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RANDOM EUCLIDEAN SECTIONS OF SOME CLASSICAL BANACH SPACES

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Abstract

Using probabilistic arguments, we give precise estimates of the Banach-Mazur distance of subspaces of the classical ℓ_q^n spaces and of Schatten classes of operators S_q^n for $q \geq 2$ to the Euclidean space. We also estimate volume ratios of random subspaces of a normed space with respect to subspaces of quotients of ℓ_q. Finally, the preceeding methods are applied to give estimates of Gelfand numbers of some linear operators.

1. Introduction

In this work we present a new method which may be employed in a variety of problems in convex analysis, such as:

(I) Giving tight asymptotic estimates on the existence of spherical sections of dimension k, for all $1 \leq k \leq n$, in n-dimensional convex bodies. We study this problem for the classical ℓ_q^n spaces for $2 \leq q \leq \infty$ and the mn-dimensional vector space $\mathscr{M}_{m\times n}(\mathsf{R})$ of all $m \times n$-matrices ($m \geq n$) with real entries equipped with a rotation invariant norm (associated with a 1-symmetric norm on the singular values of $\sqrt{M^*M}$).

(II) Investigating the volume ratio of a centrally symmetric convex body K in R^n with respect to the body of largest volume contained in K which is obtained by applying a linear map on the unit ball of a subspace of a quotient of ℓ_q.

(III) Computing upper bounds by random methods for the k-th Gelfand number of an operator T between two Banach spaces. These results improve previous estimates because they are in some cases tight and they also hold with positive probability.

The first topic (I) is related to the recent studies of Milman and Schechtman ([18] [19]) and [10], [14]. We investigate here the "large" Euclidean sections of centrally symmetric convex bodies in R^n, or equivalently, the Banach-Mazur distances of subspaces with "big dimensions" k of an n-dimensional normed

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space to the Euclidean spaces ℓ_2^k . We give first a general result about subspaces of a normed space which possesses a system of vectors satisfying a (C, s)-estimate (see the definition below), and apply these results to give sharp estimates of the distance to ℓ_2^k of k-dimensional subspace of ℓ_q^n, for $q \geq 2$. We treat then the same problem for subspaces of some normed spaces of operators from R^m to R^n, and in particular the Schatten classes, S_q^n for $q \geq 2$.

In part (II), we present a method to obtain a lower bound for the volume ratio of a random k-dimensional subspace F of a given space X with respect to the class $SQ(\ell_q)$, $q \geq 2$, consisting of all k-dimensional subspaces of quotients of ℓ_q. This gives an estimate from below of the distance of a random subspace F of X to the class $SQ(\ell_q)$, and in particular to Hilbert space.

In the final section we apply the previous methods to obtain upper bounds for the Gelfand numbers of operators from ℓ_q^n and S_q^n $(q \geq 2)$ to a space Y.

Denote by $e_1, \ldots, e_n$ the canonical basis of R^n. The spaces ℓ_q^n for $1 \leq q \leq \infty$ are defined as R^n equipped with the norm $|\cdot|_q$: for $x = x_1e_1 + \cdots + x_ne_n$,

$$|x|_q = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q} \quad \text{when} \quad q \neq \infty \quad \text{and} \quad |x|_\infty = \max_{1\leq i\leq n} |x_i|.$$

Let E be an n-dimensional normed space. We say that a family $u_1, \ldots, u_N$ of vectors of E, with $N \leq n$, satisfies a (C, s)-*estimate* for $C > 0$ and $s > 0$, if for all $(t_i)_{i=1}^N \in \mathsf{R}^N$ and all $m = 1, \ldots, N$, one has

$$\text{(1)} \qquad \frac{C}{m^{1/s}}\left(\sum_{i=1}^m (t_i^\star)^2\right)^{1/2} \leq \left\|\sum_{i=1}^N t_iu_i\right\| \leq \left(\sum_{i=1}^N t_i^2\right)^{1/2},$$

where $(t_i^\star)_{i=1}^N$ denotes the decreasing rearrangement of the sequence $(|t_i|)_{i=1}^N$. By a result of Bourgain and Szarek [2], there exists a constant $C > 0$ such that for any n, any n-dimensional normed space contains a sequence $u_1, \ldots, u_N$, with $N \geq \frac{n}{2}$, satisfying a $(C, 2)$-estimate. We shall be interested here with $s \geq 2$. It is easy to see that for $q \geq 2$, the canonical basis of ℓ_q^n satisfies a $(1, s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It may be also observed that if $s' > 0$ satisfies $\frac{1}{s'} = \frac{1}{s} - \frac{1}{\ln(n)}$, and if $(u_1, \ldots, u_N)$ satisfies a (C, s)-estimate, then it satisfies also a $(C/e, s')$-estimate; so one can restrict the study to the case when $s \leq \ln(n)$. It is important to notice that s (and so q) may depend on the dimension of the space. In particular, the canonical basis of ℓ_∞^n satisfies a (C, s)-estimate with $\frac{1}{s} = \frac{1}{2} - \frac{1}{\ln(n)}$ and the case of ℓ_∞^n is described up to a constant by taking $q = \ln n$. Finally, we denote by $d(E, F)$ the Banach-Mazur distance between two normed spaces E and F:

$$d(E, F) = \inf\big\{\|T\|\|T^{-1}\|, T : E \to F \text{ isomorphism onto}\big\}.$$

For two positive sequences $\{a_n\}$ and $\{b_n\}$ we say that $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$.

Let us recall the following estimates for the norm of Gaussian operators: if E is a Banach space and $(v_j)_{j=1}^N \in E$, we define a Gaussian operator $G_\omega : \ell_2^k \to E$ by

$$G_\omega = \sum_{i=1}^{k} \sum_{j=1}^{N} g_{ij}(\omega) e_i \otimes v_j : \ell_2^k \to E,$$

where, for $1 \le i \le k$ and $1 \le j \le N$, g_{ij} are i.i.d. $\mathcal{N}(0,1)$ real Gaussian variables. Let $g_1, \dots, g_N$ be i.i.d. $\mathcal{N}(0,1)$ real Gaussian variables and $a_k = \mathbb{E}\big(\sum_{i=1}^k g_i^2\big)^{1/2}$ then we have the following inequalities [8]:

$$\mathbb{E}\Big\|\sum_{j=1}^N g_j v_j\Big\| - a_k \sup_{\sum_{1\le j\le N} t_j^2 = 1} \Big\|\sum_{j=1}^N t_j v_j\Big\| \le \mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| \tag{2}$$

and

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \le \mathbb{E}\Big\|\sum_{j=1}^N g_j v_j\Big\| + a_k \sup_{\sum_{1\le j\le N} t_j^2 = 1} \Big\|\sum_{j=1}^N t_j v_j\Big\|. \tag{3}$$

One has $a_k = \sqrt{2}\,\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})} \le \sqrt{k}$ and $a_k \sim \sqrt{k}$.

From now on, (g_i) and (g_{ij}) will denote i.i.d. $\mathcal{N}(0,1)$ real Gaussian variables.

