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CORRIGENDUM TO ON THE SIMPLICITY OF SOME CUNTZ-PIMSNER ALGEBRAS

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We are grateful to Dr. Jürgen Schweizer for pointing out to us that Theorem 1 in [1] is incorrect. It rests upon a faulty application in Lemma 8 of hypothesis (H5). Hypothesis (H5) asserts that $\Omega(1_A) \neq 1_A$ and we concluded at the end of the proof of Lemma 8 that $\tau_0(\Omega(1_A)) \neq 1$ for a particular faithful tracial state τ_0. This inequality is a critical point of the proof. However, the following example, due to Dr. Schweizer, shows that this conclusion is faulty. Let E be the identity correspondence over $A = M_2(\mathsf{C})$, and let

$$\eta_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \eta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The map

$$a \to \begin{pmatrix} \langle a, \eta_1 \rangle_A \\ \langle a, \eta_2 \rangle_A \end{pmatrix}$$

embeds $E = A = M_2(\mathsf{C})$ into $C^2(M_2(\mathsf{C}))$ as a direct summand, and

$$\Omega(a) = \langle \eta_1, a\eta_1 \rangle + \langle \eta_2, a\eta_2 \rangle = \begin{pmatrix} 0 & 0 \\ 0 & a_{11} + a_{22} \end{pmatrix}, \qquad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Therefore,

$$\Omega(1_A) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

On the other hand, $M_2(\mathsf{C})$ has only one tracial state, τ_0, the normalized trace, and $\tau_0(\Omega(1_A)) = 1$. Note, too, that all the other hypotheses of Theorem 1 in [1] are satisfied, but that $\mathcal{O}(E) \simeq M_2(\mathsf{C}) \rtimes_{\mathrm{id}} \mathsf{Z} \simeq M_2(\mathsf{C}) \otimes C(S^1)$ is not simple.

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We offer the following stronger hypothesis that bridges the gap in the proof of [1, Theorem 1] *and* which is still satisfied in all of the examples where Theorem 1 is applied.

HYPOTHESIS (H5*). Among all tracial states τ on A that are scaled by Ω, i.e., that satisfy the equation $\tau \circ \Omega = c\tau$, for some $c \in \mathsf{R}$, there is at least one with $c \neq 0, 1$.

Of course, if Ω scales τ, then $c = \tau(\Omega(1_A))$. Also, evidently, hypothesis (H5*) implies hypothesis (H5). Note that Lemma 6 of [1], which uses only hypotheses (H1)–(H4), shows that the set of faithful traces that are scaled by Ω is non-empty. Further, under hypothesis (H4), any trace that is scaled by Ω must be faithful. Thus, under hypothesis (H4), one can never have $c = \tau(\Omega(1_A)) = 0$.

It is hypothesis (H5*) that is used on page 63 at the end of the proof Lemma 8 and this is the only place hypothesis (H5*) is used anywhere in the proof of Theorem 1. Thus, the details that we present in [1] prove

THEOREM 0.1. *If the C^*-algebra A and the correspondence E satisfy hypotheses $(H1) - (H4)$ of [1] and hypothesis (H5*), then $\mathcal{O}(E)$ is simple.*

Observe that if Ω satisfies the condition that $\Omega(1_A)$ is comparable with 1_A, but not equal to 1_A, then necessarily hypothesis (H5*) is satisfied under hypotheses (H1)–(H4). The reason, as we just noted, is that Lemma 6 implies that every trace that is scaled by Ω necessarily is faithful, under hypotheses (H1)–(H4), and every faithful trace satisfies the equation $\tau(\Omega(1_A)) \neq 1$ if $\Omega(1_A)$ is comparable with, but different from 1_A. In Corollary 2, the map Ω satisfies the condition $\Omega(1_A) \lneqq 1_A$, while in Corollary 14, $\Omega(1_A) = n1_A$, $n \geq 2$. Thus, these two corollaries remain valid with no change in hypotheses. It remains to prove Corollary 3.

