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SEMIPERFECT FINITELY GENERATED ABELIAN SEMIGROUPS WITHOUT INVOLUTION

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1. Introduction

With the possible exception of finite abelian groups, the oldest example of a semiperfect semigroup is the group Z of integers. Herglotz' Theorem of 1911 asserts that a two-sided sequence $(s_n)_{n=-\infty}^{\infty}$ of complex numbers is a trigonometric moment sequence, in the sense that

$$s_n = \int_{\mathsf{T}} z^n \, d\mu(z), \qquad n \in \mathsf{Z}$$

for some measure μ on the complex unit circle T, if and only if (s_n) is positive semidefinite in the sense that

$$\sum_{j,k=0}^{n} c_j \overline{c_k} s_{j-k} \geq 0$$

for every choice of n in N_0 (the nonnegative integers) and $c_0, \dots, c_n$ in the complex field C. When the condition is satisfied, there is only one such measure μ.

Hamburger's Theorem [21] asserts that a sequence $(s_n)_{n=0}^{\infty}$ of reals is a moment sequence, in the sense that

$$s_n = \int_{\mathsf{R}} x^n \, d\mu(x), \qquad n \in \mathsf{N}_0 \tag{1}$$

for some measure μ on the real line R, if and only if (s_n) is positive semidefinite in the sense that

$$\sum_{j,k=0}^{n} c_j c_k s_{j+k} \geq 0$$

for every choice of $n \in \mathsf{N}_0$ and $c_0, \dots, c_n \in \mathsf{R}$.

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Sz.-Nagy [29] showed that a sequence $(s_n)_{n=0}^{\infty}$ of self-adjoint bounded linear operators on a Hilbert space H admits the integral representation (1) for some measure μ on R with values that are positive operators on H if and only if (s_n) is of positive type in the sense that

$$\sum_{j,k=0}^{n} \langle s_{j+k}\xi_j, \xi_k\rangle \geq 0$$

for every choice of $n \in \mathsf{N}_0$ and $\xi_0, \ldots, \xi_n \in H$ where $\langle\cdot,\cdot\rangle$ is the inner product on H.

The moment problems solved by Herglotz, Hamburger, and Sz.-Nagy can be generalized to arbitrary abelian $*$-semigroups. Suppose $(S, +, *)$ is an abelian semigroup equipped with an *involution*, that is, a mapping $s \mapsto s^*: S \to S$ satisfying $(s^*)^* = s$ and $(s+t)^* = s^* + t^*$ for all $s, t \in S$. Such a structure will be called a $*$-*semigroup*, even abbreviated 'semigroup' when confusion is unlikely, such as when applying an adjective which makes sense only in the presence of an involution (e.g., 'semiperfect semigroup'). For subsets X and Y of S, define $\check{X} = \{x^* \mid x \in X\}$ and $X + Y = \{x + y \mid x \in X,\ y \in Y\}$, abbreviated $x + Y$ in case $X = \{x\}$ for some $x \in S$. Suppose D is a complex vector space and denote by $\mathscr{S}(D)$ the set of all sesquilinear forms on D. A function $\varphi: X + \check{X} \to \mathscr{S}(D)$ is of *positive type with respect to* X if

$$\sum_{j,k=1}^{n} \varphi(x_j + x_k^*)(\xi_j, \xi_k) \geq 0$$

for every choice of n in N (the natural numbers), $x_1, \ldots, x_n \in X$, and $\xi_1, \ldots, \xi_n \in D$. The words 'with respect to X' will be omitted if X is a $*$-subsemigroup of S, but are necessary in general since there might be a set Y distinct from X such that $X + \check{X} = Y + \check{Y}$. Denote by $\mathscr{P}(X, D)$ the set of all such functions. Make it a convention that if the symbol 'D' is part of the notation for an entity in the definition of which a complex vector space D occurs then that symbol is omitted (together with any comma immediately preceding it) in case $D = \mathsf{C}$. We also identify $\mathscr{S}(\mathsf{C})$ with C itself by identifying $a \in \mathsf{C}$ with the sesquilinear form $(\xi, \eta) \mapsto a\xi\overline{\eta}$ on C. Thus, $\mathscr{P}(X)$ is the set of those functions $\varphi: X + \check{X} \to \mathsf{C}$ which are *positive semidefinite with respect to* X in the sense that

$$\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(x_j + x_k^*) \geq 0 \tag{2}$$

for every choice of $n \in \mathsf{N}$, $x_1, \ldots, x_n \in X$, and $c_1, \ldots, c_n \in \mathsf{C}$. Say that φ is *positive definite with respect to* X if the sum in (2) is positive whenever the x_j

are pairwise distinct and the c_j are not all zero. The terms 'positive definite' and 'strictly positive definite' instead of 'positive semidefinite' and 'positive definite' are also in use.

A *character* on S is a function $\sigma\colon S \to \mathsf{C}$, not identically zero, such that $\sigma(s^*) = \overline{\sigma(s)}$ and $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S.

Suppose X is a subset of S which generates S as a $*$-semigroup. Define $p_X\colon S^* \to \mathsf{C}^X$ by $p_X(\sigma) = \sigma|X$ for $\sigma \in S^*$. Define $X^\circ = p_X(S^*) \setminus \{0\}$. Elements of X° will be called 'characters on X'. Suppose X' is a subset of X°. Denote by $\mathscr{A}(X')$ the least σ-ring of subsets of X' rendering measurable for each $x \in X$ the function $\widehat{x}\colon X' \to \mathsf{C}$ defined by $\widehat{x}(\sigma) = \sigma(x)$ for $\sigma \in X'$. For $x \in X$ and $n \in \mathsf{N}$ define $G_{s,n} = \{\sigma \in X' \mid |\sigma(x)| > 1/n\}$. Denote by $\mathscr{A}_0(X')$ the subring of $\mathscr{A}(X')$ consisting of those sets which are contained in the union of finitely many $G_{x,n}$.

The set of all subsets of X' which are contained in the union of countably many $G_{x,n}$ is a σ-ring of subsets rendering $\widehat{x}$ measurable for each $x \in X$ and so contains $\mathscr{A}(X')$ by the definition of the latter. It follows that the subring $\mathscr{A}_0(X')$ generates $\mathscr{A}(X')$ as a σ-ring. Hence, every measure μ on $\mathscr{A}_0(X')$ which is *finite* in the sense that $\mu(A) < \infty$ for all $A \in \mathscr{A}_0(X')$ extends to a unique measure on $\mathscr{A}(X')$ ([20], Theorem A p. 53). (When nothing else is said, a measure is positive by definition.)

For every mapping $\mu\colon \mathscr{A}_0(X') \to \mathscr{S}(D)$ and for each $\xi \in D$, define a mapping $\mu(\cdot)(\xi,\xi)\colon \mathscr{A}_0(X') \to \mathsf{C}$ by $\mu(\cdot)(\xi,\xi)(A) = \mu(A)(\xi,\xi)$ for $A \in \mathscr{A}_0(X')$. The mapping μ is a *measure* if $\mu(\cdot)(\xi,\xi)$ is a measure for each $\xi \in D$. For $p \in \mathsf{N}$ denote by $F^p_+(X', D)$ the set of all measures μ in this sense such that for each $\xi \in D$ the scalar-valued measure $\mu(\cdot)(\xi,\xi)$ integrates the function $|\widehat{x}|^p$ for each $x \in X$. Denote by $F^p(X', D)$ the complex linear hull of $F^p_+(X', D)$. For $\mu \in F^2_+(X', D)$ define $\mathscr{L}^2\mu\colon X + \check{X} \to \mathscr{S}(D)$ by

$$\mathscr{L}^2\mu(x+y^*)(\xi,\xi) = \int_{X'} \sigma(x)\overline{\sigma(y)}\, d\mu(\cdot)(\xi,\xi)(\sigma)$$

for $x, y \in X$ and $\xi \in D$. The integral is understood as one with respect to the unique measure on $\mathscr{A}(X')$ to which the finite measure $\mu(\cdot)(\xi,\xi)$ extends; it exists by Hölder's inequality. For $a \in X + \check{X}$ and $\sigma \in X^\circ$ the value of $\sigma(x)\overline{\sigma(y)}$ is independent of the choice of $x, y \in X$ such that $a = x + y^*$; this is because σ is the restriction to X of a character on S. The remaining values of the sesquilinear form $\mathscr{L}^2\mu(x+y^*)$ follow by polarization.

A function $\varphi\colon X + \check{X} \to \mathscr{S}(D)$ is an *X'-moment function* if there is some $\mu \in F^2_+(X', D)$ which *represents* φ in the sense that $\varphi = \mathscr{L}^2\mu$. An X'-moment function is *X'-determinate* if it is represented by only one measure on X'. The prefix 'X'' is omitted in case X is a $*$-semigroup and $X' = X^*$. Denote by $\mathscr{H}(X, X', D)$ the set of all X'-moment functions and by $\mathscr{H}_D(X, X', D)$ the subset of X'-determinate ones. In these notations we again omit 'X'' (together with the comma) in case X is a $*$-semigroup and $X' = X^*$. Using Lemma 1.1 in the paper of Schmüdgen [27] one can show $\mathscr{H}(X, X', D) \subset \mathscr{P}(X, D)$. The set X is *X'-semiperfect of order d* if $\mathscr{H}(X, X', \mathsf{C}^d) = \mathscr{P}(X, \mathsf{C}^d)$, *completely X'-semiperfect* if this is so for each $d \in \mathsf{N}$, and *X'-perfect* if $\mathscr{H}_D(X, X') = \mathscr{P}(X)$. The words 'of order d' are omitted in case $d = 1$.

If S is a perfect semigroup then, as observed by Christian Berg in the late 1980's, we have $\mathscr{H}_D(S, D) = \mathscr{P}(S, D)$ for every complex vector space D. In particular, S is completely semiperfect. If S is a completely semiperfect semigroup then $\mathscr{H}(S, D) = \mathscr{P}(S, D)$ for every complex vector space D [15]. Every semigroup which has ever (to our knowledge) been shown to be semiperfect has even been shown to be completely semiperfect. NOTE ADDED IN PROOF. We now have a counterexample. Indeed, if G is an infinite binary vector space with the involution $x^* = -x = x$ then the $*$-subsemigroup $\{(0, 0)\} \cup \big((G \setminus \{0\}) \times \{1\}\big) \cup (G \times \{2, 3, \ldots\})$ of the product $*$-semigroup $G \times \mathsf{N}_0$ is semiperfect of order 1 but not of order 2. See our paper *On the relation between the scalar moment problem and the matrix moment problem on $*$-semigroups*, submitted to Annals of Math. (2002).

The group Z with the *inverse involution* ($n^* = -n$) is perfect by Herglotz' Theorem. More generally, every abelian group G carrying the inverse involution is perfect by the discrete version of the Bochner-Weil Theorem. Even more generally, S is perfect if S is an *abelian inverse semigroup* in the sense that $s + s^* + s = s$ for all $s \in S$. (*Warning*: The 'Bochner-Weil Theorem for Locally Compact Abelian Inverse Semigroups' is false, even in the compact metrizable case. Berg, Christensen, and Ressel ([2], p. 143) write: 'On the compact semigroup $S = [0, 1]$ with maximum as semigroup operation there is only one continuous semicharacter [i.e., character], namely the constant semicharacter.' On this semigroup the function φ defined by $\varphi(s) = 1 - s$ for $s \in S$ is a continuous positive definite function such that the unique measure μ on S^* such that $\varphi = \mathscr{L}^2\mu$ is concentrated on the set of all discontinuous characters. The same authors continue: 'Perhaps the right dual object to look at might be the set of semicharacters that are continuous at zero.' However, there exist a compact metrizable semigroup S with zero, such that $s = s^* = s + s$ for all $s \in S$, and a continuous positive semidefinite function φ on S such that the unique measure μ on S^* such that $\varphi = \mathscr{L}^2\mu$ is concentrated on the set of those characters which are discontinuous at the zero [10].

The 'discrete version of the Bochner-Weil Theorem for abelian inverse semigroups' has found two generalizations: One stating that a $*$-semigroup S is perfect if $2(s + s^*) = s + s^*$ for all $s \in S$ (so far unpublished), and one stating that S is perfect if S is $*$-*divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $m, n \in \mathsf{N}_0$ such that $s = mt + nt^*$. For the case of semigroups with zero, see the paper by Ressel and the author [16]; for the general case, the paper by Sakakibara and the author [17]. We know of no natural result that generalizes both of these facts.