2. Euclidean sections of Banach spaces

The main result of this section is

THEOREM 2.1. *Let E be an n-dimensional normed space, and for $n \ge N \ge n/2$, let $(u_i)_{i=1}^N \in E$ satisfy a (C, s)-estimate for $s > 2$ and $C > 0$. Define $q > 2$ by $\frac{1}{q} = \frac{1}{2} - \frac{1}{s}$. Then for some universal positive constants c_1, c_2 and d_1, d_2, d_3, for all integers k, $1 \le k \le N$, there exists a k-dimensional subspace F^k of E such that*

(i) *If $k \le \frac{1}{4}\Big(\mathbb{E}\Big\|\sum_{j=1}^N g_j u_j\Big\|\Big)^2$, then $d(F^k, \ell_2^k) \le 3$.*

(ii) *If $\frac{1}{4}\Big(\mathbb{E}\Big\|\sum_{j=1}^N g_j u_j\Big\|\Big)^2 \le k \le c_1 C^2 q e^{-q} n$, then $d(F^k, \ell_2^k) \le \dfrac{d_1\sqrt{k}}{C\sqrt{q}\,n^{1/q}}$.*

(iii) *If $c_1 C^2 q e^{-q} n \le k \le c_2 n$, then $d(F^k, \ell_2^k) \le \dfrac{d_2 k^{1/2-1/q}}{C\sqrt{\ln(1+n/k)}}$.*

(iv) *If* $c_2 n \le k \le N$, *then* $d(F^k, \ell_2^k) \le \dfrac{d_3}{C}\, k^{1/s}$.

The space F^k, $1 \le k \le N$, *can be chosen randomly with positive probability as subspace of the linear span of* $(u_i)_{i=1}^N$.

PROOF. Remark that when $\frac{1}{2} - \frac{1}{s} \le \frac{1}{\ln(n)}$, the family $(u_i)_{i=1}^N$ satisfies a $(C/e, s')$-estimate where $\frac{1}{s'} = \frac{1}{2} - \frac{1}{\ln(n)}$. Up to replace C by C/e, we can assume that $q \le \ln(n)$.

Let $U = \operatorname{span}\{u_1, \dots, u_N\}$; we define a Gaussian operator $G_\omega : \ell_2^k \to U$ by

$$G_\omega = \sum_{i=1}^{k} \sum_{j=1}^{N} g_{ij}(\omega) e_i \otimes u_j.$$

Observe that $\sup_{t_1^2+\cdots+t_N^2=1} \left\| \sum_{j=1}^N t_j u_j \right\| \le 1$. Applying (2) and (3), we get

$$\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| \ge \mathbb{E}\left\| \sum_{j=1}^{N} g_j u_j \right\| - a_k \sup_{t_1^2+\cdots+t_N^2=1} \left\| \sum_{j=1}^{N} t_j u_j \right\|$$

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \le \mathbb{E}\left\| \sum_{j=1}^{N} g_j u_j \right\| + a_k \sup_{t_1^2+\cdots+t_N^2=1} \left\| \sum_{j=1}^{N} t_j u_j \right\|.$$

To find a set Ω of positive probability such that for all $\omega \in \Omega$, $G_\omega : \ell_2^k \to U$ is one to one, it is enough to have $\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| > 0$. We distinguish between the different values of k, $1 \le k \le N$:

1. If $k \le \frac{1}{4}\left(\mathbb{E}\left\| \sum_{j=1}^N g_j u_j \right\|\right)^2$, then

$$\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} \le \left(1 + \frac{a_k}{\mathbb{E}\left\| \sum_{j=1}^N g_j u_j \right\|}\right) \Big/ \left(1 - \frac{a_k}{\mathbb{E}\left\| \sum_{j=1}^N g_j u_j \right\|}\right) \le 3.$$

So, there exists a set of positive probability Ω such that for all $\omega_0 \in \Omega$, $\dim(\operatorname{Im} G_{\omega_0}) = k$ and

$$\frac{\sup_{|x|_2=1} \|G_{\omega_0}(x)\|}{\inf_{|x|_2=1} \|G_{\omega_0}(x)\|} \le 3.$$

See [9] for a precise estimate of the measure of this set. Let $F^k = \operatorname{Im} G_{\omega_0}$; then $\dim F^k = k$, $d(F^k, \ell_2^k) \le 3$ and case (i) is proved (it is the classical Dvoretzky's theorem).

2. In the other cases, one has $k \ge \left(\mathbb{E}\left\| \sum_{j=1}^N g_j u_j \right\|\right)^2/4$ so that

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\| \le 3\sqrt{k}.$$

For $1 \le m \le N$, in order to get a better lower bound for $\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|$, we define a new norm $\|y\|_{(m)}$ on U by

$$\|y\|_{(m)} = \Big\| \sum_{j=1}^N y_j u_j \Big\|_{(m)} = \frac{C}{m^{1/s}} \Big(\sum_{i=1}^m (y_i^\star)^2 \Big)^{1/2}.$$

It is clear from the definition of a (C, s) estimate (see (1)) that $\|G_\omega(x)\| \ge \|G_\omega(x)\|_{(m)}$. We get thus by inequality (2) applied to $G_\omega : \ell_2^k \to (U, \|\cdot\|_{(m)})$

$$\begin{aligned}
\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\| &\ge \mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_{(m)} \\
&\ge \mathbb{E} \Big\| \sum_{j=1}^N g_j u_j \Big\|_{(m)} - a_k \sup_{t_1^2+\cdots+t_N^2=1} \Big\| \sum_{j=1}^N t_j u_j \Big\|_{(m)} \\
&\ge \frac{1}{m^{1/s}} \bigg(C\mathbb{E} \Big(\sum_{i=1}^m (g_i^\star)^2 \Big)^{1/2} - \sqrt{k} \bigg) \\
&\ge m^{1/q} \left(Cc_0 \sqrt{\ln\Big(1 + \frac{N}{m}\Big)} - \sqrt{\frac{k}{m}} \right),
\end{aligned}$$

where the last inequality is a classical estimate of $\mathbb{E}\big(\sum_{i=1}^m (g_i^\star)^2\big)^{1/2}$ (see for instance [7]) with $c_0 > 0$ a universal constant.

If $k \le c_1 C^2 q e^{-q} n$, we choose $m = [Ne^{-q}] + 1$ so that $ne^{-q}/2 \le m \le 3Ne^{-q}$ (since $N \ge n/2$ and $q \le \ln(n)$). We get

$$\begin{aligned}
\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} &\le \frac{3\sqrt{k}}{Cm^{1/q} c_0 \left(\sqrt{\ln(1 + e^q/3)} - \sqrt{c_1}\sqrt{2q} \right)} \\
&\le \frac{3\sqrt{k}}{Cm^{1/q} c_0 \left(\sqrt{q - \ln 3} - \sqrt{2qc_1} \right)} \\
&\le \frac{d_1 \sqrt{k}}{C\sqrt{q}\, n^{1/q}},
\end{aligned}$$

whenever c_1 is small enough. We conclude like in 1.

If $c_1 C^2 q e^{-q} n \le k \le c_2 n$ for c_2 small enough, we choose $m = k$. We have then

$$\frac{\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|}{\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|} \le \frac{d_2 k^{1/s}}{C\sqrt{\ln(1 + n/k)}}$$

and as before, we get (iii).

If $c_2 n \leq k \leq N$, then by the definition of the (C, s)-estimate, one has $d(U, \ell_2^N) \leq N^{1/s}/C$; thus every k-dimensional subspace F^k of U satisfies

$$d(F^k, \ell_2^k) \leq \frac{N^{1/s}}{C} \leq \frac{n^{1/s}}{C} \leq \frac{1}{C}\left(\frac{k}{c_2}\right)^{1/s}.$$

REMARKS. 1. It is easy to see that for a family $\{u_1, \dots, u_N\}$ satisfying a (C, s)-estimate, one has

$$\mathbb{E}\left\|\sum_{j=1}^N g_j u_j\right\| \geq cC\sqrt{q}\, n^{1/q}.$$

Indeed, by (1), for all $m \in \{1, \dots, N\}$, we have

$$\mathbb{E}\left\|\sum_{j=1}^N g_j u_j\right\| \geq \frac{C}{m^{1/s}}\,\mathbb{E}\left(\sum_{i=1}^m (g_i^\star)^2\right)^{1/2} \geq c' C m^{1/q}\sqrt{\ln\left(1 + \frac{N}{m}\right)},$$

and we choose $m = [Ne^{-q}] + 1$ (recall that $N \geq n/2$).

