICATION OF THE SPACE OF TWISTED CUBICS

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Abstract

We give an elementary, explicit smooth compactification of a parameter space for the family of twisted cubics. The construction also applies to the family of subschemes defined by determinantal nets of quadrics, e.g., cubic ruled surfaces in P^4, Segre varieties in P^5. It is suitable for applications of Bott's formula to a few enumerative problems.

1. Introduction

A twisted cubic curve (twc) is the image in projective 3-space of the map

$$[t, u] \mapsto [t^3, t^2u, tu^2, u^3],$$

for a suitable choice of homogeneous coordinates. According to Harris [7], "This is everybody's first example of a concrete variety that is not a hypersurface, linear space, or finite set of points".

Our aim is to give a simple, explicit smooth compactification of a parameter space for the family of twisted cubics. "Simple" means no need of GIT. "Explicit" is intended to be suitable for applications of Bott's formula (as in Ellingsrud and Strømme [3], Meurer [8]) to a few enumerative problems.

Piene and Schlessinger [9] have shown that the Hilbert scheme component H of twisted cubics is a smooth projective variety of dimension twelve. Later, Ellingsrud, Piene and Strømme [1] proved that the subvariety D of the Grassmann variety $\mathsf{G}(3, 10)$ of nets of quadrics of determinantal type (i.e., spanned by the 2×2 minors of a 2×3 matrix of linear forms) is a smooth variety. H is the blowup of D along the subvariety of nets with a fixed component. Ellingsrud and Strømme [2] have also shown that D is a geometric quotient of the set of semistable 2×3 matrix of linear forms. This description enabled them to compute the Chow rings of D and H. A major motivation was to give a mathematical treatment to the physicists prediction for the number of twisted cubics contained in certain Calabi-Yau manifolds (cf. [3]).

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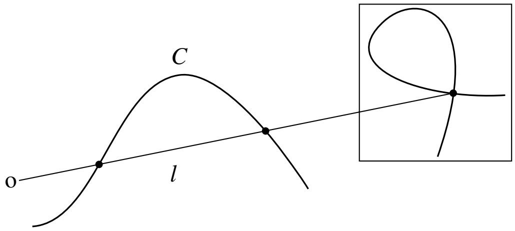
We offer an alternative approach that leads ultimately to compactification of the set of all twcs that miss a fixed point $o \in \mathsf{P}^3$. The main idea is best explained in the picture below.



If the twisted cubic C misses the point o, then there is a *unique* line $l \ni o$ that is bisecant (possibly tangent) to C. Now the configuration $l \cup C$ is a complete intersection of a pencil of quadrics.

We simply *revert* the construction.

For each line $l \subset \mathsf{P}^3$, take the Grassmann variety $\mathsf{G}(2, 7)$ of pencils of quadrics containing l. This defines a Grassmann bundle X over the Grassmannian of lines. There is a Zariski open subset of X that parametrizes a family of twcs. This family of twcs does not extend to a flat family over X. The main result is the construction of a sequence of three blowups

$$\mathsf{X}''' \longrightarrow \mathsf{X}'' \longrightarrow \mathsf{X}' \longrightarrow \mathsf{X}$$

along smooth, explicit centers that yields a flat family of twcs over X'''.

The first blowup fixes the problem of assigning a well defined net of quadrics to any pencil of quadrics as above. Precisely, we get a morphism $\mathsf{X}' \to \mathsf{G}(3, 10)$ that extends the rational map $\mathsf{X} \cdots \to \mathsf{G}(3, 10)$ given by the net of quadrics through the residual twc determined by the pencil.

The remaining two blowups are designed to resolve the indeterminacy of the rational map $\mathsf{X}' \cdots \to \mathsf{G}(10, 20)$ defined by the system of cubics through a possibly degenerate twc. This is done by studying a suitable saturation (§3) of the subsheaf of the free sheaf of cubic forms that is the image of the natural map (cf. (18)) given by multiplying by linear forms all quadrics in a net.

The construction also applies with obvious minor changes to P^n, for any $n \geq 2$. Precisely, we get a similar description for a smooth compactification of the family of subschemes of P^n defined by nets of quadrics of determinantal type.

As an amusing application, we may retrieve the number **15** of triangles in P^2 meeting 6 general lines and Schubert's **80,160** twisted cubics in P^3 meeting 12 general lines. We also find **648,151,945** (resp. **7,265,560,058,820**) rational ruled cubic surfaces in P^4 (resp. Segre varieties in P^5) meeting 18 (resp. 24) general lines. We note that a variation of a script of P. Meurer [8] adapted by A. Meireles takes a few seconds in a PC to produce these last numbers, whereas the computation of Gromov-Witten invariants implemented by J. Kock, using Kresch's FARSTA [6], took about 3 days for $n = 4$ and was too big even to get started for $n = 5$. It also reproduces the number (cf. [3]) of twisted cubics in P^4 contained in a general quintic.

2. Notation and preliminaries

Let $\mathscr{F}$ denote the space of linear forms in the variables $\underline{x} = x_1, x_2, x_3, x_4$.

Let $\mathsf{G}(2, \mathscr{F})$ be the Grassmann variety of lines in P^3, with tautological sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{F} \longrightarrow \overline{\mathscr{F}} \longrightarrow 0 \tag{1}$$

where rank $\mathscr{L} = 2$. The fiber $\mathscr{L}_l$ for $l \in \mathsf{G}(2, \mathscr{F})$ is the vector space of linear forms that vanish on the line l.

Set

$$\mathscr{Q} := \mathrm{Ker}(S_2\mathscr{F} \longrightarrow S_2\overline{\mathscr{F}}).$$

The fiber $\mathscr{Q}_l$ for $l \in \mathsf{G}(2, \mathscr{F})$ is the vector space of quadratic forms that vanish on the line l. We clearly have rank $\mathscr{Q} = 7$.

We write

$$\mathsf{X} = \mathsf{G}(2, \mathscr{Q}) \longrightarrow \mathsf{G}(2, \mathscr{F}) \tag{2}$$

for the Grassmann bundle of pencils of quadrics containining a varying line $l \in \mathsf{G}(2, \mathscr{F})$. Each element $\mathrm{x} \in \mathsf{X}$ may be thought of as a pair (π, l) such that π represents a pencil of quadrics through the distinguished line l. The latter is the image of x in $\mathsf{G}(2, \mathscr{F})$.

Denote by

$$\mathscr{R} \rightarrowtail \mathscr{Q}_{\mathsf{X}} \tag{3}$$

the rank two tautological subbundle of $\mathscr{Q}_{\mathsf{X}}$.

We omit the easy proofs of the following.

LEMMA 2.1. (i) *The orbit of the point* $\mathrm{x}_0 = (\pi_0, l_0) \in \mathsf{X}$ *given by the pencil* $\pi_0 = \langle x_1^2, x_1x_2\rangle$ *with distinguished line* $l_0 = \langle x_1, x_2\rangle$ *is the unique closed orbit of* X *under the natural action of* $\mathrm{GL}(\mathscr{F})$.

(ii) *Let* $\mathsf{P}(\mathscr{L})$ *be the projective bundle over* $\mathsf{G}(2, \mathscr{F})$ *that parametrizes the pairs* (h, l) *such that* h *is a plane containing the line* l. *Let*

$$\iota : \mathsf{G}(2, \mathscr{F}) \times \mathsf{P}(\mathscr{L}) \longrightarrow \mathsf{X}$$

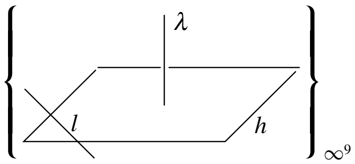
be defined by assigning to each $(\lambda, (h, l))$ *the pencil of quadrics belonging to* $\mathsf{X}_l = \mathsf{G}(2, \mathscr{Q}_l)$ *with fixed component* h *and varying part the pencil of planes with axis* λ. *Let* B *denote the image of* ι. *Then we have the following.*

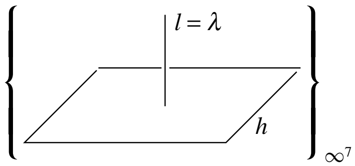
1. ι *is an equivariant embedding;*
2. B *is normally flat over* $\mathsf{G}(2, \mathscr{F})$.

We may think of a point of B as a pair $(\lambda, (h, l))$ cf. the picture below.

We also introduce now another relevant subvariety $\mathsf{Y}_1 \subset \mathsf{X}$. It is the locus of pencils with a fixed plane and moving pencil of planes with axis equal to the distinguished line.

(4)

$\mathsf{B} :=$ 

$\mathsf{Y}_1 :=$ 

REMARK. Roughly speaking, the rest of this work is designed to fill in the technicalities needed to complete these pictures in order to produce all honest (flat) degenerations of a twc.