NOTE ADDED IN PROOF. We now know such a result, viz., it suffices that S is *semi-$*$-divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $m, n \in \mathsf{N}_0$ such that $s + s^* = s^* + mt + nt^*$. This was shown by N. Sakakibara and the author in a manuscript that was submitted to Ark. Mat. in September 2001.

The semigroup N_0 with its unique involution, the identity, is semiperfect by Hamburger's Theorem and completely semiperfect by the result of Sz.-Nagy cited above. This semigroup is not perfect since there exist indeterminate moment sequences, such as the example $n \mapsto (4n+3)!$ given by Stieltjes [28], 26 years prior to the publication of Hamburger's Theorem.

The group Z, *considered with the identical involution*, is semiperfect as shown by Jones, Njåstad, and Thron [23]; see [2], 6.4.1, for a post-1920 proof. The complete semiperfectness of Z is an easy consequence of the corresponding property of N_0; see, e.g., [5]. This semigroup, like N_0, is non-perfect since there exist indeterminate two-sided moment sequences, such as $n \mapsto e^{n^2/2}$ ([2], 6.4.6).

For $k \geq 2$, the semigroups N_0^k and Z^k, considered with the identical involution, are non-semiperfect. For N_0^k, this was first shown by Berg, Christensen, and Jensen [1] and independently by Schmüdgen [26]. Each set of authors appealed to the Hahn-Banach Theorem and thus produced no example of a function $\varphi \in \mathscr{P}(\mathsf{N}_0^2) \setminus \mathscr{H}(\mathsf{N}_0^2)$. The first such example was given by Friedrich [19]. In his example,

$$\varphi(0, n) = \exp\left\{\left[\binom{n/2+2}{2} + 1\right]! \log \binom{n/2+2}{2}!\right\}$$

for even $n \geq 8$. This raised the question: How fast must $|\varphi(m, n)|$ grow as $m + n \to \infty$ if $\varphi \in \mathscr{P}(\mathsf{N}_0^2) \setminus \mathscr{H}(\mathsf{N}_0^2)$? It was shown in [8] by example that there is a function $\varphi \in \mathscr{P}(\mathsf{N}_0^2) \setminus \mathscr{H}(\mathsf{N}_0^2)$ such that $\varphi(m, n) = O\big((m+n)^{a(m+n)}\big)$ as $n \to \infty$ for each $a > 1$, and the constant 1 is the best possible. The case of Z^k is an exercise in [2].

The negative results of the preceding paragraph are subsumed in the result that the only semiperfect subsemigroups of Z^k with the identical involution are $\{0\}$ and those isomorphic to Z or N_0. For semigroups with zero, this was shown by Sakakibara [25]. The general case is a corollary of a much more general result which is published for the first time in the present paper.

A $*$-semigroup H is *$*$-archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathsf{N}$ such that $n(x + x^*) = y + z$. A *$*$-archimedean component* of a $*$-semigroup S is a $*$-archimedean *$*$-subsemigroup* (i.e., $*$-stable subsemigroup) of S which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components (see [18], Section 4.3, for the case of the identical involution). Moreover, every $*$-archimedean $*$-subsemigroup of a $*$-semigroup S is contained in a unique $*$-archimedean component of S.

An abelian semigroup H equipped with the identical involution is $*$-archimedean if and only if it is *archimedean* in the sense that for all $x, y \in H$ there exist $z \in H$ and $n \in \mathsf{N}$ such that $nx = y + z$. The $*$-archimedean components of an abelian semigroup S equipped with the identical involution are precisely the *archimedean components* of S, that is, the maximal archimedean subsemigroups.

A *$*$-homomorphism* of a $*$-semigroup S into another $*$-semigroup is a homomorphism h satisfying $h(s^*) = h(s)^*$ for all $s \in S$. Every $*$-homomorphic image of a semigroup that is semiperfect of order d has the same property ([16], Proposition 1, for $d = 1$; see also [5], Proposition 4.)

Given a subset M of C, a $*$-semigroup S is said to be *M-separative* if the M-valued characters on S separate points in S. The *greatest M-separative $*$-homomorphic image* of S is the quotient $*$-semigroup $S/{\sim}$ where $\sim$ is the congruence relation in S defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for every M-valued character σ on S. The term 'greatest' is chosen for the reason that if f is a $*$-homomorphism of S into an M-separative $*$-semigroup T then there is a unique $*$-homomorphism h of $S/{\sim}$ into T such that $f = h \circ \psi$ where ψ is the quotient mapping of S onto $S/{\sim}$. Thus $S/{\sim}$ is 'greatest' in the sense of corresponding to the smallest congruence relation. Clifford and Preston [18] use the term 'maximal' instead of 'greatest'.

Every R-separative semigroup carries the identical involution.

For every $*$-semigroup S and for every subset V of S, denote by $E(V)$ the set of those $v \in V$ such that if $s, t \in S$, $s + s^*, t + t^* \in V$, and $s + t^* = v$ then $s = t$. For every subset U of S, denote by $C(U)$ the union of all finite subsets V of S such that $E(V) \subset U$. The semigroup S is *C-finite* if $C(U)$ is a finite set for every finite subset U of S. In [7] we included in the definition of C-finiteness the condition of R-separativity, but it is better to separate the conditions. We apologize for being inconsistent.

The main result of [7] states that a countable R-separative C-finite semigroup S satisfying $S = S + S$ is semiperfect (or equivalently, completely semiperfect) if and only if the following condition is satisfied:

(B) Each archimedean component of S is isomorphic to the product of a torsion group of exponent 1 or 2 and one of the semigroups $\{0\}$, Z, N.

The conditions $S = S + S$ and (B) are together sufficient for complete semiperfectness of S even if S is not countable and not C-finite. (Note that condition (B) implies R-separativity.)

Just as [7] was going to press, it turned out that a semiperfect countable R-separative C-finite semigroup S automatically satisfies $S = S + S$. The first main purpose of the present paper is to prove this. We consider it quite important that in the necessity part of the main result of [7], the condition $S = S + S$ can be transferred from being among the assumptions to being among the conclusions. (And in the sufficiency part, it is now clear that the assumption $S = S + S$ is not an arbitrary one, but a necessary one.)

For every abelian semigroup X we denote by (G_X, g_X) the pair—uniquely determined up to isomorphism—consisting of an abelian group G_X and a homomorphism $g_X\colon X \to G_X$ such that if f is a homomorphism of X into an abelian group F then there is a unique homomorphism $h\colon G_X \to F$ such that $f = h \circ g_X$. It is well-known that the semigroup $g_X(X)$ generates G_X, i.e., $G_X = g_X(X) - g_X(X)$, and that for $x, y \in X$ we have $g_X(x) = g_X(y)$ if and only if $a + x = a + y$ for some $a \in X$.

If X is a $*$-semigroup then we consider G_X with the unique involution rendering g_X a $*$-homomorphism. With F, f, and h as in the preceding paragraph, if f is a $*$-homomorphism then so is h.

An element e of an abelian semigroup is *idempotent* if $e = e + e$. A *semilattice* is an abelian semigroup with all elements idempotent. If I is a semilattice, we consider I with the *canonical partial ordering* $\leq$ defined by the condition that $i \leq j$ if and only if $i + j = j$. For $i, j \in I$ the element $i + j$ is the least upper bound on the set $\{i, j\}$ in the partially ordered set $(I, \leq)$. See Clifford and Preston [18] around p. 25, or Berg, Christensen, and Ressel [2], Ch. 4.

For every $*$-semigroup S we denote by $\mathscr{J}(S)$ the set of all $*$-archimedean components of S. If $H, K \in \mathscr{J}(S)$ then the $*$-subsemigroup $H + K$ of S is easily seen to be $*$-archimedean, hence contained in a unique $*$-archimedean component of S, which we denote by $H \vee K$. The pair $\big(\mathscr{J}(S), \vee\big)$ is a semilattice and so carries a canonical partial ordering $\leq$.

A *face* of a $*$-semigroup S is a $*$-subsemigroup X of S such that if $x, y \in S$ and $x + y \in X$ then $x, y \in X$. Every intersection of faces of S, if nonempty, is a face of S. Hence, for every nonempty subset H of S there is a least face of S containing H, viz., the intersection of all faces of S containing H, the set of such faces being nonempty since S itself is such a face. We leave it as an exercise to verify that if $H \in \mathscr{J}(S)$ then the least face of S containing H is the set

$$X_H = \bigcup_{I \in \mathscr{J}(S) : I \leq H} I,$$

which we always denote by this symbol. Note that $X_H + H \subset H$. A *main face* of S is a face of S of the form X_H for some $H \in \mathscr{J}(S)$. Saying that a face X of S is a main face of S is equivalent to saying that there is some $H \in \mathscr{J}(S)$ such that $H \subset X$ and $X + H \subset H$. When the condition is satisfied, H is uniquely determined (being among all $*$-archimedean components of S contained in X the greatest one with respect to the canonical partial ordering on $\mathscr{J}(S)$), and $X = X_H$.

If H is a $*$-archimedean component of a $*$-semigroup S then we define a mapping of X_H into G_H, at first cautiously denoted by f, by choosing $y \in H$ and setting $f(x) = g_H(x + y) - g_H(y)$ (difference in the group G_H) for $x \in X_H$. The following two facts are easily proved in that order: (i) The definition of $f(x)$ is independent of the choice of y; (ii) the mapping f so defined is a $*$-homomorphism. Since $f|H = g_H$, it should cause no confusion that we henceforth denote the mapping f by g_H.

A $*$-semigroup S is of *class* $\mathscr{M}$ if for each $H \in \mathscr{J}(S)$ we have $0 \in g_H(X_H)$.

Every R-separative finitely generated semigroup is C-finite as shown near the end of [7]. Thus, the main result of [7] (augmented by the fact that the condition $S = S + S$ is necessary for semiperfectness) implies a characterization of semiperfect (or equivalently, completely semiperfect) R-separative finitely generated semigroups. It is the second main purpose of the present paper to extend this result to a characterization of semiperfect (or equivalently, completely semiperfect) finitely generated abelian semigroups with the identical involution. As the assumption of R-separativity is dropped, an interesting additional necessary condition crops up. For a sketch, first consider an abelian semigroup S with arbitrary involution. Denote by χ (or χ_S, if S has to be specified) the quotient mapping of S onto its greatest C-separative $*$-homomorphic image. For each $d \in \mathsf{N}$, the semigroup S is semiperfect of order d if and only if $\chi(S)$ so is and moreover, every positive semidefinite function on S factors via χ ([5], proof of Proposition 5). If S carries the identical involution then so does $\chi(S)$. This semigroup being C-separative by definition, it is then R-separative. Since $\chi(S)$, being a homomorphic image of the finitely generated semigroup S, is finitely generated then the question of whether $\chi(S)$ is semiperfect (or equivalently, completely semiperfect) can be resolved using the main result of [7] (augmented as above). It remains to settle the question of factoring. We shall show that in order for a finitely generated abelian semigroup S with arbitrary involution to be semiperfect it is necessary that S be of class $\mathscr{M}$. The factoring problem on semigroups of class $\mathscr{M}$ was considered in [12]. The main result states that if S is a semigroup of class $\mathscr{M}$ satisfying $S = S + S$ then every positive semidefinite function on S factors via χ. This is a straightforward generalization of [4], Theorem 2, which states the same for semigroups with zero. The main result of [12] is not quite enough for our purposes since a semiperfect finitely generated semigroup with the identical involution does not necessarily satisfy $S = S + S$. Fortunately, in [12] we also considered the case that S is of class $\mathscr{M}$ but $S \neq S + S$. The second main result of [12] states the following: *Define* $A = \{ s \in S \mid s + s^* \in S + S + S \}$. *Let* $\overline{A}$ *be the least subset of* S*, containing* A*, such that if* $s \in S$ *and* $s + s^* \in \overline{A} + S$ *then* $s \in \overline{A}$. *Define* $E = S \setminus \overline{A}$. *Every set that generates* S *as an abelian semigroup contains* E. *In particular, if* S *is finitely generated then* E *is finite. Define an equivalence relation* $\sim$ *in the set* $E^2 = E \times E$ *by the condition that* $(e, f) \sim (g, h)$ *if and only if* $e + f^* = g + h^*$. *In order that every positive semidefinite function on* S *factor via* χ*, it is necessary that for every nonempty subset* $\mathscr{A}$ *of* E^2 *which is a union of equivalence classes with respect to* $\sim$ *and which is itself an equivalence relation on some subset of* E*, there should exist* $(e, f) \in \mathscr{A}$ *such that* $e + f^* \in \overline{A} + S$. *The condition is sufficient if the set* E *is finite* (*in particular, if* S *is finitely generated*). This is the condition that goes into the characterization of semiperfect (or equivalently, completely semiperfect) finitely generated abelian semigroups considered with the identical involution, in addition to the conditions necessary and sufficient for semiperfectness of $\chi(S)$.