2. Observe that, up to an absolute constant, the estimates given in (ii) and (iii) coincide if $k = [c_1 C^2 q e^{-q} n]$, and these in (iii) and (iv) when $k = [c_2 n]$. Moreover, if we replace in (i) and (ii) the expression $\frac{1}{2}\mathbb{E}\left\|\sum_{j=1}^N g_j u_j\right\|$ by $cC\sqrt{q}n^{1/q}$, then they also hold and these estimates coincide for $k = [c^2 C^2 q n^{2/q}]$.

As a corollary, we get precise estimates in the case of $E = \ell_q^n$.

COROLLARY 2.2. *For some universal constants $c_i, d_i > 0$, $1 \leq i \leq 3$, for all $n \geq 1$, and all integers $k = 1, \dots, n$, there exists a k-dimensional subspace F^k of ℓ_q^n with $q \geq 2$, such that*

(i) *If $k \leq c_1 q n^{2/q}$, then $d(F^k, \ell_2^k) \leq 3$.*

(ii) *If $c_1 q n^{2/q} \leq k \leq c_2 q e^{-q} n$, then $d(F^k, \ell_2^k) \leq \dfrac{d_1\sqrt{k}}{\sqrt{q}n^{1/q}}$.*

(iii) *If $c_2 q e^{-q} n \leq k \leq c_3 n$, then $d(F^k, \ell_2^k) \leq \dfrac{d_2 k^{1/2-1/q}}{\sqrt{\ln(1 + n/k)}}$.*

(iv) *If $c_3 n \leq k \leq n$, then $d(F^k, \ell_2^k) \leq d_3 k^{1/2-1/q}$.*

Moreover, the space F^k can be chosen randomly with high probability in ℓ_q^n.

PROOF. Let $(e_1, \dots, e_n)$ be the canonical basis of ℓ_q^n; then for all $t_1, \dots, t_n$ and for all $m = 1, \dots, n$,

$$\left(\sum_{i=1}^{n} |t_i|^q\right)^{1/q} = \left|\sum_{i=1}^{n} t_i e_i\right|_q \geq \left|\sum_{i=1}^{m} t_i^\star e_i\right|_q = \left(\sum_{i=1}^{m} (t_i^\star)^q\right)^{1/q} \geq \frac{1}{m^{\frac{1}{2}-\frac{1}{q}}}\left(\sum_{i=1}^{m} (t_i^\star)^2\right)^{1/2},$$

using Hölder's inequality. Since $q \geq 2$, $(e_1, \dots, e_n)$ satisfies a $(1, s)$-estimate, with $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. It is clear from the preceeding remark that

$$\alpha_1 \sqrt{q}\, n^{1/q} \leq \mathbb{E}\left|\sum_{j=1}^{n} g_j e_j\right|_q \leq \alpha_2 \sqrt{q}\, n^{1/q},$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are universal constants. Then we apply Theorem 2.1 to get random subspaces in the whole space ℓ_q^n.

REMARKS. 1. As it is proved in [3], the result of Corollary 2.2 is optimal up to absolute constant. We include here a short proof of this optimality: Let $T : \ell_2^k \to \ell_q^n$ be a linear operator such that for all $x \in \ell_2^k$,

$$|x|_2 \leq |Tx|_q \leq d|x|_2.$$

Let G be a Gaussian random vector in R^k with i.i.d. $\mathcal{N}(0, 1)$ entries, then

$$\mathbb{E}\left(\sum_{i=1}^{n} \left|\langle G, T^\star(e_i)\rangle\right|^q\right)^{1/q} = \mathbb{E}|T(G)|_q \geq \mathbb{E}|G|_2 = a_k.$$

Since $\langle G, T^\star(e_i)\rangle$ is a $\mathcal{N}\big(0, \left|T^\star(e_i)\right|_2^2\big)$ random variable, we get by Hölder inequality,

$$\mathbb{E}\left(\sum_{i=1}^{n} \left|\langle G, T^\star(e_i)\rangle\right|^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} \mathbb{E}\left|\langle G, T^\star(e_i)\rangle\right|^q\right)^{1/q} \leq n^{1/q}\gamma(q) \sup_{1\leq i\leq n} \left|T^\star(e_i)\right|_2,$$

where $\gamma(q)$ is the L_q norm of a $\mathcal{N}(0, 1)$-variable. Since $|T^\star(e_i)|_2 \leq \|T^\star\| |e_i|_{q'} \leq d$ with $1/q + 1/q' = 1$, we get a universal constant $c > 0$ such that,

$$\sqrt{k} \leq c d n^{1/q} \sqrt{q}.$$

2. A constructive proof of a single subspace of ℓ_q^n satisfying the desired conclusion is given in [12].

3. In fact by [16], the inequality $d(F^k, \ell_2^k) \leq k^{1/2-1/q}$ is true for any k-dimensional subspace of ℓ_q^n.

3. The case of Schatten classes

We shall say that a norm τ on R^n is 1-*symmetric* if for all $(x_1, \dots, x_n)$ and for every permutation σ on $\{1, \dots, n\}$, one has

$$\tau(x_1, \dots, x_n) = \tau(|x_{\sigma(1)}|, \dots, |x_{\sigma(n)}|).$$

If τ is such a norm, it is well known that for $m \geq n \geq 1$, one defines a norm $\|\cdot\|_\tau$ on the mn-dimensional vector space $\mathscr{M}_{m\times n}(\mathsf{R})$ of all $m \times n$-matrices with real entries by setting

$$\|M\|_\tau = \tau(s_1(M), \dots, s_n(M)) \quad \text{for all} \quad M \in \mathscr{M}_{m\times n}(\mathsf{R})$$

where the $s_i(M)$, $1 \leq i \leq n$, are the eigenvalues of $\sqrt{M^*M}$. If for some $q \geq 1$ $\tau(x) = |x|_q$, we get the so called Schatten class $S_q(m \times n)$ with the norm $\|T\|_q = |\big(s_i(T)\big)_{i=1}^n|_q$.

Since τ is a 1-symmetric norm, it is clear that we can renormalize τ so that for all $x \in \mathsf{R}^n$,

$$\frac{1}{d_\tau}\,|x|_2 \leq \tau(x) \leq |x|_2, \tag{4}$$

where d_τ is the Banach-Mazur distance between (R^n, τ) and ℓ_2^n.