LEMMA 2.2. *We have a natural embedding of* $\check{\mathsf{P}}^3 \times \mathsf{G}(2, \mathscr{F})$ *in* X *defined by multiplying by a varying plane the pencil of planes through a distinguished line. Denote by* Y_1 *the image. Then the scheme-theoretic intersection* $\mathsf{Y}_1 \cap \mathsf{B}$ *is isomorphic to the incidence subvariety* $(l \subset h)$ *of* $\check{\mathsf{P}}^3 \times \mathsf{G}(2, \mathscr{F})$.

3. Saturation

Let A be an integral domain, $\mathscr{P}$ an A-module and $\mathscr{M} \subseteq \mathscr{P}$ a submodule. We define the *saturation* of $\mathscr{M}$ in $\mathscr{P}$ by

$$^s\mathscr{M} = \{m \in \mathscr{P} \mid \exists\, a \in A,\ a \neq 0,\ am \in \mathscr{M}\}.$$

Thus $^s\mathscr{M}$ is just the inverse image under the quotient map $\mathscr{P} \to \mathscr{P}/\mathscr{M}$ of the torsion submodule of $\mathscr{P}/\mathscr{M}$. The following facts are easy to check.

1. $^{ss}\mathscr{M} = {}^s\mathscr{M}$.
2. For any multiplicative system $S \subset A$ we have $S^{-1}({}^s\mathscr{M}) = {}^s(S^{-1}\mathscr{M})$.
3. For submodules $\mathscr{M}, \mathscr{M}' \subseteq \mathscr{P}$, if $\mathscr{M}_f = \mathscr{M}'_f$ for some nonzero $f \in A$ then $^s\mathscr{M} = {}^s\mathscr{M}'$.
4. If $\mathscr{M} \subseteq \mathscr{P} = A^n$ is a locally split submodule then $^s\mathscr{M} = \mathscr{M}$.

One may define the saturation of a sheaf of modules over an integral scheme in view of 2 above.

We register in the following lemma the main steps used at each blowup in order to produce the saturation of certain subsheaves of $S_m\mathscr{F}$. It is inspired by Raynaud's strategy of flattening by blowing up [10].

LEMMA 3.1. *Let* U *be an integral affine variety with coordinate ring* A *and let*

$$M = \begin{bmatrix} I_r & * & * & * \\ 0 & f_1 & \dots & f_s \end{bmatrix}$$

be a triangular $(r+1)\times n$ *matrix with entries in* A*, where* I_r *denotes an identity block of size* r*. Let* $\mathscr{M} \subseteq A^n$ *be the submodule spanned by the rows of* M*. Assume the ideal* J *of* $(r+1)$*-minors is non zero. Let* $\rho : \mathsf{U} \cdots \to \mathsf{G}(r+1, \mathsf{C}^n)$ *be the rational map defined by* M*. Let* $\mathsf{U}' \to \mathsf{U}$ *be the blowup of the scheme of zeros* $\mathsf{V} = \mathscr{Z}(J)$ *and let* V' *denote the exceptional divisor. Then we have the following.*

1. *The map* ρ *extends to a morphism* $\rho' : \mathsf{U}' \to \mathsf{G}(r+1, \mathsf{C}^n)$.
2. *Suppose* V *is a complete intersection of codimension* t *in* U *and* $J = \langle f_1, \dots, f_t\rangle$ *for some* $t \le s$*. Then* U' *is the closed subscheme of* $\mathsf{U}\times\mathsf{P}^{t-1}$ *defined by* $f_i x_j = f_j x_i$, $1 \le i,\ j \le t$*, where the* x_i *denote homogeneous coordinates for* P^{t-1}.
3. *Let* $\mathsf{U}'_0 = \mathsf{U}' \cap (\mathsf{U}\times\mathsf{C}^{t-1}) \subset \mathsf{U}\times\mathsf{P}^{t-1}$ *be the affine open subset given by* $x_0 = 1$ *and put* $\mathsf{V}'_0 = \mathsf{V}' \cap \mathsf{U}'_0$*. Then there are regular functions* $y_2, \dots, y_s$ *on* U'_0 *such that* $f_i = y_i f_1$ *for* $2 \le i \le s$ *and the coordinate ring of* V'_0 *is the polynomial ring* $(A/J)[y_2, \dots, y_t]$.
4. *The restriction of* ρ' *to the open subset* $\mathsf{V}'_0 \cong \mathsf{V}\times\mathsf{C}^{t-1}$ *of the exceptional divisor is given by*

   $$\rho'(z, a) = \mathscr{M}_z + \langle e_{r+1} + a_2 e_{r+2} + \cdots + a_t e_{r+t} \\ + y_{t+1}(a) e_{r+t+1} + \cdots + y_s(a) e_{r+s}\rangle$$

   where $a = (a_2, \dots, a_t) \in \mathsf{C}^{t-1}$ *and* $\mathscr{M}_z$ *denotes the span of the first* r *rows of* M *at the closed point* $z \in \mathsf{V}$*, whereas the* e_i *are the standard unit vectors in* C^n.
5. *Put* $B := \mathscr{O}(\mathsf{U}'_0)$*. The saturation of the image of* $\mathscr{M}\otimes B$ *in* B^n *is a split, free submodule with basis given by the first* r *rows of* M *together with the "new generator",*

   $$e_{r+1} + y_2 e_{r+2} + \ldots + y_s e_n$$

   obtained by dividing the last row of M *by* f_1*, the local equation of the exceptional divisor.*

4. The associated net

Our first task will be to resolve the indeterminacies of the rational map

$$\mathsf{X} \cdots \longrightarrow \mathsf{G}(3, S_2\mathscr{F})$$

that assigns to a general pencil of quadrics through a line l the net of quadrics that cut the residual twisted cubic. For simplicity, we explain the procedure restricted to the fiber X_0 of X over the fixed line $l_0 = \langle x_1, x_2 \rangle$. The trick that has made the calculations go through is the following trivial observation. Let

$$\pi = \langle f_{11}x_1 + f_{12}x_2, f_{21}x_1 + f_{22}x_2 \rangle$$

be a general pencil of quadrics containing the line $l_0 := x_1 = x_2 = 0$. Here the f_{ij} denote linear forms. Each point on the intersection of the two quadrics that lies off that line must annihilate the determinant $f_{11}f_{22} - f_{12}f_{21}$. Also recall that the residual twisted cubic is cut out by a determinantal net. This leads us to look at the 2×2 minors of the matrix

$$\begin{pmatrix} f_{11} & f_{21} & -x_2 \\ f_{12} & f_{22} & x_1 \end{pmatrix}.$$

Thus, consider the rational map,

$$\begin{aligned} \nu : \mathsf{X} \ \cdots\longrightarrow & \quad \mathsf{G}(3, S_2\mathscr{F}) \\ \pi \ \longmapsto & \ \pi + \langle f_{11}f_{22} - f_{12}f_{21} \rangle. \end{aligned}$$

A routine check shows that ν is indeed well defined, i.e., the assigned net is independent of the choice of generators of the pencil. Moreover, the locus where ν is a morphism contains the complement of the locus of pencils with a fixed component. In fact, ν is also defined at some points representing pencils the base locus of which contain more than one line, essentially because each element of X carries a distinguished line.

5. Resolving the indeterminacies of ν

We show next that B (cf. 2.1) is the locus of indeterminacy of the rational map ν.

PROPOSITION 5.1. *Let X' be the blowup of X along B. Let E' be the exceptional divisor. Then we have the following.*

1. *the rational map ν lifts to a morphism $\nu' : \mathsf{X}' \longrightarrow \mathsf{G}(3, S_2\mathscr{F})$;*

2. *the fiber of* E' *over* $(\lambda, (h, l)) \in \mathsf{B}$ *is the projective space of the quotient vector space* $\mathscr{Q}_\lambda/(h \cdot \mathscr{L}_\lambda)$*;*
3. *the restriction of* ν' *to* E' *is given by the rule*

$$(\overline{q}, (\lambda, (h, l))) \mapsto \langle q \rangle + (h \cdot \mathscr{L}_\lambda)$$

where $\overline{q} \in \mathsf{P}(\mathscr{Q}_\lambda/(h \cdot \mathscr{L}_\lambda))$.