A $*$-semigroup S is *facially countable* if each main face of S is countable, and *facially* C*-finite* if each main face of S is C-finite. We shall show that in the characterization of semiperfect (or equivalently, completely semiperfect) countable R-separative C-finite semigroups one can replace the conditions of countability and C-finiteness with the weaker conditions of facial countability and facial C-finiteness, otherwise changing nothing. To see that a proper generalization is involved, for every set A denote by $2^{(A)}$ the set of all finite subsets of A. Then the semigroup $I = (2^{(A)}, \cup)$ is a semilattice. The archimedean components of an arbitrary semilattice I are the sets $\{i\}$, $i \in I$. It easily follows that the main faces of I are the sets $X_i = \{ j \in I \mid j \leq i \}$, $i \in I$. Now if $I = 2^{(A)}$ then the main faces of I are finite, so I is facially countable, but I itself can of course be of arbitrarily large cardinality. Also, an arbitrary semilattice I can be shown to be C-finite if and only if for each $i \in I$ the set $[i, \infty[= \{ j \in I \mid j \geq i \}$ is finite, whereas for facial C-finiteness it suffices that the partially ordered set $(I, \leq)$ is *locally finite* in the sense that for all $i, j \in I$ such that $i \leq j$ the set $[i, j] = \{ k \in I \mid i \leq k \leq j \}$ is finite. Again, the example of $I = 2^{(A)}$ for an infinite set A shows the inequivalence of the conditions. From the point of view of the desire to characterize semiperfect semigroups, semilattices are uninteresting examples since every semilattice (being, in particular, an abelian inverse semigroup) is a perfect semigroup. However, note that if I is a semilattice, if $(S_i)_{i \in I}$ is a family of $*$-semigroups, and if (g_{ij}) is a family of $*$-homomorphisms $g_{ij}\colon S_i \to S_j$ $(i, j \in I, i \leq j)$ such that g_{ii} is the identity on S_i and $g_{ik} = g_{jk} \circ g_{ij}$ for all $i, j, k \in I$ such that

$i \leq j \leq k$ then the disjoint union $S = \bigcup_{i \in I} S_i$ becomes a $*$-semigroup when considered with the addition $+$ given by $x + y = g_{i,i+j}(x) + g_{j,i+j}(y)$ (sum in the semigroup S_{i+j}) for $i, j \in I$, $x \in S_i$, and $y \in S_j$, and the unique involution which extends the given one on each S_i ([18], Theorem 4.11, where it is not necessary for our present conclusions to assume that each S_i is a group).

A $*$-semigroup S is *locally countable* if each $*$-archimedean component of S is countable, and *locally C-finite* if each $*$-archimedean component of S is C-finite. In the characterization of semiperfect countable R-separative C-finite semigroups, it is not possible to replace 'countable' by 'locally countable'. This is shown by [5], Example 3. In the last section of the present paper, we shall discuss the question whether in the result referred to in the preceding paragraph the condition of facial C-finiteness can be replaced by the even weaker condition of local C-finiteness. It seems likely that this cannot be done. However, a proof of this would be so complicated that the inclusion of it would make the paper too long. (REMARK ADDED AFTER THE REST OF THIS PAPER WAS WRITTEN. In fact, it cannot be done. See the remark at the end of the paper.)

We mention one unpublished result that is a generalization of the main result of [7]. A *$*$-group* is an abelian group with involution, and a *$*$-subgroup* is a $*$-stable subgroup. The main result of [14] states that a countable C-separative C-finite semigroup S is semiperfect (or equivalently, completely semiperfect) if and only if $S = S + S$ and the following condition is satisfied:

(C) For each $*$-archimedean component H of S there exist a semigroup P, which is $\{0\}$, Z, or N and carries the identical involution, an abelian (torsion) group D carrying the inverse involution, and a $*$-subgroup G of the $*$-group $(P - P) \times D$ such that H is isomorphic to the $*$-semigroup $G \cap (P \times D)$.

The conditions $S = S + S$ and (C) are together sufficient for complete semiperfectness even if S is not countable and not C-finite.

The intended reading of '(torsion)' is the following. For a $*$-semigroup S satisfying (C), normalize the groups D occurring in (C) by the requirement that $D = \pi_2(H)$ where π_2 is the projection of $P \times D$ onto the second factor. (This is just a matter of replacing D, if necessary, with $\pi_2(D)$.) Then, if S is C-finite then the groups D are automatically torsion groups. However, the sufficiency part has only been proved under the hypothesis that the groups D are torsion groups.

NOTE ADDED IN PROOF. It has now been proved without that hypothesis. This result, announced in [14], implies that a C-separative finitely generated abelian $*$-semigroup S is semiperfect (or equivalently, completely semiperfect) if and only if it satisfies $S = S + S$ and (C). The assumption of C-separativity can be removed, as above, at the cost of adding a condition more.)

2. The mappings E, C, and Conv

Suppose S is a $*$-semigroup. The mappings E and C of the set of all subsets of S into itself which were defined in the Introduction will be denoted by E_S and C_S if it is necessary to specify S (such as when several semigroups are involved). We first list some useful combinatorial properties of these mappings. For every subset U of S define $U^\# = \{s + s^* \mid s \in U\}$. Note that $S^\#$ is a $*$-subsemigroup of S which carries the identical involution.

THEOREM 2.1. *If U and V are subsets of S, (V_α) is a family of subsets of S, and $s, t \in S$ then*

(i) *if $U \subset V$ then $U \cap E(V) \subset E(U)$;*
(ii) $U \subset C(U)$;
(iii) *if $U \subset V$ then $C(U) \subset C(V)$;*
(iv) *every finite subset of $C(U)$ is contained in a finite set W such that $E(W) \subset U$;*
(v) $C\big(C(U)\big) = C(U)$;
(vi) *if U is finite then $C(U) = C\big(E(U)\big)$;*
(vii) $E\big(C(U)\big) \subset E(U)$;
(viii) $s + t^* \in C(\{s + s^*, t + t^*\})$;
(ix) $U + \check{U} \subset C(U^\#)$;
(x) $E(\bigcup_\alpha V_\alpha) \subset \bigcup_\alpha E(V_\alpha)$;
(xi) $E(V \cap S^\#) \subset E(V)$.

PROOF. (i) through (vi): See [6], Lemmas 4.1 through 4.3.

(vii): See [7], Theorem 2.

(viii): As [6], Lemma 4.4, replacing '$x + y$' with '$x + y^*$' whenever $x, y \in \{s, t\}$.

(ix): Immediate from (viii) and (iii).

(x): Writing $V = \bigcup_\alpha V_\alpha$, for each α we have $E(V) \cap V_\alpha \subset E(V_\alpha)$ by (i). Since $E(V) \subset V$, it follows that $E(V) = E(V) \cap V = E(V) \cap \bigcup_\alpha V_\alpha = \bigcup_\alpha\big(E(V) \cap V_\alpha\big) \subset \bigcup_\alpha E(V_\alpha)$.

(xi): Suppose $v \in E(V \cap S^\#)$; we have to show $v \in E(V)$. Suppose $s, t \in S$, $s + s^*, t + t^* \in V$, and $s + t^* = v$; we have to show $s = t$. We have $s + s^*, t + t^* \in V \cap S^\#$. Since $v \in E(V \cap S^\#)$, it follows that $s = t$, as desired.

Properties (ii), (iii), and (v) are expressed by saying that the mapping C is a *closure operation*.

An abelian semigroup S is *torsion-free* if the conditions $x, y \in S$, $k \in \mathsf{N}$, and $kx = ky$ imply $x = y$.

For a finite subset V of a $*$-semigroup S, the set $E(V)$ is usually easy to determine. The task of determining $C(U)$ for a subset U of S is harder since one has, in principle, to check for *every* finite subset V of S whether $E(V) \subset U$. We next introduce another closure operation, Conv, such that $\mathrm{Conv}(U)$ is usually much easier to determine than is $C(U)$, and such that $C(U) \subset \mathrm{Conv}(U)$ whenever U is a subset of a C-separative semigroup. For every subset U of S, denote by $\mathrm{Conv}(U)$ the set of those $u \in S$ such that there exist $k, n \in \mathsf{N}$ and $u_1, \ldots, u_n \in U$ such that $(n+k)(u+u^*) = k(u+u^*) + u_1 + u_1^* + \cdots + u_n + u_n^*$. Clearly Conv is a closure operation. If the involution is the identity then $\mathrm{Conv}(U)$ is the set of those $u \in S$ such that there exist $k, n \in \mathsf{N}$ and $u_1, \ldots, u_n \in U$ such that $(n+k)u = ku + u_1 + \cdots + u_n$. If furthermore S is cancellative then we can subtract ku from both sides of the last equation, so that if in addition S is torsion-free then it is clear that $\mathrm{Conv}(U)$ is just the intersection of S with the convex hull of U in the enveloping real vector space. Hence the notation 'Conv'.

For every $*$-semigroup S we denote by ρ (or ρ_S, if S has to be specified) the quotient mapping of S onto its greatest R_+-separative $*$-homomorphic image. For $x, y \in S$ we have

$$\rho(x) = \rho(y) \Leftrightarrow \exists n \in \mathsf{N}\colon n(x + x^*) = n(y + y^*). \tag{3}$$

To see this, first suppose S carries the identical involution. We then have to show that $\rho(x) = \rho(y)$ if and only if $kx = ky$ for some $k \in \mathsf{N}$. If S has a zero, this is the bi-implication (ii)$\Leftrightarrow$(iii) of [4], Theorem 1. For a semigroup S without zero, it follows by applying the preceding to the semigroup obtained by adjoining a zero to S. This completes the case of the identical involution. In the general case, first suppose $n(x+x^*) = n(y+y^*)$ for some $n \in \mathsf{N}$. If σ is a nonnegative character on S then $\sigma(x)^{2n} = \sigma\big(n(x+x^*)\big) = \sigma\big(n(y+y^*)\big) = \sigma(y)^{2n}$. Since $\sigma \geq 0$ then we may take $2n$'th roots, to obtain $\sigma(x) = \sigma(y)$. This being so for all such σ, we have $\rho(x) = \rho(y)$ by the definition of ρ. Conversely, suppose $\rho(x) = \rho(y)$. For clarity, write $a = x+x^*$ and $b = y+y^*$. Then a and b are in the semigroup $S^\#$, which carries the identical involution. If τ is a nonnegative character on $S^\#$ then the function σ on S defined by $\sigma(s) = \tau(s+s^*)$ for $s \in S$ is obviously a nonnegative character on S, so $\tau(a) = \tau(x+x^*) = \sigma(x) = \sigma(y) = \tau(y+y^*) = \tau(b)$. This being so for all such τ, by the case of the identical involution (dealt with already) it follows that there is some $n \in \mathsf{N}$ such that $na = nb$, that is, $n(x+x^*) = n(y+y^*)$, as desired.

We see from the preceding that a $*$-semigroup S is R_+-separative if and only if the involution is the identity and S is torsion-free. Another equivalent condition is that the involution is the identity and each archimedean component of S is cancellative and torsion-free ([24], Theorem 0.1 p. 135).

For every abelian semigroup A with zero and every set V, we denote by $A[V]$ the set of all functions $a\colon V \to A$ which are *finitely supported* in the sense that $a(v) = 0$ for all but finitely many $v \in V$. With pointwise addition, $A[V]$ is an abelian semigroup. For every subset U of V we identify $A[U]$ with a subsemigroup of $A[V]$ by identifying each element of $A[U]$ with its zero extension which is an element of $A[V]$. The preceding definition gives a sense to the symbol $\mathsf{N}_0[V]$ but not to the symbol $\mathsf{N}[V]$. For each $n \in \mathsf{N}_0(V)$ and every subset U of V we define $n(U) = \sum_{u\in U} n(u)$. Now let $\mathsf{N}[V]$ be the set of those $n \in \mathsf{N}_0[V]$ such that $n(V) > 0$ (i.e., all except the constant 0). For each $v \in V$ define $\delta_v \in \mathsf{N}[V]$ by $\delta_v(u) = \delta_{u,v}$ (the Kronecker delta) for $u \in V$. If V is a subset of an abelian semigroup S, for each $n \in \mathsf{N}[V]$ define $\pi(n) = \sum_{v\in V} n(v)v$. Note that π is a homomorphism of $\mathsf{N}[V]$ into S. Each element n of $\mathsf{N}_0[V]$ is identified with the measure μ on V defined by $\mu(\{v\}) = n(v)$ for $v \in V$.