THEOREM 3.1. *Let τ be a 1-symmetric norm on R^n, $\|\cdot\|_\tau$ be the norm on $\mathscr{M}_{m\times n}(\mathsf{R})$ associated with τ and $d_\tau = d((\mathsf{R}^n, \tau), \ell_2^n)$ such that (4) is satisfied. Denote by G a Gaussian random matrix of $\mathscr{M}_{m\times n}(\mathsf{R})$ with i.i.d. $\mathscr{N}(0, 1)$ entries. Then for every integer k, $1 \leq k \leq nm$, there exists a k-dimensional subspace F^k of $(\mathscr{M}_{m\times n}(\mathsf{R}), \|\cdot\|_\tau)$ such that*

(i) *If $k \leq (\mathbb{E}\|G\|_\tau)^2/4$, then $d(F^k, \ell_2^k) \leq 3$.*

(ii) *If $(\mathbb{E}\|G\|_\tau)^2/4 \leq k \leq nm$, then $d(F^k, \ell_2^k) \leq 1 + 12\, d_\tau \sqrt{\dfrac{k}{nm}}$.*

PROOF. By (4), one has for all $T \in \mathscr{M}_{m\times n}(\mathsf{R})$

$$\frac{1}{d_\tau}\,\|T\|_2 \leq \|T\|_\tau = \tau(s_1(T), \dots, s_n(T)) \leq \|T\|_2 \tag{5}$$

where $\|T\|_2 = \big(\mathrm{tr}(T^\star T)\big)^{1/2}$ denotes the Hilbert-Schmidt norm. For $1 \leq p \leq m$ and $1 \leq q \leq n$, let E_{pq} be the canonical basis of $\mathscr{M}_{m\times n}(\mathsf{R})$ (with entries

$(E_{pq})_{ij} = \delta_{ip}\delta_{qj}$). Let $G_\omega : \ell_2^k \to (\mathscr{M}_{m\times n}(\mathsf{R}), \|\cdot\|_\tau)$ be the Gaussian operator defined by

$$G_\omega = \sum_{l=1}^{k} \sum_{\substack{1\le p\le m\\ 1\le q\le n}} g_{lpq}(\omega)\, e_l \otimes E_{pq}$$

where $e_1, \dots, e_k$ is the canonical basis of ℓ_2^k and the g_{lpq}, $1 \le l \le k$, $1 \le p \le m$, $1 \le q \le n$, are i.i.d. $\mathscr{N}(0, 1)$ Gaussian variables. By inequalities (2) and (3), we have

$$\mathbb{E} \sup_{|x|_2=1} \|G_\omega(x)\|_\tau \le \mathbb{E}\|G\|_\tau + a_k \sup\big\{ \|T\|_\tau ; T \in \mathscr{M}_{m\times n}(\mathsf{R}), \|T\|_2 = 1 \big\}$$

and

$$\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_\tau \ge \mathbb{E}\|G\|_\tau - a_k \sup\big\{ \|T\|_\tau ; T \in \mathscr{M}_{m\times n}(\mathsf{R}), \|T\|_2 = 1 \big\}$$

where G is a Gaussian random matrix of $\mathscr{M}_{m\times n}(\mathsf{R})$ with i.i.d. $\mathscr{N}(0, 1)$ entries.

It is clear that $\sup\{\|T\|_\tau ; T \in \mathscr{M}_{m\times n}(\mathsf{R}), \|T\|_2 = 1\} = 1$. We distinguish between three cases.

1. If $\mathbb{E}\|G\|_\tau \ge 2\sqrt{k}$, since $\sqrt{k} \ge a_k$, we have

$$\frac{\mathbb{E}\sup_{|x|_2=1} \|G_\omega(x)\|_\tau}{\mathbb{E}\inf_{|x|_2=1} \|G_\omega(x)\|_\tau} \le \frac{1 + a_k/\mathbb{E}\,\|G\|_\tau}{1 - a_k/\mathbb{E}\,\|G\|_\tau} \le 3.$$

2. If $\mathbb{E}\|G\|_\tau \le 2\sqrt{k} \le \sqrt{nm}/2$, then by condition (5) and inequality (2) with $G_\omega : \ell_2^k \to (\mathscr{M}_{m\times n}(\mathsf{R}), \|\cdot\|_2)$, we get

$$\begin{aligned}\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_\tau &\ge \frac{1}{d_\tau}\, \mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_2 \\ &\ge \frac{1}{d_\tau}\, (\mathbb{E}\|G\|_2 - a_k).\end{aligned}$$

Since $\mathbb{E}\|G\|_2 = a_{nm} \ge \dfrac{\sqrt{nm}}{2}$ and $a_k \le \sqrt{k} \le \sqrt{nm}/4$, we get

$$\mathbb{E} \inf_{|x|_2=1} \|G_\omega(x)\|_\tau \ge \frac{1}{d_\tau}\left(\frac{\sqrt{nm}}{2} - a_k\right) \ge \frac{\sqrt{nm}}{4d_\tau}.$$

We get thus

$$\frac{\mathbb{E}\sup_{|x|_2=1} \|G_\omega(x)\|_\tau}{\mathbb{E}\inf_{|x|_2=1} \|G_\omega(x)\|_\tau} \le \frac{12 d_\tau \sqrt{k}}{\sqrt{nm}}.$$

3. If $\sqrt{k} \geq \sqrt{nm}/4$, it follows from (5) that for all subspaces F^k of $(\mathscr{M}_{m\times n}(\mathsf{R}), \tau)$ with $\dim F^k = k$, one has $d(F^k, \ell_2^k) \leq d_\tau$.

As a consequence of the preceeding theorem, we get:

COROLLARY 3.2. *Let $q \geq 2$ and let $S_q(m \times n)$ be the Schatten class. Assume that for some fixed $r > 1$, one has $m = rn$. Then for some universal constant $c > 0$, and for every integer k, $1 \leq k \leq nm$, there exists a k-dimensional subspace F^k of $S_q(m \times n)$ such that*

$$d(F^k, \ell_2^k) \leq 1 + \frac{c}{\sqrt{r}} n^{-1/q} \sqrt{\frac{k}{n}}.$$

REMARK. In [5] example 3.3 (i), this result is proved in the case $r = 1$ and $k \sim cn^{1+\frac{2}{q}}$ with the estimate $d(F^k, \ell_2^k) \leq 2$.

Before proving this corollary, we need to compute $\mathbb{E}\|G\|_q$ for a Gaussian matrix, $G = (g_{ij})_{m(n)\times n}$, where the g_{ij} are i.i.d. $\mathscr{N}(0, 1)$ Gaussian variables, and $m(n)/n \to r$ (≥ 1) as $n \to \infty$. To this end we need the following theorem.

THEOREM 3.3. *Let G be a $n \times m(n)$ Gaussian matrix as above. Then almost surely the empirical distribution function*

$$L_n = \frac{\#\left\{\lambda \in \textit{eigenvalues}\left(\frac{GG^*}{n}\right); \lambda \leq x\right\}}{n}$$

converges weakly to the probability law Λ_r given by:

$$\frac{d\Lambda_r(x)}{dx} = \frac{1}{2\pi x}\sqrt{(x-a)(b-x)}\, \mathbb{1}_{[a,b]}(x)$$

where $a = (\sqrt{r} - 1)^2$, $b = (\sqrt{r} + 1)^2$.

This is known as the free analog of the Poisson-distribution with free parameter $r \geq 1$ [13], [15], [20], [21]. The distribution Λ_r was first studied by Marchenko and Pastur [20], and the almost sure convergence version stated in Theorem 3.3 is due to Wachter [21].

LEMMA 3.4. *Let $\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_n^*$ denote the decreasing rearrangement of the eigenvalues of the random matrix $\sqrt{GG^*}$.*

Let $\sigma \in [\sqrt{a}, \sqrt{b}]$,

$$\rho = \rho(\sigma) = \frac{1}{2\pi}\int_{\sigma^2}^{b} \frac{\sqrt{(x-a)(b-x)}}{x}\, dx,$$

and

$$I(\sigma, q) = \left(\frac{1}{2\pi} \int_{\sigma^2}^{b} x^{q/2} \frac{\sqrt{(x-a)(b-x)}}{x}\, dx \right)^{1/q}.$$

Then we have asymptotically

$$\left(\sum_{i=1}^{n\rho} \lambda_i^{*q} \right)^{1/q} \sim n^{1/2+1/q} I(\sigma, q) \text{ a.s.} \tag{6}$$

and in particular, for all $0 < q < \infty$,

$$\mathbb{E}\|G\|_q = \mathbb{E}\left(\sum_{i=1}^{n} \lambda_i^{*q} \right)^{1/q} \sim n^{1/2+1/q} I\big(\sqrt{r} - 1, q\big). \tag{7}$$

PROOF. Geman [6] proved that a.s.

$$\frac{\lambda_1^*}{\sqrt{n}} \to \sqrt{b} \quad (= \sqrt{r} + 1)$$

and

$$\mathbb{E}\left(\frac{\lambda_1^*}{\sqrt{n}} \right)^q \to b^{q/2}$$

for all $q > 0$ follows as well from the computation there.