PROOF. Normal flatness of B over $\mathsf{G}(2, \mathscr{F})$ ensures that the formation of the blowup commutes with base change. Thus we may restrict the verification to a fiber X_0, say over the distinguished line $l_0 = \langle x_1, x_2 \rangle$. Let $\dot{\mathsf{X}}_0$ be the standard neighborhood of $\langle x_1^2, x_1x_2 \rangle$ in X_0 with coordinate functions

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5$$

so that the two quadrics

$$\begin{cases} q_1 = x_1^2 \qquad\qquad + a_1x_1x_3 + a_2x_1x_4 + a_3x_2^2 + a_4x_2x_3 + a_5x_2x_4, \\ q_2 = \qquad\quad x_1x_2 + b_1x_1x_3 + b_2x_1x_4 + b_3x_2^2 + b_4x_2x_3 + b_5x_2x_4 \end{cases}$$

give a local trivialization for the tautological rank two subbundle (3). Put

$$f_{11} = x_1 + a_1x_3 + a_2x_4, \qquad f_{12} = a_3x_2 + a_4x_3 + a_5x_4,$$

$$f_{21} = x_2 + b_1x_3 + b_2x_4, \qquad f_{22} = b_3x_2 + b_4x_3 + b_5x_4.$$

We have $q_i = \sum f_{ij}x_j$. This enables us to represent the rational map ν by a 3×10 matrix. Indeed, the subspace spanned by q_1, q_2 and $q_3 = f_{11}f_{22} - f_{21}f_{12}$ can be written as the row space of the 3×10 matrix obtained by collecting coefficients of the quadratic monomials. The ordered basis we choose is formed by the seven monomials appearing in q_1, q_2, in that order, together with x_3^2, x_3x_4, x_4^2. We find

$$M = \begin{bmatrix} 1 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 \\ 0 & 1 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 & 0 & 0 \\ 0 & b_3 & b_4 & b_5 & -a_3 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{bmatrix} \tag{5}$$

where we have set for short

$$\alpha_1 = a_1b_3 - a_3b_1 - a_4,\ \ \alpha_2 = a_2b_3 - a_3b_2 - a_5,\ \ \alpha_3 = a_1b_4 - a_4b_1,$$
$$\alpha_4 = a_2b_4 - a_4b_2 + a_1b_5 - a_5b_1,\ \ \alpha_5 = a_2b_5 - a_5b_2.$$

Adding to the third row $-b_3$ times the second row in the above matrix, we see that the ideal of 3×3 minors of M is spanned by

$$(6) \qquad b_4 - b_3b_1,\ \ b_5 - b_3b_2,\ \ a_3 + b_3^2,\ \ a_4 - a_1b_3,\ \ a_5 - a_2b_3.$$

This is the ideal in $\dot{\mathsf{X}}_0$ of the subscheme V where ν is not defined. The subgroup G_0 of $\mathrm{Aut}(\mathsf{P}^3)$ fixing l_0 acts on X_0. Since the map ν is G_0-invariant, so is V. Since G_0 is irreducible, it follows that each irreducible component of V is also invariant. Indeed, let $\mathsf{V}_1 \subseteq \mathsf{V}$ be an irreducible component. We have $\mathsf{V}_1 \subseteq \mathrm{G}_0 \cdot \mathsf{V}_1$; since the latter is irreducible, the inclusion is in fact an equality as asserted. Therefore any irreducible component must contain the unique closed orbit and must show up in the present neighborhood. Hence V is in fact smooth and irreducible.

Solving the relations (6) for b_4, b_5, a_3, a_4, a_5 and plugging into q_1, q_2, we find that the pencil degenerates to a pencil of quadrics of the form

$$\begin{cases} \overline{q}_1 = (x_1 + b_3x_2)(x_1 - b_3x_2 + a_2x_4 + a_1x_3), \\ \overline{q}_2 = (x_1 + b_3x_2)(x_2 + b_1x_3 + b_2x_4). \end{cases}$$

Notice the appearance of a fixed component, namely the plane given by $x_1 + b_3x_2$. It contains the distinguished line l_0. Thus we see that our pencil of quadrics is in fact given now by that fixed plane times the pencil of planes with axis equal to the line

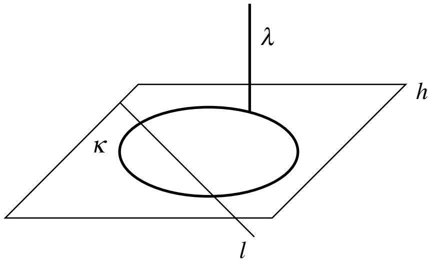
$$\lambda = \mathscr{Z}\big\langle x_1 - b_3x_2 + a_2x_4 + a_1x_3, x_2 + b_1x_3 + b_2x_4\big\rangle.$$

Recalling the definition 2.1 of B, this shows that, set-theoretically, V and B agree. Since V is smooth, it follows that $\mathsf{V} = \mathsf{B}$. By general principles (cf. 3.1), we have that the blowup X' is the closure in $\mathsf{X} \times \mathsf{G}(3, S_2\mathscr{F})$ of the graph of ν.

It remains to describe the behaviour of ν' on the exceptional divisor. This will be done in the sequel.

PROPOSITION 5.2. *Notation as above, for $q \in \mathsf{P}(\mathscr{Q}_\lambda \big/ (h \cdot \mathscr{L}_\lambda))$ we have the following.*

(i) *If the line λ is transversal to the plane h then the net of quadrics $\langle q \rangle + (h \cdot \mathscr{L}_\lambda)$ defines a degenerate twc of the form $\lambda \cup \kappa$, where κ denotes the conic $\mathscr{Z}\langle h, q \rangle$.*



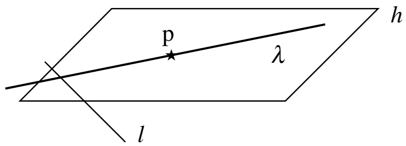
(ii) *If $\lambda \subset h$ then there are two possibilities.*

(1) *The net of quadrics $\langle q\rangle + (h\cdot\mathscr{L}_\lambda)$ still defines a degenerate twc, equal to the union of a non-planar degree two structure on λ with another line in the plane h. It is projectively equivalent to a net of the list*

$$\langle x_1^2, x_1x_3, x_1x_3 + x_2x_4\rangle,\ \ \langle x_1^2, x_1x_2, x_1x_3 + x_2^2\rangle,\\ \langle x_1^2, x_1x_2, x_2x_3\rangle,\ \ \langle x_1^2, x_1x_2, x_2^2\rangle.$$

(2) *The net of quadrics acquires a fixed component and is projectively equivalent to $\langle x_1^2, x_1x_2, x_1x_3\rangle$. This occurs along a subvariety $\mathsf{Y}'_2 \subset \mathsf{E}'$ isomorphic to the variety of flags*

$$\text{(7)}\quad \mathsf{Y}'_2 \cong \{(\mathrm{p}, l, \lambda, h) \in \mathsf{P}^3 \times \mathsf{G}(2,\mathscr{F})^{\times 2} \times \check{\mathsf{P}}^3 \mid \mathrm{p} \in \lambda \subset h \supset l\}.$$

$\mathsf{Y}'_2 = \Big\{$  $\Big\}$

Proof. For (i) we may take the line $\lambda = \mathscr{Z}\langle x_3, x_4\rangle$ and the plane $h := x_2 = 0$. Now $q = ax_3 + bx_4 \notin \langle x_2\rangle$ since $q \notin x_2\langle x_3, x_4\rangle$. Hence the quadric $\mathscr{Z}(q)$ cuts the plane h in a conic κ. It follows easily that $\mathscr{Z}\langle q, x_2x_3, x_2x_4\rangle$ is the union of κ and λ.

For (ii), we start with the line $\lambda = \mathscr{Z}\langle x_1, x_2\rangle$ and the plane $h := x_1 = 0$. Let $q = ax_1 + bx_2$ be a representative of a nonzero class in $\mathscr{Q}_{l_0}\big/x_1\langle x_1, x_2\rangle$. We may assume that the linear form a is in $\langle x_3, x_4\rangle$ and b is in $\langle x_2, x_3, x_4\rangle$. Check the cases. Suppose $a = 0$. If b is (resp. is not) a multiple of x_2 we get the last (resp. third) of the list. If $a \neq 0$ we may take $a = x_3$. Now b can't be zero lest we get a flag as in (2). If x_4 appears in b, we may take $b = x_4$ and retrieve the first net of the list. Presently the net is of the form $\langle x_1^2, x_1x_2, x_1x_3 + x_2(\alpha x_2 + \beta x_3)\rangle$. If $\beta = 0$ we get the second of the list. If

$\beta \neq 0 = \alpha$, we get the third of the list (changing $x_2 \to \beta x_2 + x_1$). Finally, if $\alpha\beta \neq 0$ the net is equivalent to the second of the list.

LEMMA 5.3. *Suppose in* (2) *above that the plane be given by a linear form h, the line λ by an additional equation h' and the point* p *by these previous two together with h''. Map the flag $\langle h\rangle \subset \langle h, h'\rangle \subset \langle h, h', h''\rangle$ to the pair*

$$\big(\langle h^2, hh'\rangle, \langle h^2, hh', hh''\rangle\big)$$

in the fiber of E' over l. Then this map is an embedding of the flag variety onto Y_2'.

Notice that Y_2' is of codimension six. Denoting by Y_2 the subvariety of B where $\lambda \subset h$ holds, we see that Y_2' sits over Y_2 as the P^1 subbundle of $\mathsf{E}'_{|\mathsf{Y}_2}$ defined by $\mathsf{P}((h \cdot \mathscr{F})/(h \cdot \mathscr{L}_\lambda))$.