THEOREM 2.2. *Suppose S is a torsion-free abelian semigroup, V is a finite subset of S, and D is a subset of V. Let $f\colon V \to \mathsf{R}$ be any function. If for each $v \in V \setminus D$ there is some $n_v \in \mathsf{N}[V]$, not supported by $\{v\}$, such that $n_v(V)v = \pi(n_v)$ and $n_v(V)f(v) \le \int f\,dn_v$ then for each $v \in V \setminus D$ there is such an n_v supported by the set $D \cup \{v\}$.*

PROOF. Let $\mathscr{A}$ be the set of all subsets A of V such that for each $v \in V \setminus D$ there is some $n_v \in \mathsf{N}[V]$, supported by $A \cup \{v\}$ but not by $\{v\}$, such that $n_v(V)v = \pi(n_v)$ and $n_v(V)f(v) \le \int f\,dn_v$. By hypothesis, $V \in \mathscr{A}$, so $\mathscr{A}$ is nonempty. Since the finite set V has only finitely many subsets, we can choose $A \in \mathscr{A}$ minimal with respect to the inclusion ordering. If $A \subset D$, we are done. Suppose $A \not\subset D$; we shall derive a contradiction. Choose $a \in A \setminus D$ and define $B = A \setminus \{a\}$. If we show $B \in \mathscr{A}$ then we shall have the desired contradiction.

Since $A \in \mathscr{A}$ then for each $v \in V \setminus D$ there is some $n_v \in \mathsf{N}[V]$, supported by $A \cup \{v\}$ but not by $\{v\}$, such that $n_v(V)v = \pi(n_v)$ and $n_v(V)f(v) \le \int f\,dn_v$. Now suppose $v \in V \setminus D$; we have to show that there is some $m_v \in \mathsf{N}[V]$, supported by $B \cup \{v\}$ but not by $\{v\}$, such that $m_v(V)v = \pi(m_v)$ and $m_v(v)f(v) \le \int f\,dm_v$. If $v = a$ then as m_v we can use n_a. Thus we may (and do!) assume $v \ne a$.

Now

$$\begin{aligned}
&n_a(A)n_v(V)v \\
&\quad = n_a(A)\pi(n_v) \\
&\quad = n_a(A)\big(\pi(n_{v|V\setminus\{a\}}) + n_v(a)a\big) \\
&\quad = n_a(A)\pi(n_{v|V\setminus\{a\}}) + n_v(a)\pi(n_a) \\
&\quad = n_a(B)\pi(n_{v|V\setminus\{a\}}) + n_v(a)\pi(n_{a|B}) + n_a(a)\pi(n_{v|V\setminus\{a\}}) + n_v(a)n_a(a)a \\
&\quad = n_a(B)\pi(n_{v|V\setminus\{a\}}) + n_v(a)\pi(n_{a|B}) + n_a(a)\pi(n_v) \\
&\quad = n_a(B)\pi(n_{v|V\setminus\{a\}}) + n_v(a)\pi(n_{a|B}) + n_a(a)n_v(V)v.
\end{aligned}$$

This shows that if as m_v we take

$$m_v = n_a(B)n_{v|V\setminus\{a\}} + n_v(a)n_{a|B} + n_a(a)n_v(V)\delta_v \tag{4}$$

then m_v satisfies the requirement $m_v(V)v = \pi(m_v)$. A similar computation shows that the requirement $m_v(V)f(v) \leq \int f\,dm_v$ is also satisfied. It remains to be shown that m_v is not supported by $\{v\}$. Suppose it is; we shall derive a contradiction. Since n_a is, by hypothesis, supported by the set $A \cup \{a\} = A$ but not by $\{a\}$ then $n_a(B) > 0$. Since m_v is supported by $\{v\}$ then so is each term in (4). In particular, so is the first term. Since $n_a(B) > 0$ then $n_{v|V\setminus\{a\}}$ is supported by $\{v\}$. Thus n_v is supported by $\{a, v\}$. Since n_v is, by hypothesis, not supported by $\{v\}$ then $n_v(a) > 0$. Since the second term in (4) is supported by $\{v\}$ it follows that so is $n_{a|B}$. Since n_a is supported by the set $A = B \cup \{a\}$ it follows that n_a is likewise supported by $\{a, v\}$. Since (as we saw) $n_a(B) > 0$ then $n_a(v) > 0$.

We now have the following facts: The elements n_a and n_v are both supported by $\{a, v\}$, $n_a(v) > 0$, and $n_v(a) > 0$. This means, in more plain language, that there exist $p, q, r, s \in \mathsf{N}_0$ such that

$$(p+q)a = pa + qv \qquad \text{and} \qquad (r+s)v = ra + sv \tag{5}$$

and moreover, $q, r > 0$. We now invoke the hypothesis that S is torsion-free, which means that the nonnegative multiplicative functions on S separate points in S. For every such function σ, from (5) we get

$$\sigma(a)^{p+q} = \sigma(a)^p\sigma(v)^q \qquad \text{and} \qquad \sigma(v)^{r+s} = \sigma(a)^r\sigma(v)^s. \tag{6}$$

We wish to infer $\sigma(a) = \sigma(v)$. If one of these numbers is 0 then it is immediate from (6) that so is the other. In particular, they are equal. Thus we may assume that both are nonzero. Then from, say, the first equation in (6) we get $\sigma(a)^q = \sigma(v)^q$. Since $q > 0$ and since σ is nonnegative then we may take q'th roots, to obtain $\sigma(a) = \sigma(v)$. Thus this equality holds in every case. This being so for all such σ, we can infer $a = v$, the desired contradiction.

COROLLARY 2.3. *Suppose S is a torsion-free abelian semigroup, V is a finite subset of S, and D is a subset of V. If for each $v \in V \setminus D$ there is some $n_v \in \mathsf{N}[V]$, not supported by $\{v\}$, such that $n_v(V)v = \pi(n_v)$ then for each $v \in V \setminus D$ there is such an n_v supported by $D \cup \{v\}$.*

PROOF. Apply the Theorem with $f = 0$.

REMARK. Since the Corollary is all that will be used in the following, one may ask why in the Theorem we added the complication of the function f. Now in [3], proof of Lemma 4, we encountered the following situation. We had a finite subset E of a rational vector space, a function $\varphi: E \to \mathsf{R}$, and a family $(\pi_t)_{t\in E}$ of probability measures on E such that $t = \int u\, d\pi_t(u)$ and $\varphi(t) \leq \int \varphi\, d\varphi_t$ for all $t \in E$. A subset S of E was given by $t \in S \leftrightarrow \pi_t = \varepsilon_t$ where ε_t is the Dirac measure at t. The problem was to construct a family of measures with properties analogous to those of the family (π_t), but concentrated on the set S. In fact, the measures π_t were rational-valued, so the problem could be solved by the above Theorem. (Note that the solution in [3] was faulty, as pointed out in [11] where a solution similar to the present one was also indicated.)

It is well-known that a $*$-semigroup S is $(\mathsf{T} \cup \{0\})$-separative if and only if S is an abelian inverse semigroup. See Warne and Williams [30]. For every $*$-semigroup S, denote by π (or π_S if S has to be specified) the quotient mapping of S onto its greatest $(\mathsf{T} \cup \{0\})$-separative $*$-homomorphic image. If G is a $*$-group then $\pi(G)$ can be identified with the quotient $*$-group $G/\{\, x + x^* \mid x \in G \,\}$, which is the greatest $*$-homomorphic image of G which carries the inverse involution. For every $*$-semigroup S, denote by ι (or ι_S if S has to be specified) the mapping $x \mapsto \bigl(\pi(x), \rho(x)\bigr): S \to \pi(S) \times \rho(S)$.

THEOREM 2.4. *For a $*$-group G, the following three conditions are equivalent:*

(i) *G is C-separative*;

(ii) *the conditions $x \in G$, $k \in \mathsf{N}$, and $k(x + x^*) = 0$ imply $x + x^* = 0$;*

(iii) *the mapping ι is one-to-one.*

PROOF. (i)$\Rightarrow$(ii): Suppose $x \in G$, $k \in \mathsf{N}$, and $k(x + x^*) = 0$. For $\sigma \in G^*$ we have $|\sigma(x)|^{2k} = \sigma\bigl(k(x + x^*)\bigr) = \sigma(0) = 1$, hence $|\sigma(x)| = 1$, so $\sigma(x + x^*) = |\sigma(x)|^2 = 1 = \sigma(0)$. This being so for all such σ, since G is C-separative it follows that $x + x^* = 1$, as desired.

(ii)$\Rightarrow$(iii): Suppose $x \in G$ is such that $\iota(x) = 0$; we have to show that $x = 0$. The fact that $\iota(x) = 0$ means that $\pi(x) = 0$ and $\rho(x) = 0$. Since $\pi(x) = 0$

then there is some $y \in G$ such that $x = y + y^*$. Since $\rho(x) = 0$ then by (3) there is some $n \in \mathsf{N}$ such that $n(x+x^*) = 0$. Now $0 = n(x+x^*) = 2n(y+y^*)$. By the hypothesis it follows that $0 = y + y^* = x$, as desired.

(iii)⇒(i): The mapping ι is an embedding of G into the product of two C-separative semigroups, so G is C-separative.

REMARK. On the strength of the preceding Theorem, it might be thought that a $*$-semigroup S would be C-separative if and only if the conditions $x, y \in S$, $k \in \mathsf{N}$, and $k(x+x^*) = k(y+y^*)$ imply $x+x^* = y+y^*$. A counterexample is afforded by the semigroup $S = \{u, v\} \cup \{2, 3, 4, \ldots\}$ where u and v are two distinct elements, $2u = u + v = 2v = 2$, $u + n = v + n = n + 1$ for $n \geq 2$, and addition of integers is the usual one.

For every abelian semigroup S, define $2S = \{\, 2s \mid s \in S \,\}$.

THEOREM 2.5. *If S is a C-separative semigroup then the mapping ρ is one-to-one on $S^\#$. It follows that $S^\#$ is isomorphic to $\rho(S)$. In particular, $S^\#$ is R_+-separative.*

PROOF. Suppose $a, b \in S^\#$ and $\rho(a) = \rho(b)$; we have to show $a = b$. Choose $x, y \in S$ such that $a = x + x^*$ and $b = y + y^*$. If $\sigma \in S^*$ then $|\sigma|$ is a nonnegative character on S, so $|\sigma(a)| = |\sigma(b)|$. But $\sigma(a) = \sigma(x + x^*) = |\sigma(x)|^2 \geq 0$ and similarly for b, so $\sigma(a) = \sigma(b)$. This being so for all such σ, since S is C-separative it follows that $a = b$, as desired. This proves that ρ is one-to-one. Now $\rho|S^\#$ is an isomorphism between $S^\#$ and the semigroup $\rho(S^\#) = 2\rho(S)$, which is isomorphic to $\rho(S)$ since $\rho(S)$, being R_+-separative, is torsion-free.

A $*$-semigroup S is *$*$-separative* if the conditions $x, y \in S$ and $x + x^* = y + x^* = x + y^* = y + y^*$ imply $x = y$. The conditions is equivalent to each $*$-archimedean component of S being cancellative [12]. It is easy to see that every C-separative semigroup is $*$-separative.

THEOREM 2.6. *If S is a C-separative semigroup then for every finite subset V of S we have $V \subset \operatorname{Conv}\big(E(V)\big)$. Hence, $C(U) \subset \operatorname{Conv}(U)$ for every subset U of S.*

PROOF. First suppose V is a finite subset of S; we have to show $V \subset \operatorname{Conv}\big(E(V)\big)$. If $v \in E(V)$ then trivially $v \in \operatorname{Conv}\big(E(V)\big)$. Thus we only have to show $V \setminus E(V) \subset \operatorname{Conv}\big(E(V)\big)$. Suppose $v \in V \setminus E(V)$. By the definition of $E(V)$, there exist $s, t \in S$ such that $s+s^*, t+t^* \in V$, $s+t^* = v$, and $s \neq t$. From $s + t^* = v$ we get $v + v^* = s + s^* + t + t^*$, so writing $x = s+s^* \in V \cap S^\#$ and $y = t+t^* \in V \cap S^\#$ we have $2(v+v^*) = x+x^*+y+y^*$, which shows $v \in \operatorname{Conv}(\{x, y\}) \subset \operatorname{Conv}(V \cap S^\#)$. It thus suffices to show

$\mathrm{Conv}(V \cap S^{\#}) \subset \mathrm{Conv}\big(E(V)\big)$, and for this it obviously suffices to show $V \cap S^{\#} \subset \mathrm{Conv}\big(E(V)\big)$. If we show $V \cap S^{\#} \subset \mathrm{Conv}\big(E(V \cap S^{\#})\big)$ then the desired inclusion follows since $E(V \cap S^{\#}) \subset E(V)$ (Theorem 2.1 (xi)).