Setting $\mu = \sqrt{\lambda}$, we get by Theorem 3.3

$$\frac{\#\left\{\mu \in \text{eigenvalues}\left(\sqrt{\frac{GG^*}{n}}\right); \mu \le \sigma\right\}}{n} \longrightarrow \frac{1}{2\pi} \int_a^{\sigma^2} \frac{\sqrt{(x-a)(b-x)}}{x}\, dx$$

and the convergence is a.s. as $n \to \infty$. Then by the above, if $\mu_1, \dots, \mu_n$ denote the eigenvalues of the matrix $\sqrt{\frac{GG^*}{n}}$, one has $n\rho \sim \#\{i; \mu_i \ge \sigma\}$. We have then, for all $0 < q < \infty$ a.s.

$$\left(\frac{1}{n} \sum_{i=1}^{n\rho} \mu_i^{*q} \right)^{1/q} = \left(\int_{\sigma^2}^{\infty} x^{q/2} dL_n(x) \right)^{1/q}$$

and by Geman's result above, a.s. there exists $n_0(\omega)$ such that $L_n(\omega)$ is supported on $[a-1, b+1]$ for all $n \ge n_0(\omega)$. Therefore the last integral is a.s. asymptotically equivalent to

$$\left(\int_{\sigma^2}^{b+1} x^{q/2}\, d\Lambda_r(x) \right)^{1/q},$$

and since $x^{q/2}$ is a continuous bounded function on $[\sigma^2, b+1]$, the last integral is equal to

$$\left(\int_{\sigma^2}^{b+1} x^{q/2}\, d\Lambda_r(x)\right)^{1/q} = \left(\frac{1}{2\pi}\int_{\sigma^2}^{b} x^{q/2}\frac{\sqrt{(x-a)(b-x)}}{x}\, dx\right)^{1/q}.$$

In order to prove equation (7) it suffices to prove that $\mathbb{E}\left(\frac{\|G\|_q}{n^{1/2+1/q}}\right)^{1+\varepsilon}$ is bounded for some $\varepsilon > 0$. This follows easily from the concentration property of $\|G\|_q$ around its mean [9], or using

$$\mathbb{E}\left(\frac{\|G\|_q}{n^{1/2+1/q}}\right)^{1+\varepsilon} \le \mathbb{E}\left(\frac{n^{1/q}\lambda_1^*}{n^{1/2+1/q}}\right)^{1+\varepsilon} = \mathbb{E}\left(\frac{\lambda_1^*}{\sqrt{n}}\right)^{1+\varepsilon}$$

which is known to be bounded by Geman's result above.

REMARKS. 1. When $q = 2$ and $r = 1$, we obtain the well known result

$$\mathbb{E}\|G\|_2 = \mathbb{E}\left(\sum_{i=1}^{n}\lambda_i^{*2}\right)^{1/2} = \mathbb{E}\left(\sum_{1\le i,j\le n} g_{ij}^2\right)^{1/2} = \frac{\sqrt{2}\,\Gamma\left(\frac{n^2+1}{2}\right)}{\Gamma\left(\frac{n^2}{2}\right)} \sim n$$

and $I(0, 2) = 1$.

If we take $q = 1$, $r = 1$ we have for the 1-nuclear norm of the Gaussian $n \times n$ matrix G

$$\mathbb{E}\|G\|_1 = \mathbb{E}\left(\sum_{i=1}^{n}\lambda_i^*\right) \sim n^{3/2} I(0,1) = \frac{n^{3/2}}{2\pi}\int_0^4 \sqrt{4-x}\, dx = \frac{8n^{3/2}}{3\pi}.$$

2. In fact, one has for all $q \ge 2$ and all $n \ge 1$,

$$\frac{\sqrt{r}}{2} n^{\frac{1}{q}+\frac{1}{2}} \le \mathbb{E}\|G\|_q \le n^{\frac{1}{q}+\frac{1}{2}}(\sqrt{r}+1).$$

Indeed for $\tau(x) = \left(\sum_{i=1}^{n}|x_i|^q\right)^{1/q}$ one has $d_\tau = n^{\frac{1}{2}-\frac{1}{q}}$, and we get

$$n^{\frac{1}{q}-\frac{1}{2}}\,\mathbb{E}\|G\|_2 \le \mathbb{E}\|G\|_q \le n^{1/q}\,\mathbb{E}\|G\|_\infty.$$

Since $\mathbb{E}\|G\|_2 = a_{nm} \ge \sqrt{nm}/2$ and $\mathbb{E}\|G\|_\infty \le a_n + a_m \le \sqrt{n} + \sqrt{m}$, this gives the result.

3. Haagerup and Thorbjornsen [15] have studied the more general case of Gaussian matrices with operator entries. They obtain an upper bound for $\mathbb{E}(\lambda_1^*)$ and a lower bound for $\mathbb{E}(\lambda_n^*)$.

PROOF OF COROLLARY 3.2. For $q \geq \ln(n)$, the norm on $S_q(m \times n)$ is equivalent up to universal constant to the norm on $S_\infty(m \times n)$; so we reduce to the case when $2 \leq q \leq \ln(n)$. We have $\tau(x) = \left(\sum_{i=1}^n |x_i|^q\right)^{1/q}$ so that $d_\tau = n^{\frac{1}{2}-\frac{1}{q}}$. The result follows now from Remark 2 after Lemma 3.4 and Theorem 3.1, because the estimates (i) and (ii) of this theorem coincide up to a constant when $k = (\mathbb{E}\|G\|_q)^2/4$.

REMARK. As for ℓ_q^n, we can prove the optimality of Corollary 3.2.

Let $\Theta : \ell_2^k \to S_q(m \times n)$ be a linear operator such that for all $x \in \ell_2^k$,

$$|x|_2 \leq \|\Theta x\|_q \leq d\,|x|_2.$$

If $T_i = \Theta(e_i)$ and $G = (g_1, \ldots, g_k)$ is a Gaussian vector, we have

$$a_k = \mathbb{E}|G|_2 \leq \|T(G)\|_q \leq n^{1/q}\mathbb{E}\|T(G)\|_\infty = n^{1/q}\mathbb{E}\left\|\sum_{i=1}^k g_i T_i\right\|_\infty.$$

But

$$\left\|\sum_{i=1}^k g_i T_i\right\|_\infty = \sup_{|x|_2=1}\sup_{|y|_2=1}\sum_{i=1}^k g_i\langle T_i x, y\rangle.$$

Let $h_1, \ldots, h_n, h_1', \ldots, h_m'$ be $n+m$ i.i.d. $\mathcal{N}(0,1)$ real Gaussian variables and define for $x \in \mathsf{R}^n$ and $y \in \mathsf{R}^m$ the two Gaussian processes:

$$X_{x,y} = \sum_{i=1}^k g_i\langle T_i x, y\rangle \qquad \text{and} \qquad Y_{x,y} = \sqrt{2}\,d\left(\sum_{i=1}^m h_i x_i + \sum_{i=1}^n h_i' y_i\right).$$