There is no problem with the verifications in [1] that hypotheses (H1)–(H4) are satisfied. We show that there is a faithful trace τ_0 on A that is scaled by Ω with $\tau_0(\Omega(1_A)) \neq 1$. To this end, let M be the set of all tracial states on B that are Φ-invariant. One of the hypotheses of Corollary 3 is that M is non-empty. We define the completely positive map β on B by the formula $\beta(b) = \sum \Phi(u_i^* b u_i)$ where $\{u_i\}$ is the quasi-basis, $u_i = W^{-1}P\varepsilon_i$ used in [1]. (Recall that for $x \in B$, $Wx = (\Phi(u_1^*x), \Phi(u_2^*x), \ldots, \Phi(u_n^*x))^t$.) Observe that for $b \in A$, $\beta(b) = \sum\langle u_i, bu_i\rangle = \sum\langle P\varepsilon_i, W\varphi(b)W^{-1}P\varepsilon_i\rangle = \Omega(b)$. For $\rho \in M$, write

$$\rho'(b) = (\rho(\beta(1_B)))^{-1}\rho(\beta(b)),$$

$b \in B$. (Observe that if we apply the argument in the first paragraph of the proof of [1, Lemma 6] to the restriction $\rho|A$ and note that $1_B = 1_A$, we find that $\rho(\beta(1_B)) = \rho(\beta(1_A)) = \rho(\Omega(1_A)) \neq 0$. Thus ρ' is well defined.) Then ρ' is a Φ-invariant *state* on B. To show that ρ' is a trace, let $b, c \in B$. Then because ρ is a Φ-invariant trace, we have $\rho(\beta(bc)) = \sum \rho(\Phi(u_i^* bcu_i)) = \rho(\sum u_i^* bcu_i) = \rho(\sum u_i u_i^* bc) = \rho((\operatorname{ind}\Phi)bc)$, where $\operatorname{ind}\Phi = \sum u_i u_i^*$. Likewise, we see that $\rho(\beta(cb)) = \rho((\operatorname{ind}\Phi)cb)$. Since ρ is a trace and $(\operatorname{ind}\Phi)$ lies in the center of B, we see that $\rho \circ \beta$ is tracial. Therefore ρ' is a trace; i.e., $\rho' \in M$. By the Schauder fixed point theorem, there is a $\rho_0 \in M$ such that $\rho_0 = \rho_0'$, i.e., $\rho_0(b)\rho_0(\beta(1_B)) = \rho_0(\beta(b))$. If we set $\tau_0 = \rho_0|A$, then since $\Omega = \beta|A$, we see that Ω scales τ_0. To verify that hypothesis (H5*) is satisfied, we need to show $\tau_0(\Omega(1_A)) \neq 0, 1$. To show $\tau_0(\Omega(1_A)) \neq 0$, it suffices to show that ρ_0 is faithful. Observe that since ρ_0 is Φ-invariant, we may write $\rho_0 = \tau_0 \circ \Phi$. Set $N_{\rho_0} = \{b \in B \mid \rho_0(b^*b) = 0\}$. Then N_{ρ_0} is a two-sided ideal in B which Φ maps into the two-sided ideal $N_{\tau_0} := \{a \in A \mid \tau_0(a^*a) = 0\}$ in A. (Indeed, for $b \in N_{\rho_0}$, $\tau_0(\Phi(b)^*\Phi(b)) \leq \tau_0(\Phi(b^*b)) = \rho_0(b^*b) = 0$, by the Cauchy-Schwarz-Kadison inequality.) However, N_{τ_0} is Ω-invariant, since Ω scales τ_0 (see the proof of [1, Lemma 6]). By hypothesis (H4), $N_{\tau_0} = \{0\}$, and since Φ is faithful by [4, Proposition 2.1.5], we see that $N_{\rho_0} = \{0\}$, too. Thus ρ_0 is faithful. To see that $\tau_0(\Omega(1_A)) \neq 1$, observe that $\tau_0(\Omega(1_A)) = \sum \tau_0(\Phi(u_i^* u_i)) = \sum \rho_0(u_i^* u_i) = \sum \rho_0(u_i u_i^*) = \rho_0(\operatorname{ind}\Phi)$. By [4, Lemma 2.3.1], $\operatorname{ind}\Phi$ is an element of the center of B that dominates 1. Further, since $A \neq B$, by hypothesis, $\operatorname{ind}\Phi \neq 1$ by [4, Proposition 2.3.7]. Therefore, $\tau_0(\Omega(1_A)) = \rho_0(\operatorname{ind}\Phi) \gneqq 1$.

Finally, we note that Theorem 5 of [1], and the results related to it, are unaffected by the change from hypothesis (H5) to hypothesis (H5*).

For what appears to be the final word on the simplicity issue for $\mathcal{O}(E)$, please consult [2], [3].

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