We have reduced the problem concerning V to the corresponding problem for the subset $V \cap S^{\#}$ of the semigroup $S^{\#}$, which is R_+-separative (hence torsion-free) by the preceding Theorem. Replacing S by $S^{\#}$ and V by $V \cap S^{\#}$, it might thus seem that we could assume that S is torsion-free. This argument is not quite correct since the mappings E and C for the semigroup $S^{\#}$ may be different from those for S. Thus we have to be more careful. We have to show $V \cap S^{\#} \subset \mathrm{Conv}\big(E(V \cap S^{\#})\big)$. Since trivially $E(V \cap S^{\#}) \subset \mathrm{Conv}\big(E(V \cap S^{\#})\big)$, it suffices to show $(V \cap S^{\#}) \setminus E(V \cap S^{\#}) \subset \mathrm{Conv}\big(E(V \cap S^{\#})\big)$. This amounts to showing that for each $v \in (V \cap S^{\#}) \setminus E(V \cap S^{\#})$ there is some $n_v \in \mathsf{N}[V]$, supported by $E(V \cap S^{\#}) \cup \{v\}$ but not by $\{v\}$, such that $n_v(V)v = \pi(n_v)$. By Corollary 2.3 it suffices to show that there is such an n_v, not necessarily supported by $E(V \cap S^{\#}) \cup \{v\}$. Since $v \notin E(V \cap S^{\#})$, there exist $s, t \in S$ such that $s + s^*, t + t^* \in V \cap S^{\#}$, $s + t^* = v$, and $s \neq t$. Writing $x = s + s^*$ and $y = t + t^*$, we infer $v + v^* = x + y$. Since $v \in S^{\#}$ then $v = v^*$, so $2v = x + y$. Thus we can use $n_v = \delta_x + \delta_y$, provided that this function is not supported by $\{v\}$. Suppose it is. This means that $x = y = v$, so $s + s^* = t + t^* = v$. Since we also have $s + t^* = v$, by $*$-separativity it follows that $s = t$, a contradiction.

A $*$-semigroup S is *normal* if $C(\emptyset) = \emptyset$.

COROLLARY 2.7. *Every* C*-separative semigroup is normal.*

PROOF. $C(\emptyset) \subset \mathrm{Conv}(\emptyset) = \emptyset$.

3. A criterion for semiperfectness

We have already introduced the semigroup $A[S]$ for an abelian semigroup A with zero and a set S. If A is a group, so is $A[S]$. If A is a vector space, so is $A[S]$. If A is a ring and S is an abelian semigroup then $A[S]$ is made into a ring by introducing the multiplication $*$ (convolution) defined by

$$a * b(u) = \sum_{s,t \in S: s+t=u} a(s)b(t)$$

for $a, b \in A[S]$ and $u \in S$. The symbol $\sum_{s,t \in S: s+t=u}$ can be abbreviated $\sum_{s+t=u}$ if S is understood. If the ring A carries an *involution*, that is, an involutory anti-automorphism written $x \mapsto x^*$, and if S is a $*$-semigroup, then the ring $A[S]$ is considered with the involution $a \mapsto \widetilde{a}$ defined by $\widetilde{a}(s) = a(s^*)^*$ for $a \in A[S]$ and $s \in S$. If A is a $*$*-algebra*, that is, a complex algebra with involution, then so is $A[S]$.

We review the generalization to arbitrary semigroups of the criterion of Haviland [22] for a multisequence to be a moment sequence. Suppose S is a $*$-semigroup. For $d \in \mathsf{N}$ let $M_d(\mathsf{C})$ be the algebra of square complex matrices of order d with the adjoint operation as involution. For every subset T of S, define a bilinear form $\langle\cdot,\cdot\rangle$ on $M_d(\mathsf{C})[T] \times M_d(\mathsf{C})^T$ by

$$\langle a, \varphi\rangle = \sum_{s\in S} \operatorname{tr}\bigl(a(s)\varphi(s^*)\bigr)$$

for $a \in M_d(\mathsf{C})[T]$ and $\varphi \in M_d(\mathsf{C})^T$ where $\operatorname{tr}(x)$ denotes the trace of $x \in M_d(\mathsf{C})$. Under this bilinear form, the spaces $M_d(\mathsf{C})[T]$ and $M_d(\mathsf{C})^T$ are in duality, cf. [2], Ch. 1. The finest locally convex topology on $M_d(\mathsf{C})[T]$, and the topology of pointwise convergence on $M_d(\mathsf{C})^T$, are compatible with the duality. (We always consider these spaces with these topologies.) For every subset A of $M_d(\mathsf{C})[T]$ we define a closed convex cone $A^\perp$ in $M_d(\mathsf{C})^T$ as the set of those $\varphi \in M_d(\mathsf{C})^T$ such that $\langle a, \varphi\rangle \geq 0$ for all $a \in A$. For every subset U of S, define a convex cone $\Sigma_d(U)$ in $M_d(\mathsf{C})[U + \check{U}]$ by

$$\Sigma_d(U) = \{\,\widetilde{a}_1 * a_1 + \cdots + \widetilde{a}_n * a_n \mid a_1, \ldots, a_n \in M_d(\mathsf{C})[U]\,\}.$$

Then

$$\mathscr{P}(S, \mathsf{C}^d) = \Sigma_d(S)^\perp, \tag{7}$$

cf. [5], Proposition 3 (in which the last statement contains an error, to be corrected in [9]). For $a \in M_d(\mathsf{C})[S]$ define $\widehat{a}\colon S^* \to M_d(\mathsf{C})$ by

$$\widehat{a}(\sigma) = \sum_{s\in S} \sigma(s)a(s)$$

for $\sigma \in S^*$. Note that $(a*b)\widehat{\ } = \widehat{a}\cdot\widehat{b}$ (pointwise multiplication) and $(\widetilde{a})\widehat{\ } = (\widehat{a})^*$ (pointwise adjoint operation) for $a, b \in M_d(\mathsf{C})[S]$. For every subset A of $M_d(\mathsf{C})[S]$, denote by A_+ the set of those $a \in A$ such that $\widehat{a}(\sigma)$ is a positive semidefinite matrix for each $\sigma \in S^*$. Denote by A^{sa} the set of those $a \in A$ such that $a = a^*$, and write $A_+^{\mathrm{sa}} = A_+ \cap A^{\mathrm{sa}}$. Then

$$\mathscr{H}(S, \mathsf{C}^d) \subset \bigl(M_d(\mathsf{C})[S + S]_+^{\mathrm{sa}}\bigr)^\perp, \tag{8}$$

cf. [5], Proposition 3, where the presence of a zero is not used in the proof of this inclusion. The $*$-semigroup S is *adapted* if for each $x \in S$ there exist $n \in \mathsf{N}$ and $y_1, \ldots, y_{n+1} \in S$ such that $n(x + x^*) = y_1 + y_1^* + \cdots + y_{n+1} + y_{n+1}^*$. Every semiperfect semigroup is adapted [15]. In the inclusion (8), equality holds if and only if S is adapted [15]. Using the Hahn-Banach Theorem and

(7), one sees that a $*$-semigroup S is semiperfect of order d if and only if S is adapted and $\Sigma_d(S)$ is dense in $M_d(\mathsf{C})[S+S]_+^{\mathrm{sa}}$, cf. [15]. (The error in the last statement of [5], Proposition 3, consisted in writing $M_d(\mathsf{C})[S+S]_+$ instead of $M_d(\mathsf{C})[S+S]_+^{\mathrm{sa}}$.) For this reason, it is very interesting if one can show that $\Sigma_d(S)$ is closed in the finest locally convex topology. Our next aim is to show that this is so if S is a countable normal C-finite semigroup.

For $a \in M_d(\mathsf{C})$, write $a > 0$ if a is positive semidefinite and nonzero. Denote by $\operatorname{supp} a$ the set of those $s \in S$ such that $a(s) \neq 0$.

THEOREM 3.1. *If S is a $*$-semigroup, $a_1, \ldots, a_n \in M_d(\mathsf{C})[S]$, and $v \in E\big((\bigcup_{j=1}^n \operatorname{supp} a_j)^\#\big)$ then $\sum_{j=1}^n \widetilde{a}_j * a_j(v) > 0$.*

PROOF. Write $T = \bigcup_{j=1}^n \operatorname{supp} a_j$. We have

$$\sum_{j=1}^n \widetilde{a}_j * a_j(v) = \sum_{j=1}^n \sum_{s,t\in T: s+t^*=v} a_j(t)^* a_j(s)$$

since the terms excluded are zero. Writing $V = T^\#$, if $s, t \in T$ and $s + t^* = v$ then $s + s^*, t + t^* \in V$, and since $v \in E(V)$ it follows that $s = t$. Thus the above formula reduces to

$$\sum_{j=1}^n \widetilde{a}_j * a_j(v) = \sum_{j=1}^n \sum_{s\in T: s+s^*=v} a_j(s)^* a_j(s).$$

Each term is positive semidefinite, and since $v \in E(T^\#) \subset T^\#$ then at least one is nonzero. Since a sum of nonzero positive semidefinite matrices cannot be zero, the claim follows.

THEOREM 3.2. *If S is a $*$-semigroup and $a_1, \ldots, a_n \in M_d(\mathsf{C})[S]$ then*

$$C\Big(\operatorname{supp} \sum_{j=1}^n \widetilde{a}_j * a_j\Big) = C\Big[\Big(\bigcup_{j=1}^n \operatorname{supp} a_j\Big)^\#\Big].$$

PROOF. Write $a = \sum_{j=1}^n \widetilde{a}_j * a_j$ and $T = \bigcup_{j=1}^n \operatorname{supp} a_j$; we have to show $C(\operatorname{supp} a) = C(T^\#)$. We have $\operatorname{supp} a \subset T + \check{T} \subset C(T^\#)$ by Theorem 2.1 (ix). Hence $C(\operatorname{supp} a) \subset C\big(C(T^\#)\big) = C(T^\#)$ by Theorem 2.1 (iii) and (v). For the converse inclusion, note that by the preceding Theorem we have $E(T^\#) \subset \operatorname{supp} a$. Since $T^\#$ is a finite set then $C(T^\#) = C\big(E(T^\#)\big) \subset C(\operatorname{supp} a)$ by Theorem 2.1 (vi) and (iii).

THEOREM 3.3. *Every normal semigroup is $*$-separative.*

PROOF. Suppose S is a normal semigroup; we have to show that S is $*$-separative. Suppose $s, t \in S$ and $s + s^* = t + s^* = s + t^* = t + t^* = v$, say; we have to show that $s = t$. Let V be the finite nonempty set $\{v\}$. If we had $E(V) = \emptyset$ then it would follow that $V \subset C(\emptyset)$, contradicting normality. Thus $E(V) \neq \emptyset$. Since $E(V)$ is a subset of V it follows that $v \in E(V)$. Since $s + s^*, t + t^* \in V$ and $s + t^* = v$ it follows that $s = t$, as desired.

A $*$-semigroup S is *2-finite* if for each $t \in S$ there are only finitely many $s \in S$ such that $t = s + s^*$.

THEOREM 3.4. *Every $*$-separative C-finite semigroup is 2-finite.*

PROOF. Suppose S is a $*$-separative C-finite semigroup; we have to show that S is 2-finite. Suppose $u \in S$; we have to show that the set $A = \{ s \in S \mid u = s + s^* \}$ is finite. If A is empty, we are done. Suppose A is nonempty. Choose $a \in A$. For $s \in A$ we have $s + a^* \in C(\{s + s^*, a + a^*\}) = C(\{u\})$ by Theorem 2.1 (viii). The right-hand side is a finite set since S is C-finite. Thus the set $A + a^*$ is finite. Let H be the unique $*$-archimedean component of S containing u. For $s \in A$, if I is the unique $*$-archimedean component of S containing s then $u = s + s^* \in H \cap I$. Since distinct $*$-archimedean components are disjoint it follows that $I = H$. Thus $A \subset H$. Since S is $*$-separative then H is cancellative, so the mapping $x \mapsto x + a^* \colon H \to H$ is one-to-one. Since the set $A + a^*$ is finite, it follows that so is A.