By definition of T_i, one has

$$\left\|\sum_{i=1}^k \alpha_i T_i\right\|_\infty = \left\|\Theta(\sum_{i=1}^k \alpha_i e_i)\right\|_\infty \leq d\left(\sum_{i=1}^k \alpha_i^2\right)^{1/2}.$$

If $|x|_2 = 1$ and $|y|_2 = 1$, one has

$$\left(\sum_{i=1}^k \left|\langle T_i x, y\rangle\right|^2\right)^{1/2} = \sup_{|\alpha|_2\leq 1}\sum_{i=1}^k \alpha_i\langle T_i x, y\rangle \leq \sup_{|\alpha|_2\leq 1}\left\|\sum_{i=1}^k \alpha_i T_i\right\|_\infty \leq d,$$

and thus

$$\begin{aligned}\mathbb{E}|X_{x,y}-X_{x',y'}|^2 &= \sum_{i=1}^{k}\big(\big|\langle T_i x, y-y'\rangle + \langle x-x', T_i^\star y'\rangle\big|\big)^2\\ &\le 2\sum_{i=1}^{k}\big(\big|\langle T_i x, y-y'\rangle\big|^2 + \big|\langle x-x', T_i^\star y'\rangle\big|^2\big)\\ &\le 2d^2\big(|y-y'|_2^2+|x-x'|_2^2\big) = \mathbb{E}|Y_{x,y}-Y_{x',y'}|^2.\end{aligned}$$

Then by Fernique's lemma [4], we obtain

$$\mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} X_{x,y} \le \mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} Y_{x,y}$$

and since $\mathbb{E}\sup_{|x|_2=1}\sup_{|y|_2=1} Y_{x,y} = \sqrt{2}\,d(a_n+a_m)$, we get a universal constant $c>0$ such that

$$\sqrt{k} \le cd\big(\sqrt{r}+1\big)\,n^{1/2+1/q}.$$

If $F^k \subset S_q(m\times n)$ of dimension k satisfies $d(F^k,\ell_2^k) \le \frac{c}{\sqrt{r}}\,n^{-1/q}\sqrt{\frac{k'}{n}}$ then $k \le ck'$, which proves the optimality of Corollary 3.2.

4. Volume ratios with respect to quotients of subspaces of L_q

In this section we introduce volume ratios of random k-dimensional subspaces F of an n-dimensional normed space X with respect to the class of all k-dimensional subspaces of quotients of ℓ_q, $2\le q\le\infty$. Among other things, these volume ratios yield in the case $q=2$, a lower bound for the distance $d(F,\ell_2^k)$ for random subspaces F of X.

Let us consider the following concept of volume ratios introduced in [11], [12]. Given an n-dimensional normed space $X=(\mathsf{R}^n,\|\cdot\|)$ with unit ball B_X, and a Banach space Z with unit ball B_Z, we define the volume ratios

$$\mathrm{vr}(X,Z) := \inf\left\{\left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(T(B_Z))}\right)^{1/n};\ T(B_Z)\subset B_X\right\},$$

$$\mathrm{vr}(X,S(Z)) := \inf\left\{\left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(T(B_F))}\right)^{1/n};\ F\subset Z, \dim F = n, T(B_F)\subset B_X\right\},$$

$$\mathrm{vr}(X,S_p) := \mathrm{vr}(X,S(\ell_p)),$$

and

$$\mathrm{vr}(X,SQ(\ell_p)) := \inf_{Q\text{ quotient of }\ell_p}\mathrm{vr}(X,S(Q)).$$

As in [12] the n-th volume number of an operator $T : X \to Y$ is defined by

$$v_n(T) = \sup\left\{ \left(\frac{\mathrm{vol}(T(B_E))}{\mathrm{vol}(B_F)} \right)^{1/n} ; E \subset X, T(E) \subset F \subset Y, \dim E = \dim F = n \right\}$$

We shall also need the definition of the *p-nuclear norm* of an operator $T : X \to Y$ between two finite dimensional Banach spaces, which is defined by

$$\nu_p(T) = \inf\left\{ \|A_N\| \|\sigma_N\| \|B_N\|; T = B_N \sigma_N A_N,\ N \geq 1 \right\}$$

where $A_N : X \to \ell_\infty^N$, $\sigma_N : \ell_\infty^N \to \ell_p^N$ is a diagonal operator, and $B_N : \ell_p^N \to Y$.

THEOREM 4.1. *Let $X = (\mathsf{R}^n, \|\cdot\|)$ be an n-dimensional normed space, $\{b_i, b_i^*\}_{i=1}^n$ be a biorthogonal basis for X and $J = \sum_{j=1}^n e_j^\star \otimes b_j : \mathsf{R}^n \to X$. For all $u \in \mathcal{O}_n$, let $u_k : \mathsf{R}^k \to \mathsf{R}^n$ be the linear operator defined by $u_k(e_j) = u(e_j)$ for all $1 \leq j \leq k$ and let $A_u = J \circ u_k : \ell_2^k \to X$.*

Then for some universal constant $c > 0$ and for all $2 \leq q \leq \infty$, the k-dimensional random subspace $F_u = A_u(\ell_2^k) \subset X$ satisfies

$$\mathbb{E}_u \,\mathrm{vr}(F_u, SQ(\ell_q)) \geq \frac{c\sqrt{k}}{\left(\sqrt{q} + \frac{\sqrt{k}}{n^{1/q}}\right) \max\limits_{1\leq i\leq n} \|b_i^*\| \,\mathbb{E}\left\| \sum_{i=1}^n g_i\, b_i \right\|}$$

where $\mathbb{E}_u$ denotes the expectation with respect to the Haar measure on $\mathcal{O}_n$.

PROOF. For $u \in \mathcal{O}_n$, define $B_u : X \to \ell_2^k$ by $B_u = u_k^\star \circ J^{-1}$ where $u_k^\star : \mathsf{R}^n \to \mathsf{R}^k$ is the adjoint of u_k. Clearly $B_u A_u = id_{\ell_2^k}$.

Claim. Let q' be the conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\mathbb{E}_u \nu_{q'}(B_u : X \to \ell_2^k) \leq c\sqrt{n}\left(\sqrt{q} + \frac{\sqrt{k}}{n^{1/q}}\right) \max_{1\leq j\leq n} \|b_j^*\|. \tag{8}$$

To show this, we write $B_u = u_k^*|_{\ell_{q'}^n \to \ell_2^k} I J^{-1}$ where $I = \sum_{i=1}^n e_i \otimes e_i : \ell_\infty^n \to \ell_{q'}^n$ is the identity map, and $J^{-1} = \sum_{i=1}^n b_i^* \otimes e_i : X \to \ell_\infty^n$. Then clearly

$$\nu_{q'}\big(B_u|_{X\to\ell_2^k}\big) \leq \|J^{-1}\| \|I\| \left\| u_k^*|_{\ell_{q'}^n \to \ell_2^k} \right\| = \max_{1\leq i\leq n} \|b_i^*\|\, n^{1/q'} \left\| u_k^*|_{\ell_{q'}^n \to \ell_2^k} \right\|.$$

Let $G = \sum_{ij} g_{ij} e_i \otimes e_j$ denote the Gaussian operator which maps $\ell_{q'}^n$ into ℓ_2^k; we have by [1]

$$\begin{aligned}\mathbb{E}_u \big\| u_k^* |_{\ell_{q'}^n \to \ell_2^k} \big\| &\le \frac{c_0}{\sqrt{n}}\, \mathbb{E} \big\| G : \ell_{q'}^n \to \ell_2^k \big\| \\ &\le \frac{c_1}{\sqrt{n}} \big(c n^{1/q} \sqrt{q} + \sqrt{k}\, \big)\end{aligned}$$

hence

$$\mathbb{E}_u \nu_{q'} \big(B_u |_{X \to \ell_2^k} \big) \le c_0\, n^{1/2} \big(c\sqrt{q} + n^{-1/q} \sqrt{k}\, \big) \max_{1 \le i \le n} \|b_i^*\|$$

and () is proved.