6. Tangent maps and normal bundles

We describe in this section the normal bundles of the embeddings

$$\begin{array}{ccccc} \mathsf{B} & \subset & \mathsf{X} & \supset & \mathsf{Y}_1 \\ \| & & \| & & \| \\ \mathsf{P}(\mathscr{L}) \times \mathsf{G}(2,\mathscr{F}) & & \mathsf{G}(2,\mathscr{Q}) & & \check{\mathsf{P}}^3 \times \mathsf{G}(2,\mathscr{F}). \end{array}$$

We consider $\mathsf{B}, \mathsf{X}, \mathsf{Y}_1$ as schemes over $\mathsf{G}(2,\mathscr{F})$ (cf. §2 for notation). Care must be taken with B as there are two maps $p_l, p_\lambda : \mathsf{B} \to \mathsf{G}(2,\mathscr{F})$. We take p_l as the structure map $\mathsf{B} \to \mathsf{G}(2,\mathscr{F})$; it factors through $\mathsf{P}(\mathscr{L})$.

In view of the formula for the relative tangent bundle of a Grassmann bundle,

$$T\mathsf{X}/\mathsf{G}(2,\mathscr{F}) = \mathrm{Hom}(\mathscr{R}, \mathscr{Q}/\mathscr{R}),$$

we must compute the restrictions of the tautological rank two subbundle (3) over B and over Y_1. We have

$$\begin{aligned} T\mathsf{B}/\mathsf{G}(2,\mathscr{F}) &= \mathrm{Hom}(\mathscr{O}_{\mathscr{L}}(-1), \mathscr{L}/\mathscr{O}_{\mathscr{L}}(-1)) \bigoplus \mathrm{Hom}(\mathscr{L}, \mathscr{F}/\mathscr{L}),\\ \mathscr{R}_{|\mathsf{B}} &= \mathscr{O}_{\mathscr{L}}(-1) \bigotimes p_\lambda^* \mathscr{L},\\ \mathscr{R}_{|\mathsf{Y}_1} &= \mathscr{O}_{\mathscr{F}}(-1) \bigotimes \mathscr{L}. \end{aligned}$$

PROPOSITION 6.1. *Notation as above, we have the formulae for the normal bundles,*

$$\mathscr{N}_{\mathsf{Y}_1/\mathsf{X}} \cong \big(\mathscr{O}_{\mathscr{F}}(-1) \bigotimes \overset{2}{\wedge} \mathscr{L}\big)\check{}\bigotimes S_3\mathscr{F}^{(2)}/\big(\mathscr{O}_{\mathscr{F}}(-1) \bigotimes S_2\mathscr{F}^{(2)}\big), \tag{8}$$

where $S_m\mathscr{F}^{(2)}$ stands for the vector bundle $/\,\mathsf{G}(2,\mathscr{F})$ with fiber over l equal to the subspace $S_m\mathscr{F}^{(l^2)}$ of $S_m\mathscr{F}$ of forms of degree m contained in the square of the homogeneous ideal of l.

(9)

$$\mathscr{N}_{\mathsf{B}/\mathsf{X}} \cong \big(p_\lambda^\star \overset{2}{\wedge} \check{\mathscr{L}} \bigotimes p_l^*(\mathscr{O}_{\mathscr{L}}(2) \bigotimes \overset{2}{\wedge} \mathscr{L})\big) \bigotimes p_\lambda^* \mathscr{Q}/(\mathscr{O}_{\mathscr{L}}(-1) \otimes \mathscr{L}).$$

Before starting the proof, we need a quick review of multilinear algebra.

Given vector bundles E, F, recall the standard isomorphism $\check{E} \otimes F \cong \operatorname{Hom}(E, F)$ given by $\check{e} \otimes f \mapsto (e \mapsto \check{e}(e) f)$. In terms of (local) basis $(e_i) \subset E$, with dual basis $(\check{e}_i) \subset \check{E}$ and basis $(f_j) \subset F$, $(\varphi_{i,j}) \subset \operatorname{Hom}(E, F)$ such that $\varphi_{ij}(e_i) = f_j$ and $\varphi_{ij}(e_k) = 0$, $i \neq k$, the isomorphism maps $\check{e}_i \otimes f_j$ to φ_{ij}.

We also have for rank $E = m$, the isomorphism

$$\overset{m-1}{\wedge} E \cong \operatorname{Hom}(E, \overset{m}{\wedge} E)$$

$$v_2 \wedge \cdots \wedge v_m \mapsto (v_1 \mapsto v_1 \wedge \cdots \wedge v_m).$$

In particular, for rank $E = 2$ we get

(10)
$$\begin{gathered} E \cong \operatorname{Hom}(E, \overset{2}{\wedge} E) \cong \check{E} \otimes \overset{2}{\wedge} E \\ e' \mapsto (e \mapsto e' \wedge e). \end{gathered}$$

In terms of a pair of local dual basis e_1, e_2 and $\check{e}_1, \check{e}_2$, we have

(11)
$$\begin{aligned} e_1 &\mapsto \begin{Bmatrix} e_1 \mapsto 0 \\ e_2 \mapsto e_1 \wedge e_2 \end{Bmatrix} \mapsto \check{e}_2 \otimes e_1 \wedge e_2, \\ e_2 &\mapsto \begin{Bmatrix} e_1 \mapsto -e_1 \wedge e_2 \\ e_2 \mapsto 0 \end{Bmatrix} \mapsto -\check{e}_1 \otimes e_1 \wedge e_2, \end{aligned}$$

whence $\qquad a_1 e_1 + a_2 e_2 \mapsto (a_1 \check{e}_2 - a_2 \check{e}_1) \otimes e_1 \wedge e_2.$

We may now proceed to the proof of the proposition.

6.2. $\mathscr{N}_{\mathsf{B}/\mathsf{X}}$. *The relative tangent map*

$$T\mathsf{B}/\mathsf{G}(2,\mathscr{F}) \rightarrowtail T\mathsf{G}(2,\mathscr{Q}_l)/\mathsf{G}(2,\mathscr{F})_{|\mathsf{B}}$$

at a point $(l, h, \lambda) \in \mathsf{P}(\mathscr{L}) \times \mathsf{G}(2,\mathscr{F}) = \mathsf{B}$ is given by

(12)
$$\begin{gathered} \operatorname{Hom}(\langle h\rangle, l/\langle h\rangle) \oplus \operatorname{Hom}(\lambda, \mathscr{F}/\lambda) \longrightarrow \operatorname{Hom}(h \cdot \lambda, \mathscr{Q}_l \big/ h \cdot \lambda) \\ (\theta_1, \theta_2) \mapsto \theta' \end{gathered}$$

where

$$\theta'(h \cdot h') := \theta_1(h)\lambda + \theta_2(h')h + h \cdot \lambda.$$

The partial derivative corresponding to the first factor may be written in the form,

$$\begin{aligned} l/\langle h\rangle &\longrightarrow \operatorname{Hom}(\lambda, \mathscr{Q}_l/h \cdot \lambda) \\ \bar{h}' &\mapsto (h'' \mapsto h'h'' + h \cdot \lambda). \end{aligned} \tag{13}$$

We compose the above map with the isomorphisms

$$\operatorname{Hom}(\lambda, \mathscr{Q}_l/(h \cdot \lambda)) \cong (\lambda)\check{}\otimes \big(\mathscr{Q}_l/(h \cdot \lambda)\big) \cong \overset{2}{\wedge}\, \check{\lambda} \otimes \lambda \otimes \big(\mathscr{Q}_l/(h \cdot \lambda)\big). \tag{14}$$

Now notice that the space $\lambda \otimes l \otimes \mathscr{F}$ maps onto $\langle h\rangle\check{}\otimes \overset{2}{\wedge}\, l \otimes \mathscr{Q}_\lambda/(h \cdot \lambda)$. Consequently, we get a natural surjection

$$\begin{array}{ccc} T\mathsf{G}(2, \mathscr{Q}_l)/\mathsf{G}(2, \mathscr{F})_{(l,h,\lambda)} & \cong & \langle h\rangle\check{}\otimes \overset{2}{\wedge}\, \check{\lambda} \otimes \lambda \otimes \big(\mathscr{Q}_l/(h \cdot \lambda)\big) \\ & & \big\downarrow \\ & & \langle h\rangle^{-2}\otimes \overset{2}{\wedge}\, l \otimes \mathscr{Q}_\lambda/(h \cdot \lambda). \end{array}$$

Indeed, first replace the factor l by $\check{l}\otimes \overset{2}{\wedge}\, l$. Then use the surjections

$$\check{l} \twoheadrightarrow \langle h\rangle\check{} \qquad \text{and} \qquad \lambda \otimes \mathscr{F} \twoheadrightarrow \mathscr{Q}_\lambda.$$