THEOREM 3.5. *If S is a countable normal C-finite semigroup and $d \in \mathbb{N}$ then the convex cone $\Sigma_d(S)$ is closed in the finest locally convex topology on $M_d(\mathbb{C})[S + S]$. Hence, S is semiperfect of order d if and only if $\Sigma_d(S) = M_d(\mathbb{C})[S + S]^{\mathrm{sa}}_{+}$.*

PROOF. By [2], 6.3.3, it suffices to show that $\Sigma_d(S) \cap M_d(\mathbb{C})[U]$ is closed, in the canonical topology on a finite-dimensional space, for every finite subset U of S. It even suffices to show that $\Sigma_d(S) \cap M_d(\mathbb{C})[V]$ is closed for every finite subset V of S satisfying $V = C(V)$. Indeed, every finite subset U of S is contained in such a set V, viz., the set $V = C(U)$. (Use Theorem 2.1 (ii) and (v).)

So suppose V is a finite subset of S satisfying $V = C(V)$; we have to show that $\Sigma_d(S) \cap M_d(\mathbb{C})[V]$ is closed. Let U be the set of those $s \in S$ such that $s + s^* \in V$. Since S is normal then S is $*$-separative by Theorem 3.3. Since S is also C-finite, by Theorem 3.4 it follows that S is 2-finite. Thus the set U is finite. Now

$$\Sigma_d(V) \cap M_d(\mathbb{C})[V] = \Sigma_d(U). \tag{9}$$

To see this, first note that for $a \in \Sigma_d(U)$ we have $\operatorname{supp} a \subset U + \check{U} \subset C(U^{\#}) \subset C(V) = V$ by Theorem 2.1 (ix) and (iii). For the converse inclusion, suppose

$a \in \Sigma_d(S) \cap M_d(\mathsf{C})[V]$; we have to show $a \in \Sigma_d(U)$. Choose $a_1, \ldots, a_n \in M_d(\mathsf{C})[S]$ such that $a = \widetilde{a}_1 * a_1 + \cdots + \widetilde{a}_n * a_n$. Since $\operatorname{supp} a \subset V$ then, writing $T = \bigcup_{j=1}^n \operatorname{supp} a_j$, we have $V = C(V) \supset C(\operatorname{supp} a) = C(T^\#) \supset T^\#$ by Theorem 2.1 (iii), Theorem 3.2, and Theorem 2.1 (ii). By the definition of U it follows that $T \subset U$, that is, the a_j are supported by U, so $a \in \Sigma_d(U)$, as desired. This proves (9).

It now suffices to show that if U is a finite subset of S then the convex cone $\Sigma_d(U)$ is closed in the canonical topology on the finite-dimensional space $M_d(\mathsf{C})[U + \check{U}]$. Choose a compact subset B of $M_d(\mathsf{C})[U] \backslash \{0\}$ which intersects every ray from the origin. Then the set $\{\, \widetilde{b} * b \mid b \in B \,\}$ is again compact, and so, therefore, is its convex hull K. Now $\Sigma_d(U) = \{\, \lambda k \mid k \in K,\ \lambda \geq 0 \,\}$, so it suffices to show $0 \notin K$. Given $c \in K$, we have to show $c \neq 0$. But $c = \widetilde{c}_1 * c_1 + \cdots + \widetilde{c}_n * c_n$ for some $c_1, \ldots, c_n \in M_d(\mathsf{C})[U] \backslash \{0\}$ $(n \geq 1)$. Writing $T = \bigcup_{j=1}^n \operatorname{supp} c_j$, we have $C(\operatorname{supp} c) = C(T^\#) \supset T^\# \neq \emptyset$ by Theorem 3.2 and Theorem 2.1 (ii). Since S is normal it follows that $\operatorname{supp} c \neq \emptyset$, that is, $c \neq 0$, as desired.

4. Semiperfect facially countable R-separative facially C-finite semigroups

In this section we shall solve the characterization problem indicated in the title. Indeed, we shall show that a facially countable R-separative facially C-finite semigroup S is semiperfect (or equivalently, completely semiperfect) if and only if $S = S + S$ and condition (B) is satisfied. The sufficiency part is covered by the main result of [7]. Thus we only have to show that if S is a facially countable R-separative facially C-finite semigroup then $S = S + S$ and condition (B) is satisfied.

We write $\Sigma(S) = \Sigma_1(S) \subset \mathsf{C}[S + S]_+$, identifying $M_1(\mathsf{C})$ with C by identifying a square matrix of order 1 with its unique entry.

We first note that it suffices to treat the case that S is countable and C-finite. Indeed, suppose this case has been dealt with. Suppose S is a semiperfect facially countable R-separative facially C-finite semigroup; we have to show that $S = S + S$ and that condition (B) is satisfied. It suffices to verify that if $H \in \mathscr{J}(S)$ then $H \subset S + S$ and H has the structure specified in (B). Let X be the least face of S containing H (i.e., $X = X_H$ in the notation of the Introduction). Then X is a main face of S. Since S is facially C-finite then X is C-finite. Since S is R-separative, so is X. Since S is facially countable then X is countable. Being a face of the semiperfect semigroup S, X is semiperfect [25]. By the supposed solution of the special case it follows that $X = X + X$ and that X satisfies condition (B). Now $H \subset X = X + X \subset S + S$. To see that H has the structure specified in (B), it suffices to verify that H is an archimedean component of X. But this follows from the following result.

LEMMA 4.1. *If H is a $*$-archimedean component of a $*$-semigroup S, and if H is contained in a $*$-subsemigroup X of S, then H is a $*$-archimedean component of X.*

PROOF. Being a $*$-archimedean $*$-subsemigroup of X, H is contained in a unique $*$-archimedean component of X, say K. Being a $*$-archimedean $*$-subsemigroup of S, K is contained in a unique $*$-archimedean component of S, say L. Since $H \subset L$, by the definition of a $*$-archimedean component it follows that $H = L$. Now $H \subset K \subset L = H$, so $H = K$, which shows that H is a $*$-archimedean component of X.

Thus it suffices to show that if S is a semiperfect countable R-separative C-finite semigroup then $S = S + S$ and condition (B) is satisfied. Since S is R-separative, it is, in particular, C-separative. By Corollary 2.7 it follows that S is normal. By Theorem 3.5 it now follows that $\Sigma(S) = \mathsf{C}[S + S]_+$.

THEOREM 4.2. *If S is a C-separative semigroup satisfying $\Sigma(S) = \mathsf{C}[S + S]_+$ then for every subset U of $S^\#$ we have $C(U) \cap (S+S) = \mathrm{Conv}(U) \cap (S+S)$.*

PROOF. One inclusion is immediate from Theorem 2.6. For the converse, suppose $v \in \mathrm{Conv}(U) \cap (S + S)$; we have to show $v \in C(U)$. Since $v \in \mathrm{Conv}(U)$, we can choose $n, k \in \mathsf{N}$ and $u_1, \dots, u_n \in U$ such that

$$(n + k)(v + v^*) = k(v + v^*) + u_1 + u_1^* + \cdots + u_n + u_n^*. \tag{10}$$

Note that $U \subset S^\# \subset S + S$. Define $a \in \mathsf{C}[S + S]$ by

$$a = \delta_{u_1} + \cdots + \delta_{u_n} - \frac{n}{2}(\delta_v + \delta_{v^*}).$$

We claim that $a \in \mathsf{C}[S + S]_+$. To see that this is so, suppose $\sigma \in S^*$; we have to show $\langle a, \sigma\rangle \geq 0$. We have

$$\langle a, \sigma\rangle = \sigma(u_1) + \cdots + \sigma(u_n) - n \operatorname{Re} \sigma(v).$$

Thus we have to show

$$\operatorname{Re} \sigma(v) \leq \frac{1}{n}\big(\sigma(u_1) + \cdots + \sigma(u_n)\big). \tag{11}$$

This is trivial if $\sigma(v) = 0$ since we have $\sigma(u_j) \geq 0$ for each j because of $u_j \in U \subset S^\#$. Thus we may assume $\sigma(v) \neq 0$. From (10) we get $|\sigma(v)|^{n+k} = |\sigma(v)|^k \sigma(u_1) \dots \sigma(u_n)$. Dividing by the nonzero number $|\sigma(v)|^k$ and taking n'th roots, we obtain

$$|\sigma(v) \leq \sqrt[n]{\sigma(u_1) \dots \sigma(u_n)},$$

and (11) follows by the arithmetic-geometric inequality. This proves $a \in \mathsf{C}[S+S]_+$. By the hypothesis it follows that $a \in \Sigma(S)$. Thus we can choose $a_1, \ldots, a_m \in \mathsf{C}[S]$ such that $a = \widetilde{a}_1 * a_1 + \cdots + \widetilde{a}_m * a_m$. Define $T = \bigcup_{j=1}^{n} \operatorname{supp} a_j$ and $V = T^{\#}$. By Theorem 3.1, if $u \in E(V)$ then $a(u) > 0$, so $u \in \{u_1, \ldots, u_n\} \subset U$. Thus $E(V) \subset U$. Since V is a finite set it follows that $V \subset C(U)$. Therefore $C(V) \subset C\big(C(U)\big) = C(U)$ by Theorem 2.1 (iii) and (v). Thus it suffices to show $v \in C(V)$. This is trivial if $v \in \{u_1, \ldots, u_n\}$. Otherwise, we have $a(v) < 0$, so $v \in \operatorname{supp} a \subset T + \check{T} \subset C(T^{\#}) = C(V)$ by Theorem 2.1 (ix).

In the following, we first consider a torsion-free cancellative C-finite semigroup S carrying the identical involution and satisfying $C(U) \cap (2S) = \operatorname{Conv}(U) \cap (2S)$ for every subset U of the set $S^{\#} = 2S$. Denote by R the enveloping rational vector space of S. A subset of R consists of *equidistant points* if it is a finite, one-sided infinite, or two-sided infinite sequence of points which are equidistant in the sense that the difference between consecutive points in the sequence is a constant. If Y is a 1-dimensional affine subspace of R then the set $Y \cap (2S)$ consists of equidistant points ([7], Lemma 3). If Y is a 1-dimensional linear subspace of R which intersects S then the semigroup $Y \cap T$ is isomorphic to a subsemigroup of Z ([7], Lemma 5). We note that this semigroup, consisting as it does of equidistant points, must be isomorphic to $\{0\}$, Z, or $\mathsf{N}_p := \{n \in \mathsf{N}_0 \mid n \geq p\}$ for some $p \in \mathsf{N}_0$. Now the dimension of R is at most 1. The easiest way to see this is to follow the proof of [7], Lemma 6, making the necessary changes. Note that the result does not follow directly from [7], Lemma 6, since in that result it was assumed that $\Sigma(S) = \mathsf{C}[S]_+$. The first change concerns the space Y introduced in [7] on p. 153 l. 1 from below. Instead of finding $t \in S$ such that

$$Y \cap S = \{mt \mid m \in M\} \tag{12}$$

with $M = \mathsf{Z}$ or $M = \mathsf{N}$, we find some $t \in S - S$ such that (12) holds with $M = \mathsf{Z}$ or $M = \mathsf{N}_p$ for some $p \in \mathsf{N}_0$. Now in [7], equation (4), N should be replaced by N_q for some $q \in \mathsf{N}$ (equal to p if $p > 0$, but equal to 1 if $p = 0$). When it comes to proving that the point $y = v + s$ is in $2S$, we cannot conclude this immediately from the fact that $2S$ is a semigroup. This is because the point s is now not necessarily in $2S$. Instead, note that the set $v + Y$ is a 1-dimensional affine subspace of R which intersects $2S$ in a set B that contains the set $A = \{v, v + (k-1)s, ks\}$. Since B consists of equidistant points, it must also contain the point $v + s$ which is an affine combination of two points of A with integer coefficients and at the same time is in the convex hull of A (i.e., the set of convex combinations of points of A with coefficients that are rationals). This argument must be repeated once. Otherwise, everything is as in [7], Lemma 6.

An *ideal* of a $*$-semigroup S is a $*$-subsemigroup H such that $S+H \subset H$.