For $T : \ell_2^k \to X$, define $\mathrm{rad}(T) := \int_0^1 \big\| \sum_{i=1}^k r_i(t) T(e_i) \big\|_X \, dt$ where $(r_i)_{i=1}^k$ are independent Rademacher variables. Now we use a method from [1]. By the Marcus-Pisier inequality [17], we get

$$\begin{aligned}\sqrt{n}\, \mathbb{E}_u \, \mathrm{rad}(A_u : \ell_2^k \to X) &= \sqrt{n}\, \mathbb{E}_u \int_0^1 \Big\| \sum_{j=1}^k r_j(t) A_u(e_j) \Big\| \, dt \\ &\le c\, \mathbb{E} \int_0^1 \Big\| \sum_{j=1}^k \sum_{i=1}^n r_j(t) g_{ij} b_i \Big\| \, dt \\ &\le c \sqrt{k}\, \mathbb{E} \Big\| \sum_{i=1}^n g_i b_i \Big\|.\end{aligned}$$

By [3] one has

$$\mathbb{E}_u \sqrt{k}\, v_k(A_u) \le c_1 \mathbb{E}_u \, \mathrm{rad}(A_u) \le \frac{c\sqrt{k}}{\sqrt{n}}\, \mathbb{E} \Big\| \sum_{i=1}^n g_i b_i \Big\|.$$

By [12] Lemma 1.3, we have for $2 \le q \le \infty$ and $k = 1, 2, \ldots$, and any operator T from a Banach space Z to ℓ_2

$$\frac{\sqrt{k}\, v_k(T)}{\nu_{q'}(T)} \le c_0 \sup_{F \subset Z,\, \dim(F) = k} \mathrm{vr}(F, SQ(\ell_q)).$$

Applying this to $B_u |_{F_u \to \ell_2^k}$ we get

$$\sqrt{k}\, v_k \big(B_u |_{F_u} \big) \le c_0 \nu_{q'}(B_u)\, \mathrm{vr}(F_u, SQ(\ell_q)).$$

Since $B_u A_u = id_{\ell_2^k}$, we have $1 = v_k(B_u A_u) = v_k(A_u) v_k(B_u|_{F_u})$. Hence we obtain

$$1 \leq c_0 v_k(A_u) \frac{\nu_{q'}(B_u)}{\sqrt{k}} \operatorname{vr}(F_u, SQ(\ell_q))$$

and taking the 3-rd root we get by Hölder's inequality

$$\begin{aligned} 1 &\leq c_0 \mathbb{E}_u v_k(A_u)\, \mathbb{E}_u \left(\frac{\nu_{q'}(B_u)}{\sqrt{k}} \right) \mathbb{E}_u \operatorname{vr}(F_u, SQ(\ell_q)) \\ &\leq \frac{c}{\sqrt{n}} \mathbb{E} \left\| \sum_{i=1}^n g_i b_i \right\| \frac{c\sqrt{n}}{\sqrt{k}} \left(\sqrt{q} + \frac{\sqrt{k}}{n^{1/q}} \right) \max_i \|b_i^*\|\, \mathbb{E}_u \operatorname{vr}(F_u, SQ(\ell_q)). \end{aligned}$$

This concludes the proof.

REMARKS. 1. It was proved in [12] that

$$\begin{aligned} \operatorname{vr}(X, SQ(\ell_p)) &\leq \operatorname{vr}(X, S(\ell_p)) \\ &\leq c_o \sqrt{p + p'} \operatorname{vr}(X, SQ(\ell_p)) \end{aligned}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.

2. We obtain for $2 \leq p \leq \ln n$ and $X = \ell_p^n$,

$$\mathbb{E}_u \operatorname{vr}(F_u, \ell_2) \geq \frac{c\sqrt{k}}{\sqrt{p}\, n^{1/p}},$$

and

$$\mathbb{E}_u \operatorname{vr}(F_u, SQ(\ell_q)) \geq \frac{c\sqrt{k}}{\sqrt{p}\, n^{1/p} \max\left(\sqrt{q}, \frac{\sqrt{k}}{n^{1/q}}\right)},$$

Now when $p \geq q \geq 2$ and k similar to $qn^{2/q}$, we have

$$\mathbb{E}_u \operatorname{vr}(F_u, SQ(\ell_q)) \geq \frac{n^{1/q-1/p}}{\sqrt{p}}.$$

This estimate is sharp because for all k-dimensional subspaces F^k of R^n, if F_p^k denotes F^k endowed with the norm of ℓ_p^n, one has

$$\operatorname{vr}(F_p^k, SQ(\ell_q)) \leq d(F_p^k, F_q^k) \leq d(\ell_p^n, \ell_q^n) = n^{1/q-1/p}.$$

3. For $2 \leq p \leq \ln n$ and $X = S_p(m \times n)$ where $m = rn$, $r \geq 1$, it follows from Theorem 4.1 that

$$\mathbb{E}_u \operatorname{vr}(F_u, \ell_2) \geq \frac{c\sqrt{k}}{\sqrt{r}\, n^{1/2+1/p}},$$

and

$$\mathbb{E}_u \operatorname{vr}(F_u, SQ(\ell_q)) \geq \frac{c\sqrt{k}}{\sqrt{r}\, n^{1/p+1/2} \max\left(\sqrt{q}, \frac{\sqrt{k}}{n^{1/q}}\right)}.$$

Indeed, by Lemma 3.4, one has

$$\mathbb{E}\left\| \sum_{i,j} g_{ij} E_{ij} \right\|_p \leq c n^{1/p+1/2} \sqrt{r}.$$

4. In particular Theorem 4.1 gives, in average, an optimal lower bound for the Banach Mazur distance between random k-dimensional subspaces of X to ℓ_2^k (see parts 2 and 3 above).

5. Application to Gelfand numbers

The k-th Gelfand number of a linear operator $T : X \to Y$ is defined to be

$$c_k(T) = \inf\left\{ \|T|_L\|; L \subset X, \operatorname{codim} L = k-1 \right\}.$$

In this section, we will study particularly the Gelfand numbers (c_k) for large values of k, in terms of the dimension of X.

THEOREM 5.1. *Let X be an n-dimensional normed space with a basis $\{x_i\}_{i=1}^n$ satisfying a (C, s)-estimate for $s > 2$ and $C > 0$. Let q be defined by $\frac{1}{s} = \frac{1}{2} - \frac{1}{q}$. Let $T : X \to Y$ and denote $T(x_i) = y_i$ for all $i = 1, \dots, n$. Then for some universal constants $c_1, c_2, d_1, d_2 > 0$, for all integers k, we have*

(i) *if $n - k \leq \frac{1}{4}\left(\mathbb{E}\left\|\sum_{j=1}^n g_j x_j\right\|\right)^2$,*

$$c_{k+1}(T) \leq \frac{\mathbb{E}\left\| \sum_{j=1}^n g_j y_j \right\|_Y + a_{n-k} \sup_{t_1^2+\cdots+t_n^2=1} \left\| \sum_{j=1}^n t_j y_j \right\|_Y}{\frac{1}{2}\mathbb{E}\left\| \sum_{j=1}^n g_j x_j \right\|_X},$$