In terms of a pair of dual basis $\{x_1, x_2\} \subset l$, $\{\check{x}_1, \check{x}_2\} \subset \check{l}$ the map is given by the rule (11),

$$\begin{array}{ccc} \lambda \otimes l \otimes \mathscr{F} \cong \lambda \otimes \check{l}\otimes \overset{2}{\wedge}\, l \otimes \mathscr{F} & \twoheadrightarrow & \langle h\rangle\check{}\otimes \overset{2}{\wedge}\, l \otimes \big(\mathscr{Q}_\lambda/(h \cdot \lambda)\big) \\ \overbrace{\overset{\Cup}{a \otimes x_1 \otimes b}} & \longmapsto & \check{x}_{2|\langle h\rangle} \otimes x_1 \wedge x_2 \otimes (\overline{ab}), \\ a \otimes x_2 \otimes b & \longmapsto & -\check{x}_{1|\langle h\rangle} \otimes x_1 \wedge x_2 \otimes (\overline{ab}), \end{array} \tag{15}$$

the bar now indicating class mod $h \cdot \lambda$. To see that the map above factors through the natural surjection

$$\lambda \otimes l \otimes \mathscr{F} \twoheadrightarrow \lambda \otimes \big(\mathscr{Q}_l\big/(h \cdot \lambda)\big)$$

we must show that

$$\check{x}_{2|\langle h\rangle} \otimes (\overline{ax_2}) = -\check{x}_{1|\langle h\rangle} \otimes (\overline{ax_1})$$

holds in $\langle h\rangle^\vee\otimes\big(\mathcal{Q}_\lambda/(h\cdot\lambda)\big)$. This is equivalent to the condition

$$\check{x}_2(h)(\overline{ax_2}) = -\check{x}_1(h)(\overline{ax_1})$$

which is in turn a trivial consequence of the identity (multiplying by $a\in\lambda$)

$$h = \check{x}_1(h)x_1 + \check{x}_2(h)x_2.$$

Next we check that the image of (13) goes to zero in $\overset{2}{\wedge}\,\lambda^\vee\otimes\langle h\rangle^\vee\otimes\overset{2}{\wedge}\,l\otimes\big(\mathcal{Q}_\lambda/(h\cdot\lambda)\big)$. Pick $x\in l$ and $a_1, a_2\in\lambda$. Now $x+\langle h\rangle$ is sent to the map in $\mathrm{Hom}(\overset{2}{\wedge}\,\lambda, \lambda\otimes\big(\mathcal{Q}_l\big/(h\cdot\lambda)\big))$ defined by

$$a_1\wedge a_2\mapsto a_2\otimes(\overline{a_1x}) - a_1\otimes(\overline{a_2x}). \tag{16}$$

To show (16) goes to zero, we use the rule (15). For this, write $x = c_1x_1 + c_2x_2$. Then we see that $a_2\otimes x\otimes a_1$ goes to $(c_1\check{x}_2 - c_2\check{x}_1)_{|h}\otimes(\overline{a_1a_2})$. Therefore $a_2\otimes x\otimes a_1 - a_1\otimes x\otimes a_2$ goes to zero as desired.

The image of the partial derivative,

$$\begin{aligned}&\mathrm{Hom}(\lambda, \mathcal{F}/\lambda)\\ &\quad\longrightarrow \mathrm{Hom}(h\cdot\lambda, \mathcal{Q}_l\big/(h\cdot\lambda)) \cong \overset{2}{\wedge}\,(\lambda\otimes\langle h\rangle)^\vee\otimes\langle h\rangle\otimes\lambda\otimes\big(\mathcal{Q}_l\big/(h\cdot\lambda)\big)\end{aligned}$$

goes to zero in $\overset{2}{\wedge}\,\lambda^\vee\otimes\langle h\rangle^{-2}\otimes\overset{2}{\wedge}\,l\otimes\big(\mathcal{Q}_\lambda/(h\cdot\lambda)\big)$ as well.

Indeed, letting $\varphi\in\mathrm{Hom}(\lambda, \mathcal{F}/\lambda)$ and $a_1, a_2\in\lambda$, set $\alpha_i = \varphi(a_i)\bmod\lambda$. Then φ is mapped to the homomorphism $\langle h\rangle^2\otimes\overset{2}{\wedge}\,\lambda\longrightarrow\langle h\rangle\otimes\lambda\otimes\big(\mathcal{Q}_l\big/(h\cdot\lambda)\big)$ given by

$$h^2\otimes a_1\wedge a_2\mapsto h\otimes a_1\otimes\overline{h\alpha_2} - h\otimes a_2\otimes\overline{h\alpha_1}.$$

We check that $a_1\otimes h\otimes\alpha_2\in\lambda\otimes l\otimes\mathcal{F}$ goes to zero in $\langle h\rangle^\vee\otimes\overset{2}{\wedge}\,l\otimes\big(\mathcal{Q}_\lambda/(h\cdot\lambda)\big)$. Employing (15), we see that

$$a_1\otimes\underbrace{(\check{x}_1(h)x_1 + \check{x}_2(h)x_2)}_{\displaystyle\Vert\atop\displaystyle h}\otimes\alpha_2 \text{ goes to } (\check{x}_1(h)\check{x}_2(h) - \check{x}_2(h)\check{x}_1(h))(\overline{a_1\alpha_2}) = 0.$$

Summarizing, we have shown the formula (9). The proof of (8) is similar and will be omitted.

6.3 (End of proof of 5.1). Let us show that the the restriction of ν' to the fibers of $\mathsf{E}'\to\mathsf{B}$ are as described in 5.1. We must show that the normal directions to B, say at $(\lambda, (h, l))$ correspond to elements $\overline{q}\in\mathsf{P}(\mathcal{Q}_\lambda\big/(h\cdot\mathcal{L}_\lambda))$

and moreover, the assigned net is as prescribed there. Note that $\mathsf{P}(\mathscr{Q}_\lambda/(h\cdot\mathscr{L}_\lambda))$ is naturally isomorphic to the projectivization of the corresponding fiber in (9). The idea is to calculate suitable one parameter families of pencils. Take $(\lambda,(h,l))$ in the open orbit of B. We may assume $\lambda=\langle x_3,x_4\rangle$, $h=x_1$, $l=\langle x_1,x_2\rangle$. Write $q=x_3\alpha+x_4\beta$, with $\alpha,\beta\in\mathscr{F}$. Form the one-parameter family by considering the matrix,

$$\begin{bmatrix} x_1 & \beta & -\alpha \\ tx_2 & x_3 & x_4 \end{bmatrix}, \qquad t\in\mathsf{C}.$$

We get a one-parameter family of pencils,

$$\pi_t=\langle x_1x_3-tx_2\beta,\ x_1x_4+tx_2\alpha\rangle.$$

Note that each quadric in π_t does contain the distinguished line. One checks that for general t we have $\pi_t\notin\mathsf{B}$. Indeed, if $\pi_t\in\mathsf{B}$ for all t, we must have

$$x_1x_3-tx_2\beta=ac, \qquad x_1x_4+tx_2\alpha=bc$$

for some polynomials a, b, c in the variables x, t. We may write $c=x_1+t\bar{c}$ and similarly $a=x_3+t\bar{a}$, $b=x_4+t\bar{b}$. We get $-acx_4+bcx_3=tx_2q$. Canceling t, we obtain $c(-\bar{a}x_4+\bar{b}x_3)=x_2q$. Setting $t=0$ we have that x_1 divides q. This is contrary to the choice of q.

We have that ν maps π_t to the net $\pi_t+\langle q\rangle$ whenever $\pi_t\notin\mathsf{B}$. Letting $t\to 0$ we see that the normal direction defined by π_t is the image of $\overline{q}$ in $\mathsf{G}(3,S_2\mathscr{F})$. Since the fibers of E' and $\mathsf{P}(\mathscr{Q}_\lambda/(h\cdot\mathscr{L}_\lambda))$ are projective spaces of the same dimension, the desired equality follows over the open orbit at first, thence everywhere. This completes the proof of 5.1

COROLLARY 6.4. *There are precisely two closed orbits in* X'. *One is represented by the point*

$$\mathrm{o}_1'=(\langle x_1^2,x_1x_2\rangle,\langle x_1^2,x_1x_2,x_1x_3\rangle)\in\mathsf{G}(2,\mathscr{Q}_l)\times\mathsf{G}(3,S_2\mathscr{F}). \tag{17}$$

The other is represented by

$$\mathrm{o}_2'=(\langle x_1^2,x_1x_2\rangle,\langle x_1^2,x_1x_2,x_2^2\rangle)\in\mathsf{G}(2,\mathscr{Q}_l)\times\mathsf{G}(3,S_2\mathscr{F}).$$

PROOF. We may restrict the search to the fiber X_0' (acted on by the stabilizer of l_0, of course). We may also start by setting $\lambda=l_0=\langle x_1,x_2\rangle$, $h=x_1$. The general element $\overline{q}\in\mathsf{P}(\mathscr{Q}_\lambda/(h\cdot\mathscr{L}_\lambda))$ may be written in the form

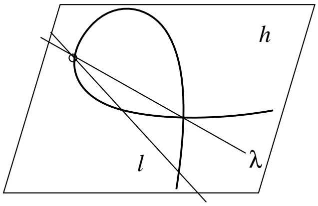
$$\overline{q}=c_1\overline{x_1x_3}+c_2\overline{x_1x_4}+c_3\overline{x_2^2}+c_4\overline{x_2x_3}+c_5\overline{x_2x_4}, \quad [c_1,\ldots,c_5]\in\mathsf{P}^4.$$

If $c_3 \neq 0$ we may act with $x_3 \mapsto tx_3$, $x_4 \mapsto tx_4$ and get o_2' in the orbit's closure. If $c_4 \neq 0$ or $c_5 \neq 0$ we change coordinates, $x_i \mapsto x_i + x_2$ ($i = 3$ or 4) and reduce to the previous case. Finaly, if $\overline{q} = c_1\overline{x_1x_3} + c_2\overline{x_1x_4}$ we may take a coordinate change (always fixing x_1 and $\langle x_1, x_2\rangle$) such that $\overline{q} = \overline{x_1x_3}$. This produces o_1'. A simple argument shows that the two orbits are closed.