LEMMA 4.3. *Every cancellative $*$-semigroup which has a C-finite ideal is C-finite.*

PROOF. Suppose S is a cancellative $*$-semigroup having a C-finite ideal H; we have to show that S is C-finite. Since two semigroups are involved, the mappings E and C will be given the name of the semigroup as subscript. Suppose U is a finite subset of S; we have to show that the set $C_S(U)$ is finite. Choose $a \in H$. Since S is cancellative then the mapping $x \mapsto a + a^* + x$ is one-to-one. Thus it suffices to show that the set $a+a^*+C_S(U)$ is finite. Since the set U is finite, so is the set $a + a^* + U \subset H$. Since H is C-finite then the set $C_H(a + a^* + U)$ is finite. Thus it suffices to show that $a + a^* + C_S(U) \subset C_H(a + a^* + U)$. Suppose V is a finite subset of S such that $E_S(V) \subset U$; we have to show $a + a^* + V \subset C_H(a + a^* + U)$. Since the set $a + a^* + V$ is a finite subset of H, it suffices to show $E_H(a+a^*+V) \subset a+a^*+U$. Suppose $z \in E_H(a + a^* + V)$; we have to show $z \in a + a^* + U$. Since $z \in a + a^* + V$ then there is some $v \in V$ such that $z = a + a^* + v$. (In fact, v is unique, by cancellativity.) It now suffices to show $v \in U$. Since $E_S(V) \subset U$, it suffices to show $v \in E_S(V)$. Suppose $s, t \in S$, $s + s^*, t + t^* \in V$, and $s + t^* = v$; we have to show $s = t$. With $x = a + s$ and $y = a + t$ we have $x, y \in H$, $x + x^* = a + a^* + s + s^* \in a + a^* + V$, $y + y^* \in a + a^* + V$ (similarly), and $x + y^* = a + a^* + s + t^* = a + a^* + v = z$. Since $z \in E_H(a + a^* + V)$, it follows that $x = y$, that is, $a + s = a + t$. Since S is cancellative it follows that $s = t$, as desired.

THEOREM 4.4. *If S is a semiperfect locally countable torsion-free cancellative locally C-finite semigroup carrying the identical involution then S is isomorphic to $\{0\}$, Z, or N_0.*

PROOF. First suppose S is countable and C-finite. As we have seen, it follows that $\Sigma(S) = \mathsf{C}[S + S]_+$, and by Theorem 4.2 it follows that $C(U) \cap (S + S) = \mathrm{Conv}(U) \cap (S + S)$ for every subset U of $2S$. In particular, $C(U) \cap (2S) = \mathrm{Conv}(U) \cap (2S)$ for every subset U of $2S$. As we just saw, it follows that the dimension of the space R is at most 1. If the dimension is 0 then $S = \{0\}$, and we are done. Suppose the dimension is 1. Applying [7], Lemma 5, to the 1-dimensional space $Y = R$, we see that S is isomorphic to a subsemigroup of Z which, since it consists of equidistant points, must be isomorphic to $\{0\}$, Z, or N_p for some $p \in \mathsf{N}_0$. The case of $\{0\}$ is excluded since the dimension is 1. If S is isomorphic to Z, we are done. Suppose S is isomorphic to N_p for some $p \in \mathsf{N}_0$. If $p = 0$, we are done. Thus it suffices to show that for $p \in \mathsf{N}$ the semigroup N_p is not semiperfect. Since every semiperfect semigroup is adapted, it suffices to show that N_p is not adapted, that is, there is some $x \in \mathsf{N}_p$ for which there do not exist $n \in \mathsf{N}$ and $y_1, \ldots, y_{n+1} \in \mathsf{N}_p$ such that $nx = y_1 + \cdots + y_{n+1}$. It obviously suffices to take $x = p$.

(Though it is not needed, we give a proof of the non-semiperfectness of N_p which is somewhat longer but has the merit of not referring to an unpublished source. Let φ be the indicator function of the set $\{2p\}$ as a subset of the set $\mathsf{N}_p + \mathsf{N}_p = \mathsf{N}_{2p}$. If $n \geq p$ and $c_p, \ldots, c_n \in \mathsf{R}$ then

$$\sum_{j,k=p}^{n} c_j c_k \varphi(j+k) = c_p^2 \geq 0.$$

Thus φ is positive semidefinite. Supposing that N_p is semiperfect, we infer that φ is a moment function, that is, there is a measure μ on N_p^* such that $\varphi(n) = \int_{\mathsf{N}_p^*} \sigma(n)\, d\mu(\sigma)$ for $n \in \mathsf{N}_p$. Now $1 = \varphi(2p) = \int \sigma(p)^2\, d\mu(\sigma)$, so $\mu(\{\sigma \in \mathsf{N}_p^* \mid \sigma(p) \neq 0\}) > 0$, hence $0 < \int \sigma(p)^4\, d\mu(\sigma) = \varphi(4p) = 0$, a contradiction.)

Now consider the general case. If $H \in \mathscr{J}(S)$ then the semigroup X_H is a semiperfect countable torsion-free cancellative semigroup, which has the C-finite ideal H and therefore is C-finite by the Lemma. Hence X_H is isomorphic to $\{0\}$, Z, or N_0. If S has only finitely many archimedean components then it has a greatest one with respect to the canonical ordering on the semilattice $\mathscr{J}(S)$ (the 'sum' of all of them with respect to the operation $\vee$). Assume that S has infinitely many archimedean components; we shall derive a contradiction. In fact, a contradiction will follow already from the hypothesis that S has at least 3 archimedean components. Choose 3 distinct elements $H, K, L \in \mathscr{J}(S)$. Define $M = H \vee K \vee L \in \mathscr{J}(S)$. Then X_M is isomorphic to $\{0\}$, Z, or N_0 and so has at most 2 archimedean components, contradicting the fact that it has at least 3 (viz., H, K, and L).

REMARK. It is worth contemplating whether in the preceding Theorem one could omit the words 'locally countable'. It is unknown whether one can do so, even if S is assumed to be a group. A countable $*$-group is semiperfect if and only if the $*$-group $(\mathsf{Z}^2, n^* = n)$ is not a $*$-homomorphic image of it (and perfect if and only if $(\mathsf{Z}, n^* = n)$ is not a $*$-homomorphic image; see [3]). Suppose G is an abelian group which is *countably free* in the sense that every countable subgroup of G is a free abelian group. Consider G with the identical involution. If G admits Z^2 as a homomorphic image then, since Z^2 is non-semiperfect, it follows that G is not semiperfect. However, as far as we have been able to ascertain, there exist (uncountable) countably free groups that do not even admit Z as a homomorphic image. For such groups, the problem of semiperfectness is unsolved. This question is of interest since an arbitrary $*$-group G is perfect if and only if the greatest countably free identical-involution

$*$-homomorphic image of G is perfect [13].

THEOREM 4.5. *A $*$-semigroup S is of class $\mathscr{M}$ if and only if for each $x \in S$ there exist $e \in S$ and $n \in \mathsf{N}$ such that $n(x + x^*) = e + n(x + x^*)$.*

PROOF. 'If': Suppose H is a $*$-archimedean component of S; we have to show that $0 \in g_H(X_H)$. Choose $x \in H$. By hypothesis there exist $e \in S$ and $n \in \mathsf{N}$ such that with $y = n(x + x^*)$ we have $y = e + y$. Since x is in H, so is y. From $e + y = y \in H \subset X_H$ we get $e \in X_H$ by the definition of a face. Applying g_H to the equation $y = e + y$ we get $g_H(y) = g_H(e) + g_H(y)$. Since G_H is a group, we can infer $g_H(e) = 0$, which shows $0 \in g_H(X_H)$, as desired.

'Only if': Let $x \in S$ be given; we have to show that there exist $e \in S$ and $n \in \mathsf{N}$ such that $n(x + x^*) = e + n(x + x^*)$. Let H be the $*$-archimedean component of S containing x. Since S is of class $\mathscr{M}$ then $0 \in g_H(X_H)$, so we can choose $e \in X_H$ such that $g_H(e) = 0$. It follows that $g_H(x) = 0 + g_H(x) = g_H(e) + g_H(x) = g_H(e + x)$. Since x and $e + x$ are both in H (because of $X_H + H \subset H$), this shows that there is some $a \in H$ such that with $b = a + x$ we have $b = b + e$. Since H is $*$-archimedean, there exist $c \in H$ and $n \in \mathsf{N}$ such that $n(x + x^*) = b + c$. Now $n(x + x^*) = b + c = (b + e) + c = e + (b + c) = e + n(x + x^*)$, as desired.

COROLLARY 4.6. *Suppose S is a $*$-separative semigroup. Then S is of class $\mathscr{M}$ if and only if for each $x \in S$ there is some $e \in S$ such that $x = e + x$. In particular, if S is of class $\mathscr{M}$ then $S = S + S$.*

PROOF. If $x = e + x$ then $x + x^* = e + (x + x^*)$, so the condition of the Theorem is satisfied with $n = 1$. Conversely, if $n(x + x^*) = e + n(x + x^*)$ then the elements x, $e + x$, and $x^* + (n - 1)(x + x^*)$ belong to the same $*$-archimedean component of S. That component is cancellative since S is $*$-separative, so we can infer $x = e + x$. This proves the first statement. For the second statement, suppose S is of class $\mathscr{M}$. Given $x \in S$, we have to show $x \in S + S$. Choose $e \in S$ such that $x = e + x$. Then obviously $x \in S + S$.

A *Z-semigroup* is a $*$-semigroup S such that each archimedean component of $\rho(S)$ is isomorphic to a subsemigroup of Z.

THEOREM 4.7. *For a $*$-separative Z-semigroup S, the following three conditions are equivalent:*

(i) *S is adapted*;

(ii) *S is of class $\mathscr{M}$*;

(iii) *$S = S + S$.*

PROOF. The implication (iii)$\Rightarrow$(i) is easily seen to be valid for an arbitrary $*$-semigroup S. Thus we only have to verify (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

(ii)$\Rightarrow$(iii): Corollary 4.6. Note that here we do not use the assumption that S is a Z-semigroup.

(i)$\Rightarrow$(ii): First suppose S is R_+-separative. Then $S = \rho(S)$, so each archimedean component of S is isomorphic to a subsemigroup of Z. Suppose H is an archimedean component of S; we have to show $0 \in g_H(X_H)$, that is, there is some $e \in X_H$ such that $g_H(e) = 0$. If H is a group then as e we can use the zero of H. Suppose H is not a group. By hypothesis, H is isomorphic to a subsemigroup of Z. Identify H with such a semigroup. If H intersects both N and the set $-\mathsf{N} = \{-n \mid n \in \mathsf{N}\}$ then H is a group, a contradiction. Thus $H \subset \mathsf{N}_0$ or $H \subset -\mathsf{N}_0$. Applying the automorphism $n \mapsto -n$ of Z, if necessary, we may assume $H \subset \mathsf{N}_0$. Since $g_H(X_H) \subset G_H = H - H \subset \mathsf{Z}$ and $\mathsf{N}_0 \supset H \supset X_H + H = g_H(X_H) + H$ then $g_H(X_H) \subset \mathsf{N}_0$. Let e be the least element of $g_H(X_H)$; we have to show $e = 0$. Choose $x \in X_H$ such that $e = g_H(x)$. Since S is adapted, we can choose $n \in \mathsf{N}$ and $y_1, \dots, y_{n+1} \in S$ such that $nx = y_1 + \cdots + y_{n+1}$. Since x is in X_H, so is nx, and so, therefore, are the y_j. Applying g_H to the last equality, we obtain $ne = f_1 + \cdots + f_{n+1}$ where $f_j = g_H(y_j)$ for $j = 1, \dots, n+1$. Since $f_j \geq e$ (by the definition of e) then $ne \geq (n+1)e$, that is, $e \leq 0$. Since $e \in \mathsf{N}_0$, it follows that $e = 0$, as desired.

THEOREM 4.8. *For a facially countable* R*-separative facially* C*-finite semigroup* S*, the following three conditions are equivalent:*

(i) *S is completely semiperfect*;

(ii) *S is semiperfect*;

(iii) *$S = S + S$ and condition* (B) *holds.*

PROOF. (iii)$\Rightarrow$(i): See [7]. Note that this implication is true without the assumptions that S is facially countable and facially C-finite.

(i)$\Rightarrow$(ii): Trivial.