(ii) *if $\frac{1}{4}\left(\mathbb{E}\left\|\sum_{j=1}^n g_j x_j\right\|\right)^2 \leq n - k \leq c_1 C^2 q e^{-q} n$,*

$$c_{k+1}(T) \leq \frac{\mathbb{E}\left\| \sum_{j=1}^n g_j y_j \right\|_Y + a_{n-k} \sup_{t_1^2+\cdots+t_n^2=1} \left\| \sum_{j=1}^n t_j y_j \right\|_Y}{d_1 C \sqrt{q}\, n^{1/q}},$$

(iii) *if* $c_1 C^2 q e^{-q} n \le n-k \le c_2 n$,

$$c_{k+1}(T) \le \frac{\mathbb{E}\Big\|\sum_{j=1}^n g_j y_j\Big\|_Y + a_{n-k} \sup_{t_1^2+\cdots+t_n^2=1}\Big\|\sum_{j=1}^n t_j y_j\Big\|_Y}{d_2 C(n-k)^{1/q}\sqrt{\ln\big(1+\frac{n}{n-k}\big)}}.$$

The following theorem will treat the case of spaces of operators. For the notation, see section 3. In particular G denotes a random Gaussian matrix of $\mathscr{M}_{m\times n}(\mathsf{R})$ with i.i.d. $\mathscr{N}(0,1)$ entries.

THEOREM 5.2. *Let τ be a* 1*-symmetric norm on R^n, $\|\cdot\|_\tau$ the norm on $\mathscr{M}_{m\times n}(\mathsf{R})$ associated with τ and $d_\tau = d((\mathsf{R}^n,\tau),\ell_2^n)$ such that* (4) *is satisfied. Let E_{ij} denote the canonical basis of $\mathscr{M}_{m\times n}(\mathsf{R})$, T an operator from $\mathscr{M}_{m\times n}(\mathsf{R})$ to Y, and for $i=1,\dots,n$, $j=1,\dots m$, let $T(E_{ij})=y_{ij}$. The following estimates hold:*

(i) *if* $nm-k \le \frac{1}{4}\big(\mathbb{E}\|G\|_\tau\big)^2$, *then*

$$c_{k+1}(T) \le \frac{\mathbb{E}\Big\|\sum_{i=1}^n\sum_{j=1}^m g_{ij} y_{ij}\Big\|_Y + a_{nm-k}\sup_{|t|_2=1}\Big\|\sum_{i=1}^n\sum_{j=1}^m t_{ij}y_{ij}\Big\|_Y}{\frac{1}{2}\mathbb{E}\|G\|_\tau},$$

(ii) *if* $\frac{1}{4}\big(\mathbb{E}\|G\|_\tau\big)^2 \le nm-k \le nm/16$, *then*

$$c_{k+1}(T) \le \frac{4d_\tau}{\sqrt{nm}}\Bigg(\mathbb{E}\Big\|\sum_{i=1}^n\sum_{j=1}^m g_{ij}y_{ij}\Big\|_Y + a_{nm-k}\sup_{|t|_2=1}\Big\|\sum_{i=1}^n\sum_{j=1}^m t_{ij}y_{ij}\Big\|_Y\Bigg).$$

PROOFS. The beginning of the proof is the same for both theorems. We denote by $(x_i)_{i=1}^M$ the basis of X and by $y_i = Tx_i$, for $1\le i\le M$, and $M=n$ (for theorem 5.1), and $M=nm$ (for theorem 5.2).

If (g_{ij}), $i,j=1,\dots,M$ is a sequence of $\mathscr{N}(0,1)$ i.i.d. Gaussian variable then $L_\omega = \operatorname{span}\big\{\sum_{j=1}^M g_{ij}(\omega)x_j\big\}_{i=1}^{M-k}$ is a random subspace of X of dimension $M-k$ almost everywhere. Hence, a.e.,

$$c_{k+1}(T) \le \|T|_{L_\omega}\| \le \sup_{t_1^2+\cdots+t_{M-k}^2=1}\left(\frac{\Big\|\sum_{i=1}^{M-k}\sum_{j=1}^M g_{ij}(\omega)t_i y_j\Big\|_Y}{\Big\|\sum_{i=1}^{M-k}\sum_{j=1}^M g_{ij}(\omega)t_i x_j\Big\|_X}\right),$$

and by integration, we obtain

$$c_{k+1}(T) \leq \frac{\mathbb{E} \sup_{t_1^2+\cdots+t_{M-k}^2=1} \left\| \sum_{i=1}^{M-k} \sum_{j=1}^{M} g_{ij}(\omega) t_i y_j \right\|_Y}{\mathbb{E} \inf_{t_1^2+\cdots+t_{M-k}^2=1} \left\| \sum_{i=1}^{M-k} \sum_{j=1}^{M} g_{ij}(\omega) t_i x_j \right\|_X}.$$

By (3), we have the classical upper bound for the numerator:

$$\mathbb{E} \sup_{|t|_2=1} \left\| \sum_{i,j} g_{ij}(\omega) t_i y_j \right\|_Y \leq \mathbb{E} \left\| \sum_{j=1}^{M} g_j y_j \right\| + a_{M-k} \sup_{t_1^2+\cdots+t_M^2=1} \left\| \sum_{j=1}^{M} t_j y_j \right\|.$$

We need now a lower bound of $\mathbb{E} \inf_{|t|_2=1} \|G_\omega(t)\|$, where $G_\omega : \ell_2^{M-k} \to X$ is the Gaussian operator

$$G_\omega = \sum_{i=1}^{M-k} \sum_{j=1}^{M} g_{ij}(\omega) e_i \otimes x_j.$$

End of proof of Theorem 5.1. Here $n = M = \dim X$ and the family $(x_1, \ldots, x_n)$ satisfies a (C, s)-estimate. Using the arguments of the proof of Theorem 2.1, there exist universal constants $c_1, c_2, d_1, d_2 > 0$ such that

– if $n - k \leq \left(\mathbb{E}\left\| \sum_{j=1}^{n} g_j x_j \right\|\right)^2/4$, we are in the case of Dvoretzky's theorem, then
$$\mathbb{E} \inf_{|t|_2=1} \|G_\omega(t)\| \geq \frac{1}{2} \mathbb{E} \left\| \sum_{j=1}^{n} g_j x_j \right\|,$$

– if $\left(\mathbb{E}\left\| \sum_{j=1}^{n} g_j x_j \right\|\right)^2/4 \leq n - k \leq c_1 C^2 q e^{-q} n$, then
$$\mathbb{E} \inf_{|t|_2=1} \|G_\omega(t)\| \geq d_1 C \sqrt{q}\, n^{1/q}.$$

– if $c_1 C^2 q e^{-q} n \leq n - k \leq c_2 n$, then
$$\mathbb{E} \inf_{|t|_2=1} \|G_\omega(t)\| \geq d_2 C (n-k)^{1/q} \sqrt{\ln\left(1 + \frac{n}{n-k}\right)}.$$

This proves Theorem 5.1.

End of proof of Theorem 5.2. In the case of operator spaces, we take $M = nm$ and we work with the canonical basis of $\mathscr{M}_{m\times n}(\mathsf{R})$. Using the arguments of the proof of Theorem 3.1 (cases 1 and 2), we have

– if $nm - k \le \left(\mathbb{E}\|G\|_\tau\right)^2/4$,

$$\mathbb{E}\inf_{|t|_2=1}\|G_\omega(t)\| \ge \frac{1}{2}\mathbb{E}\|G\|_\tau,$$

– if $\left(\mathbb{E}\|G\|_\tau\right)^2/4 \le nm - k \le nm/16$,

$$\mathbb{E}\inf_{|t|_2=1}\|G_\omega(t)\| \ge \frac{\sqrt{nm}}{4d_\tau}$$

and this proves Theorem 5.2.

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