REMARK 6.5. The orbit o_2' is irrelevant in the sequel. Indeed, the net of quadrics appearing in the second component of o_2' represents a degenerate twc. The multiplication map studied below is of maximal rank at o_2' and the rational map from X' to Hilb is regular there. For this reason, the succeeding blowup centers will be away from this orbit.

7. The multiplication map

It turns out that the projectivized normal bundle of Y_2' (see (7)) in X' at a general point parametrizes the linear system of plane cubics passing through the point of intersection of the two lines and singular at the distinguished point. Look at the picture below.



This will be shown by considering the natural multiplication map,

$$\mathscr{A} \otimes \mathscr{F} \xrightarrow{\mu} S_3\mathscr{F}, \tag{18}$$

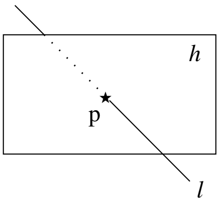
where $\mathscr{A}$ denotes the pullback of the rank three tautological bundle of $\mathsf{G}(3, S_2\mathscr{F})$ via ν'.

The generic rank of μ is ten. It drops rank to nine along a subscheme Y' containing Y_2' as one of its two components (see (23)). Blowing up Y_2' in X' brings us closer to the desired flat family of twcs. One further blowup is still necessary, essentially due to the locus where $\lambda = l$ holds (whence the point of intersection is no longer determined).

Actually, in order to ensure that the abovementioned point in the intersection $\lambda \cap l$ becomes everywhere well defined, a different blowup strategy will be pursued below.

The other component of Y' is a subvariety $\mathsf{Y}_1' \subset \mathsf{X}'$. Its fiber $\mathsf{Y}_1'{}_l$ over a distinguished line $l \in \mathsf{G}(2, \mathscr{F})$ is isomorphic to the incidence variety

$$\mathsf{Y}_1'{}_l \cong \left\{(\mathrm{p}, h) \in \mathsf{P}^3 \times \check{\mathsf{P}}^3 \mid \mathrm{p} \in h \cap l \subset \mathsf{P}^3\right\}.$$



Clearly, Y'_1 is of codimension seven in X'. It will be shown below that $\mathsf{Y}'_{1\,l}$ is isomorphic to $\check{\mathsf{P}}^3$ blownup along the pencil $\mathsf{P}(\mathscr{L}_l)$ of planes containing the distinguished line. Thus, Y'_1 is the strict transform of Y_1 (see (4)) in X'.

8. Local calculations

The construction of the flat family of twcs involves the calculation of Fitting ideals of suitable modifications of the sheaf coker (μ) (cf. 18). For this we need to compute a local matrix representation for the multiplication map.

We pick appropriate coordinate charts for the fibers X_0 of X (as in the proof of 5.1) and X'_0 of X' over the line $l_0 \in \mathsf{G}(2, \mathscr{F})$ given by $x_1 = x_2 = 0$.

8.1. Local chart for X'_0

Recalling (6), it follows that the blowup of $\dot{\mathsf{X}}_0$ along $\dot{\mathsf{B}}_0 = \mathsf{B} \cap \dot{\mathsf{X}}_0$ is covered by five affine pieces, one for each generator of the ideal of the blowup center $\dot{\mathsf{B}}_0$. Since flatness is an open condition, it suffices to restrict to an affine chart $\dot{\mathsf{X}}'_0$ containing the point o'_1 which represents a closed orbit (cf. 17). Take the local equation of the exceptional ideal to be given by

$$\epsilon' = b_4 - b_3 b_1. \tag{19}$$

This choice is guided by the blueprint (3.1). Observe that in the matrix representation for ν (after suitable row and column operations), the entry ϵ' appears in the column corresponding to the monomial x_1x_3. Dividing the third row of that matrix by ϵ', we see that it will correspond to a quadric of the form x_1x_3+ terms vanishing at the origin of the coordinate neighborhood, cf. (21) below. The coordinate functions may be chosen as

$$a_1,\ a_2,\ b_1,\ b_2,\ b_3,\ b_4,\ c_2,\ c_3,\ c_4,\ c_5.$$

where the c_i are the ratios to ϵ' of the remaining four generators of the ideal of $\dot{\mathsf{B}}_0$. Precisely, the map $\dot{\mathsf{X}}'_0 \to \dot{\mathsf{X}}_0$ is given by the inclusion of affine coordinate rings

$$\mathsf{C}[\dot{\mathsf{X}}_0] = \mathsf{C}[a_1, \ldots, b_5] \hookrightarrow \mathsf{C}[\dot{\mathsf{X}}'_0] = \mathsf{C}[a_1, a_2, b_1, \ldots, c_5]$$

defined by

$$
\begin{aligned}
&a_3 = -b_3^2 - c_3\epsilon', \qquad && a_4 = a_1b_3 - c_4\epsilon', \\
&a_5 = a_2b_3 - c_5\epsilon', \qquad && b_5 = b_3b_2 + c_2\epsilon'.
\end{aligned} \tag{20}
$$

Presently a local basis for $\mathscr{A}$ is formed by the three quadrics

$$
\begin{cases}
q_1 = x_1^2 + a_1x_1x_3 + a_2x_1x_4 - (b_3^2 + c_3\epsilon_1)x_2^2 + (a_1b_3 - c_4\epsilon_1)x_2x_3 \\
\qquad\quad + (a_2b_3 - c_5\epsilon_1)x_2x_4, \\
q_2 = x_1x_2 + b_1x_1x_3 + b_2x_1x_4 + b_3x_2^2 + (b_1b_3 + \epsilon_1)x_2x_3 + (b_3b_2 \\
\qquad\quad + c_2\epsilon_1)x_2x_4, \\
q_3 = x_1x_3 + c_2x_1x_4 + c_3x_2^2 + (c_4 - b_3 + c_3b_1)x_2x_3 \\
\qquad\quad + (c_5 - c_2b_3 + c_3b_2)x_2x_4 + (b_1c_4 + a_1)x_3^2 \\
\qquad\quad + (b_1c_5 + a_1c_2 + b_2c_4 + a_2)x_4x_3 + (a_2c_2 + b_2c_5)x_4^2
\end{cases} \tag{21}
$$

where q_1, q_2 yield a local basis for the rank two tautological subbundle $\mathscr{R} \rightarrowtail \mathscr{Q}$ (cf. 2) pulled back to X_0'. A local representation of the multiplication map μ may now be computed as a 12×20 matrix in the form

$$
\begin{pmatrix} I_9 & \star \\ 0 & R \\ 0 & 0 \end{pmatrix} \tag{22}
$$

where I_9 denotes an identity block of size 9 and R is a row matrix that spans the Fitting ideal $\mathscr{J}$ of 10×10 minors of μ. We find that $\mathscr{J}$ is equal to the sum of the two ideals

$$
\mathscr{J}_0 = \langle a_1 + 2b_3b_1 - \epsilon',\ a_2 + 2b_3b_2 - c_2\epsilon', c_3, c_4 - 2b_3, c_5 - 2c_2b_3 \rangle
$$

and $\epsilon' \cdot \langle b_1, b_2 \rangle$, with $\epsilon' = b_4 - b_3b_1$ as in (19). Put

$$
\begin{aligned}
\mathscr{J}_1 &= \mathscr{J}_0 + \langle b_1, b_2 \rangle, \\
\mathscr{J}_2 &= \mathscr{J}_0 + \langle \epsilon' \rangle.
\end{aligned} \tag{23}
$$

Hence, $\mathscr{J} = \mathscr{J}_1 \cap \mathscr{J}_2$ holds. So the locus where μ drops rank is the union of the two smooth pieces $\mathsf{Y}_1', \mathsf{Y}_2'$ given locally by the respective ideals $\mathscr{J}_1,\ \mathscr{J}_2$.

8.1.1. Description of Y_1'

Here are, freshly picked from the generators of $\mathscr{J}_1$, the seven relations that define Y_1' locally,

$$a_1 = b_4,\ b_1 = 0,\ b_2 = 0,\ a_2 = b_4c_2,\ c_3 = 0,\ c_4 = 2b_3,\ c_5 = 2c_2b_3. \tag{24}$$

Substituting in (21), it can be seen that the net of quadrics acquires a fixed component, namely the plane given by

$$x_1 + b_3x_2 + b_4x_3 + b_4c_2x_4. \tag{25}$$

Note that, in general, this plane does not contain the distinguished line. The moving part of the net, namely the net of planes

$$\langle x_1 - b_3x_2, x_2, x_3 + c_2x_4\rangle,$$

defines the point

$$\mathrm{p} = [0 : 0 : -c_2 : 1] \in \mathsf{P}^3.$$

It is the point of intersection of the fixed plane with the distinguished line. Moreover, looking at the coefficients of the equation (25) that defines the plane, we recognize $\mathsf{Y}_{1,l}'$ as the dual space $\check{\mathsf{P}}^3$ blown up along the pencil of planes through the distinguished line $l_0 := x_1 = x_2 = 0$. Equivalently, $\mathsf{Y}_{1,l}'$ is the closure of the graph of the rational map $\check{\mathsf{P}}^3 \cdots \to l_0 \cong \mathsf{P}^1$ produced by intersecting a moving plane with the distinguished line. The intersection of $\mathsf{Y}_{1,l}'$ with the exceptional divisor E_l' is equal to the exceptional divisor of the blowup $\mathsf{Y}_{1,l}' \to \check{\mathsf{P}}^3$. It corresponds to all choices of a plane through the distinguished line, together with a marked point *on* the line.

We summarize the above discussion as follows.

LEMMA 8.1. *Let* $\mathsf{Y}_1' \subset \mathsf{P}(\overline{\mathscr{F}}) \times \check{\mathsf{P}}^3$ *consist of all* (p, l, h) *such that* p *is a point in the intersection of the line l with the plane h. Then* Y_1' *maps isomorphically onto the strict transform of* Y_1 *in* $\mathsf{X}' \subset \mathsf{X} \times \mathsf{G}(3, S_2\mathscr{F})$.

8.1.2. Description of Y_2'

This is just the flag variety (5.3) introduced earlier. Indeed, solving the equations defined by the generators of $\mathscr{J}_2$, we find

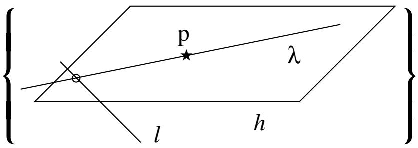
$$a_1 = -2b_1b_3,\ a_2 = -2b_2b_3,\ b_4 = b_1b_3,\ c_3 = 0,\ c_4 = 2b_3,\ c_5 = 2c_2b_3.$$

Plugging these relations in (21) we get the net of quadrics with fixed plane $x_1 + b_3x_2$ and moving part $\langle x_1 - b_3(b_2x_4 + b_1x_3), x_2 + b_1x_3 + b_2x_4, x_3 + c_2x_4\rangle$ fitting the prescription 5.3. We note that Y_2' is contained in the exceptional divisor E', as the local equation (19) of the latter is contained in the ideal of the former. The line λ is given by the first two linear forms. We see that λ coincides with the distinguished line $\langle x_1, x_2\rangle$ if and only if $b_1 = b_2 = 0$ holds. These are the local equations for $\mathsf{Y}'_1 \cap \mathsf{Y}'_2$ in Y'_2. We may summarize this as follows.

LEMMA 8.2. *The intersection of Y'_2 and Y'_1, viewed inside Y'_2, is equal to the codimension two subvariety of Y'_2 where the two lines l, λ coincide. Moreover, the strict transform Y''_2 in the blowup X'' of X' along Y'_1 is isomorphic to the closure of the graph of the rational map defined by $(l, \lambda) \mapsto l \cap \lambda$. In other words,*

(26)

$$\mathsf{Y}''_2 = \{(\mathrm{p}, \mathrm{o}, l, \lambda, h) \in (\mathsf{P}^3)^{\times 2} \times (\mathsf{G}(2,4))^{\times 2} \times \check{\mathsf{P}}^3 \mid \mathrm{p} \in \lambda \subset h \supset l,\ \mathrm{o} \in l \cap \lambda\}.$$



REMARK 8.3. As a subset of Y'_1, the intersection with Y'_2 is just the codimension one subvariety where the fixed plane swallows the distinguished line – all in keeping a marked point as a reminder of an intersection point.

Though a result of Hironaka ([5], p. 41) ensures a “commutativity” of blowups, we have chosen to blowup X' along Y'_1 first and then blowup along the strict transform Y''_2 of Y'_2 due to a nice geometrical reason.

Indeed, Y''_2 carries a locally split subbundle of $S_3\overline{\mathscr{F}}$ of cubic forms in the varying plane h that vanish at the point o and are singular at the point p.

In fact, Y''_2 embeds in a punctual relative Hilbert scheme parametrizing zero dimensional subschemes of degree four of the family of planes in P^3 through a distinguished line. A point $(p, \mathrm{o}, l, \lambda, h)$ in Y''_2 produces the subscheme $\mathrm{p}^2 + \mathrm{o}$ of the plane h. We mean by this the zero dimensional subscheme of degree four defined by squaring the ideal of p in h and intersecting with the ideal of the point o. When $\mathrm{p} = \mathrm{o}$ holds, we still get a well defined limiting subscheme isomorphic to $\langle y^2, xy, x^3\rangle$. Here we have taken h as the x, y-plane, $l = \lambda$ as the axis $y = 0$ and the point $\mathrm{o} = \mathrm{p} = (0, 0)$.

More precisely, let s be an affine coordinate in the line l. The equation of λ may be written as $y = t(x - s)$. The choice of p on λ will be provided by intersecting λ with $x = u$. Thus, s, t, u are local coordinates for $(\mathsf{Y}''_2)_{l,h}$. After homogenizing, we find (e.g., using MAPLE) the equality for $s \neq u$,

$$\langle x-uz,\ y-t(x-sz)\rangle^2 \cap \langle x-sz,\ y\rangle = [-utsz^2+t(u+s)xz-uyz+xy-tx^2,$$

$$st^2(s-2u)z^2 + 2ut^2xz + 2t(s-u)yz - t^2x^2 + y^2,\ u^2yz^2 - 2uxyz + x^2y,$$

$$su^2z^3 - u(u+2s)xz^2 + (2u+s)x^2z - x^3].$$

It can be checked that *for all* s, t, u the homogenous ideal in the right hand side imposes 4 independent conditions on quartics (as well as already on cubics). Since $\mathrm{Hilb}^4_{\mathsf{P}^2}$ embeds in the grassmannian of codimension 4 subspaces of $S_4\langle x, y, z\rangle$, it follows that $(\mathsf{Y}''_2)_{l,h}$ maps into that Hilb. In fact, one checks that the map to that grassmannian is an embedding.

Assume this picture for the moment and let's see how do the exceptional divisor $\mathsf{E}'' \to \mathsf{Y}'_1$ and the strict transform Y''_2 fit together. Now $\mathsf{Y}''_2 \cap \mathsf{E}''$ is the locus where $\lambda = l$ holds. It sits over $\mathsf{Y}'_2 \cap \mathsf{E}' = \mathsf{Y}'_2 \cap \mathsf{Y}'_1$ as the P^1-bundle defined by varying the point denoted by o on the line λ.

The fiber of the projectivized normal bundle E''' of $\mathsf{Y}''_2 \subset \mathsf{X}''$ at a point given by $(\mathrm{p}, \mathrm{o}, l, \lambda, h)$ is the linear system of cubics in the plane h containing the subscheme $\mathrm{p}^2 + \mathrm{o}$ described above. This corresponds to the picture at the beginning of §7. Details are given in the next sections.

8.2. Blowing up $\mathsf{Y}'_1 \subset \mathsf{X}'$

Let $\mathsf{X}'' \to \mathsf{X}'$ be the blowup of $\mathsf{Y}'_1 \subset \mathsf{X}'$. There are two interesting charts for this blowup, namely, those given by choosing as local equation for the exceptional divisor either b_1 or b_2 among the seven generators given in (24). We now let $\epsilon'' = b_1$ be the chosen one. (The calculations for b_2 are similar and will be omitted.) The blowup map is given by the relations,

$$\text{(27)} \quad \begin{aligned} a_1 &= (d_5 - 3b_3)b_1 + b_4, & b_2 &= d_7b_1,\ c_4 = 2b_3 + d_3b_1, \\ a_2 &= (d_6 - b_3(2d_7 + c_2))b_1 + c_2b_4, & c_3 &= d_2b_1,\ c_5 = 2c_2b_3 + d_4b_1 \end{aligned}$$

where $b_1, b_4, b_3, c_2, d_2, \ldots, d_7$ are local coordinates.

Plug the above relations into the local equations for Y'_2 (cf. 8.1.2). We find,

$$b_4 - b_1b_3 + d_5b_1,\ \ c_2(b_4 - b_3b_1) + d_6b_1,\ \ b_4 - b_3b_1,\ \ d_2b_1,\ \ d_3b_1,\ \ d_4b_1$$

for the ideal of the total transform. The strict transform is given by the saturation with respect to b_1, the local equation of the exceptional divisor. Hence Y''_2 is locally given by the ideal,

$$\text{(28)} \qquad \langle b_4 - b_3b_1, d_2, d_3, d_4, d_5, d_6\rangle.$$

We compute next the saturation of the image sheaf of the multiplication map μ pulled back to X''. Let $\mathscr{M}'$ be the submodule spanned by rows of the matrix M' obtained from (22) by pullback via (27). We are required to find the row matrices R' with entries in the coordinate ring of the present chart, such that b_1R' lies in $\mathscr{M}'$.

Now the tenth row of M' is of the form $b_1 R_0$ (i.e., all entries in that row are divisible by b_1). Hence ${}^s\mathscr{M}'$ contains R_0. The first nonzero entry of R_0 is d_2; it appears in the 11th. column. Hence ${}^s\mathscr{M}' = \mathscr{M}' + \langle R_0 \rangle$ holds since they agree upon inverting d_2. This also shows that we get a local presentation of ${}^s\mathscr{M}'$ given by replacing the row $b_1 R_0$ in M' by R_0. The Fitting ideal of ${}^s\mathscr{M}'$ defined by the 10×10 minors of the presentation is spanned by the entries of R_0. This ideal is precisely (28). It defines the last blowup center Y_2''.

Solving (28) for $b_4, d_2, \ldots, d_6$ and substituting in (21) pulled back to the present coordinate chart, we find that the net of quadrics parametrized by Y_2'' is of the form, $(x_1 + b_3 x_2)$ times the net of planes $\langle x_1 - b_1 b_3(x_3 + d_7 x_4), x_2 + b_1(x_3 + d_7 x_4), x_3 + x_4 c_2 \langle$. As in (8.1.2), the line λ is given by the first two generators of the latter net. We see that λ intersects the distinguished line $l_0 = \langle x_1, x_2 \rangle$ at the point $\mathrm{o} = [0 : 0 : -d_7 : 1]$. This point is well defined even when the lines coincide. Clearly $\lambda = l_0$ if and only if $b_1 = 0$. This is of course in agreement with (26): Y_2'' is the closure of the graph of the rational map $\mathsf{Y}_2' \cdots\rightarrow \mathsf{P}^3$ defined by $\lambda \cap l_0$. The varying linear part of the net defines the point $\mathrm{p} = [-b_3 b_1(c_2 - d_7) : b_1(c_2 - d_7) : -c_2 : 1]$.

8.3. Blowing up $\mathsf{Y}_2'' \subset \mathsf{X}''$

Let $\mathsf{X}''' \to \mathsf{X}''$ be the blowup of $\mathsf{Y}_2'' \subset \mathsf{X}''$. The discussion above shows that the saturation of the pullback of the image of μ is a rank 10, locally split subsheaf of $S_3\mathscr{F}$. Hence X''' maps to $\mathsf{G}(10, S_3\mathscr{F})$ and dominates the Hilb of twcs embedded in that grassmannian.

In the neighborhood where

$$\epsilon''' = d_3$$

is a local generator for the (last) exceptional divisor $\mathsf{E}''' \to \mathsf{Y}_2''$, the blowup is given by the relations

$$b_4 = b_3 b_1 + e_2 \epsilon''', \;\; d_2 = e_3 \epsilon''', \;\; d_4 = e_4 \epsilon''', \;\; d_5 = e_5 \epsilon''', \;\; d_6 = e_6 \epsilon'''.$$

For the sake of completeness we also list the strict transforms of the previous two exceptional divisors. They are given by $\widetilde{\epsilon'} = e_2$ and $\widetilde{\epsilon''} = b_1$. This shows that our compactification X''' is built from a suitable open subset of the Grassmann bundle X by adding three normal crossing divisors.

We also observe that the tenth cubic, i.e., the one corresponding to the last nonzero row of the local matrix presentation, is given by,

$$\begin{gathered} e_3 x_2^3 + (1 + 2e_3 b_1) x_3 x_2^2 + (2e_3 d_7 b_1 + e_4) x_4 x_2^2 + (e_5 + 2b_1 + e_3 b_1^2) x_3^2 x_2 \\ + (b_1^2 + e_2 + e_5 b_1) x_3^3 + ((2d_7 + 2e_4) b_1 + e_5 c_2 + e_6 + 2e_3 d_7 b_1^2) x_4 x_2 x_3 \\ + (d_7^2 e_3 b_1^2 + 2d_7 e_4 b_1 + e_6 c_2) x_4^2 x_2 \end{gathered}$$

$$+((2d_7+e_4)b_1^2+(e_6+e_5c_2+d_7e_5)b_1+d_7e_2+2c_2e_2)x_4x_3^2$$
$$+(d_7^2e_4b_1^2+d_7e_6c_2b_1+d_7c_2^2e_2)x_4^3$$
$$+((d_7c_2e_5+e_6c_2+d_7e_6)b_1+2d_7c_2e_2+c_2^2e_2+(2d_7e_4+d_7^2)b_1^2)x_4^2x_3.$$

Restricting to E''', we find that it cuts the fixed plane $x_1=-b_3x_2$ in a cubic passing through $\mathrm{o}=[0:0:-d_7:1]$ and singular at $\mathrm{p}=[-b_3b_1(c_2-d_7):b_1(c_2-d_7):-c_2:1]$.

9. Enumerative applications

We explain how to find the number of twcs meeting 12 general lines. First, the set of pencils of quadrics containing a distinguished line such that the residual twisted cubic meet the line $x_3=x_4=0$ form a divisor $\mathsf{L}\subset\mathsf{X}$. An easy elimination shows that in the coordinate chart for X_0, its local equation is $a_3=-b_3^2$. Its total transform in X_0' is given by c_3 times the local equation $\epsilon'=b_4-b_3b_1$ of E'. Hence the strict transform is given by $c_3=0$. We see from (27) that the total transform in X'' is given by d_2b_1 whence the strict transform is d_2. Finally, a local equation of the strict transform in X''' is found to be given by e_3. It follows that the class of the strict transform $\mathsf{L}'''\subset\mathsf{X}'''$ can be written (omitting pullbacks) as

$$[\mathsf{L}''']=[\mathsf{L}]-[\mathsf{E}']-[\mathsf{E}'']-[\mathsf{E}'''].$$

In order to cut to size the ∞^2 bisecant lines, we restrict X''' over the Schubert cycle $\mathsf{G}_{\mathbf{o}}\subset\mathsf{G}(2,4)$ of lines meeting a fixed point $\mathbf{o}$. The class $[\mathsf{L}]$ in X is easily computed as $-2c_1\mathscr{R}+c_1\mathscr{L}$. The class $[\mathsf{G}_{\mathbf{o}}]$ in $\mathsf{G}(2,4)$ is $c_2\mathscr{L}$. The number sought for is therefore given by

$$\int[\mathsf{L}''']^{12}c_2\mathscr{L},$$

and may be calculated explicitly using Bott's formula as in [8]. The script adaptaded by A. Meireles is available at the homepage [11]. It also does the job of producing the higher dimensional examples cited at the introduction.

Finally, let us sketch a verification of the enumerative significance of the above intersection theoretic computation. Let $\mathsf{H}_{\mathbf{o}}$ denote the restriction of X''' over $\mathsf{G}_{\mathbf{o}}$. Let $p:\mathsf{H}_{\mathbf{o}}\to\mathsf{H}$ be the map to the Hilbert scheme component of twcs. Let $\mathrm{GL}_{\mathbf{o}}\subset\mathrm{GL}(\mathscr{F})$ denote the stabilizer of the point $\mathbf{o}$. Clearly p is $\mathrm{GL}_{\mathbf{o}}$-invariant. Let $\Sigma\subset\mathsf{H}_{\mathbf{o}}$ be any irreducible divisor shrunk by p, i.e., such that codim $p(\Sigma)\geq 2$. We have that Σ is $\mathrm{GL}_{\mathbf{o}}$-invariant and so is its image $p(\Sigma)$. Let $\mathsf{I}\subset\mathsf{H}$ be the divisor of twcs incident to a fixed line away from $\mathbf{o}$. One checks that I contains none of the two closed $\mathrm{GL}_{\mathbf{o}}$-orbits of H. Hence I

contains no $p(\Sigma)$. Therefore $p^{-1}\mathsf{I}$ is irreducible. Hence it is equal to the closure L''' of a suitable open subset of L (restricted over $\mathsf{G_o}$). Therefore, we may write

$$\int [\mathsf{L}''']^{12} c_2 \mathscr{L} = \int [p^{-1}\mathsf{I}]^{12} = \int [\mathsf{I}]^{12}.$$

The latter is well known to be an enumeratively significant Gromov-Witten invariant (cf. [4]).

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