(ii)$\Rightarrow$(iii): We noted in the beginning of this section that we may assume that S is countable and C-finite. If we show that $S = S + S$ then it follows from the main result of [7] that (B) holds. Thus we only have to show $S = S + S$. Since S is semiperfect then S is adapted. By the preceding Theorem, it now suffices to show that S is a $*$-separative Z-semigroup. Since S is R-separative, it is, in particular, $*$-separative. Thus it suffices to verify that S is a Z-semigroup, that is, each archimedean component of $\rho(S)$ is isomorphic to a subsemigroup of Z. Since S is C-finite, so is its $*$-subsemigroup $S^{\#}$, and so, therefore, is the isomorphic semigroup $\rho(S)$, cf. Theorem 2.5. The semigroup $\rho(S)$ is R_+-separative by definition. Since S is countable, so is

$\rho(S)$. Since S is semiperfect, so is its $*$-homomorphic image $\rho(S)$. Thus $\rho(S)$ is a semiperfect countable R_+-separative C-finite semigroup. Suppose H is an archimedean component of $\rho(S)$; we have to show that H is isomorphic to a subsemigroup of Z. Let X be the least face of $\rho(S)$ containing H (i.e., $X = X_H$ in the notation of the Introduction). Being a face of the semiperfect semigroup $\rho(S)$, X is semiperfect. Since $\rho(S)$ is countable, so is X. Thus X is a semiperfect countable semigroup. Hence so is its homomorphic image $g(X)$ where $g = g_H$. (Recall from the Introduction the definition of the mapping g_H.) Being a subsemigroup of the group G_H, $g(X)$ is cancellative. Since $\rho(S)$ is R_+-separative, it is torsion-free. Hence so is its subsemigroup H. Hence so is the group $G_H = H - H$. It follows that $g(X)$ is torsion-free. Thus $g(X)$ is a semiperfect countable torsion-free cancellative semigroup. Now $g(X) + H = X + H \subset H$, that is, H is an ideal of $g(X)$. Since the cancellative semigroup $g(X)$ has a C-finite ideal, it is C-finite by Lemma 4.3. It now follows from Theorem 4.4 that $g(X)$ is isomorphic to a subsemigroup of Z. Hence so is its subsemigroup H.

5. Semiperfect finitely generated abelian semigroups with arbitrary involution: Reduction to the C-separative case

So far, semiperfect (or, presumably equivalently, completely semiperfect) finitely generated abelian semigroups with arbitrary involution have not been characterized. As indicated in the Introduction, a necessary condition is known which for all we know may be sufficient. The sufficiency proof could be completed if a suitable converse homomorphism theorem could be demonstrated.

In spite of this situation, the problem of characterizing semiperfect finitely generated abelian semigroups with arbitrary involution can be reduced to the problem of doing the same for C-separative semigroups. It is the purpose of the present section to do this. Since semiperfect (or equivalently, completely semiperfect) R-separative finitely generated semigroups are characterized in Theorem 4.8 (every R-separative finitely generated semigroup being C-finite), a complete characterization of semiperfect (or equivalently, completely semiperfect) finitely generated abelian semigroups with the identical involution (also known as 'no involution') follows.

As mentioned in the Introduction, an arbitrary $*$-semigroup S is semiperfect of order $d \in \mathsf{N}$ if and only if $\chi(S)$ so is and moreover, every positive semidefinite function on S factors via χ. The semigroup $\chi(S)$ is C-separative by definition. If S is finitely generated, so is $\chi(S)$. Thus, if some day in the future semiperfect C-separative finitely generated semigroups will be characterized then in order to remove the hypothesis of C-separativity it suffices to solve the factoring problem. This we can do already. We note that the last Theorem in [12] is a partial solution of the factoring problem for semigroups S of class

$\mathscr{M}$, which is complete in case S is finitely generated. Thus, it suffices to show that every semiperfect finitely generated semigroup is of class $\mathscr{M}$.

For an arbitrary $*$-semigroup S and for $H, K \in \mathscr{J}(S)$ such that $H \leq K$, we have $H \subset X_K$, hence $X_H \subset X_K$, so the mapping $g_K|X_H$ is well-defined. This mapping being a $*$-homomorphism of X_H into the $*$-group G_K, it has the form $g_K|X_H = g_{H,K} \circ g_H$ for a unique $*$-homomorphism $g_{H,K}\colon G_H \to G_K$. The mapping $g_{H,H}$ is the identity on G_H, and if $H, K, L \in \mathscr{J}(S)$ are such that $H \leq K \leq L$ then $g_{H,L} = g_{K,L} \circ g_{H,K}$. By [18], Theorem 4.11, it follows that the disjoint union

$$G = \bigcup_{H \in \mathscr{J}(S)} G_H$$

becomes a $*$-semigroup when considered with the addition $+$ given by

$$x + y = g_{H,H\vee K}(x) + g_{K,H\vee K}(y) \qquad \text{(sum in the group } G_{H\vee K})$$

for $H, K \in \mathscr{J}(S)$, $x \in G_H$, and $y \in G_K$, and the unique involution which extends the given one on each G_H. The mapping $g\colon S \to G$ given by $g|H = g_H|H$ for $H \in \mathscr{J}(S)$ is a $*$-homomorphism. The semigroup G is $*$-separative. In fact, the semigroup $g(S)$ is the greatest $*$-separative $*$-homomorphic image of S [12].

THEOREM 5.1. *A $*$-semigroup S is of class $\mathscr{M}$ if and only if $g(S)$ so is.*

PROOF. Suppose $H \in \mathscr{J}(S)$. Then the semigroup $g(H) = g_H(H)$ is a $*$-archimedean component of $g(S)$. The least face of $g(S)$ containing $g(H)$ will be denoted by Y_H. Now

$$Y_H = \bigcup_{I \in \mathscr{J}(S): I \leq H} I = g(X_H).$$

Since the semigroup $g(H)$ is a generating subsemigroup of the group G_H then the group $G_{g(H)}$ can be identified with G_H. The mapping $g_{g(H)}|g(H)$ is then identified with the inclusion mapping of $g(H)$ into G_H. In fact, the mapping $g_{g(H)}$ is the union of the mappings $g_{I,H}\colon G_I \to G_H$ ($I \in \mathscr{J}(S)$, $I \leq H$). We have the chain of bi-implications $0 \in g_{g(H)}(Y_H) \Leftrightarrow \exists I \in \mathscr{J}(S)\colon I \leq H,\ 0 \in g_{I,H}\big(g(I)\big) \Leftrightarrow 0 \in g_H(X_H)$.

THEOREM 5.2. *Every semiperfect finitely generated semigroup is of class $\mathscr{M}$.*

PROOF. Suppose S is a semiperfect finitely generated semigroup; we have to show that S is of class $\mathscr{M}$. First suppose S is $*$-separative. Since S is semiperfect then S is adapted. By Theorem 4.7 it now suffices to show that S is a Z-semigroup, that is, each archimedean component of $\rho(S)$ is isomorphic to a subsemigroup of Z. The semigroup $\rho(S)$, being a $*$-homomorphic image of the semiperfect finitely generated semigroup S, is finitely generated and semiperfect. By definition, $\rho(S)$ is R_+-separative. Being R-separative and finitely generated, $\rho(S)$ is C-finite. Being finitely generated, $\rho(S)$ is countable. Thus $\rho(S)$ is a semiperfect countable R_+-separative C-finite semigroup. By Theorem 4.8 it follows, in particular, that condition (B) holds for $\rho(S)$, that is, each archimedean component H of $\rho(S)$ is isomorphic to $F \times P$ for some torsion group F of exponent 1 or 2 and some semigroup P which is $\{0\}$, Z, or N. It now suffices to verify that $F = \{0\}$. But this follows from the fact that $\rho(S)$, being R_+-separative, is torsion-free.

In the general case, by Theorem 5.1 it suffices to verify that the semigroup $g(S)$ is of class $\mathscr{M}$. Since $g(S)$ is $*$-separative, by what we showed in the preceding paragraph it suffices to verify that $g(S)$ is a semiperfect finitely generated semigroup. But both facts follow from the fact that $g(S)$ is a $*$-homomorphic image of the semiperfect finitely generated semigroup S.

THEOREM 5.3. *A finitely generated abelian semigroup S with arbitrary involution is semiperfect of order $d \in \mathsf{N}$ if and only if $\chi(S)$ so is and moreover, the condition in the last Theorem in* [12] (*cited in italics in the present Introduction*) *is satisfied.*

PROOF. First suppose S is semiperfect of order d. As already noted, it follows that $\chi(S)$ is semiperfect of order d and every positive semidefinite function on S factors via χ. By Theorem 5.2, S is of class $\mathscr{M}$. It now follows that the condition from [12] must be satisfied.

Conversely, suppose the conditions hold; we have to show that S is semiperfect of order d. Since $\chi(S)$ is semiperfect of order d, it suffices to verify that every positive semidefinite function on S factors via χ. Since the condition from [12] is supposed to be satisfied, it suffices to show that S is of class $\mathscr{M}$. Now the property of being of class $\mathscr{M}$ is a property of the greatest R_+-separative $*$-homomorphic image in the sense that a $*$-semigroup T is of class $\mathscr{M}$ if and only if $\rho(T)$ so is (see [15]). Since every nonnegative character on S is, in particular, a character on S then $\rho(S)$ and $\rho\big(\chi(S)\big)$ are the same semigroup up to isomorphism. Thus S is of class $\mathscr{M}$ if and only if $\chi(S)$ so is. Thus it suffices to show that $\chi(S)$ is of class $\mathscr{M}$. Since $\chi(S)$ is semiperfect of order d, it is, in particular, semiperfect, so the desired fact follows from Theorem 5.2.

The preceding result has the demerit that a solution of the C-separative case is lacking. The next result has no such demerit.

COROLLARY 5.4. *A finitely generated abelian semigroup S is semiperfect* (*or equivalently, completely semiperfect*) *if and only if, firstly, the semigroup* $T = \chi(S)$ *satisfies* $T = T + T$ *and condition* (B)*, and secondly, the condition from* [12] *is satisfied.*

PROOF. Immediate from Theorems 5.3 and 4.8, recalling that every R-separative finitely generated semigroup is C-finite.

6. A discussion

The present section is devoted to a discussion—mostly informal—of whether some of the assumptions in Theorem 4.8 might be weakened or dropped. Firstly, note that condition (B) implies R-separativity, so that the assumption of R-separativity cannot be dropped unless every semiperfect semigroup satisfying the other assumptions is R-separative—which is obviously false, cf. the example of the semigroup $\{a, 2a\}$ where $3a = 2a$—which is a perfect semigroup. Next, we note that the assumption of facial countability cannot be replaced with local countability. Indeed, in [5], Example 3, we associated with every set Δ with at least 2 elements a semigroup S_Δ such that all archimedean components of S_Δ except two are isomorphic to N while the remaining two are isomorphic to $\{0\}$ and N_2, respectively. If Δ is uncountable then S is semiperfect. Obviously, S is R-separative (even R_+-separative) and locally countable. It was shown in [7], Remark 1, that S is C-finite. Since S_Δ has an archimedean component isomorphic to N_2 then it does not satisfy condition (B). It follows from [7], Remark 1, that in the present Theorem 4.8 the assumption of facial C-finiteness cannot simply be dropped. It remains to investigate whether it can be replaced by some weaker condition, such as local C-finiteness. Since even the R_+-separative case is hard, we may as well restrict attention to this case.

THEOREM 6.1. *If* S *is a semiperfect locally countable* R_+*-separative locally* C*-finite semigroup then* $S = S + S$ *and each archimedean component of* S *is isomorphic to* $\{0\}$*,* Z*, or* N_p *for some* $p \in \mathsf{N}$.

PROOF. Suppose H is an archimedean component of S. Let X be the least face of S containing H (i.e., $X = X_H$ in the notation of the Introduction). Being a face of the semiperfect semigroup S, X is semiperfect. Hence so is its homomorphic image $g(X)$ where $g = g_H$. Since S is R_+-separative then H is cancellative and torsion-free. Being cancellative, H can be identified with a subsemigroup of G_H such that $G_H = H - H$. Since H is torsion-free, so is G_H. Since S is locally C-finite then H is C-finite. The semigroup $g(X)$, which has the C-finite ideal H, is C-finite. Since S is locally countable then H is countable and so, therefore, are G_H and $g(X)$. Thus $g(X)$ is a semiperfect countable torsion-free cancellative C-finite semigroup, hence isomorphic to $\{0\}$, Z, or N_0. Now H is an ideal of $g(X)$. If $g(X)$ is a group, it follows that

$H = g(X)$, so H is isomorphic to $\{0\}$ or Z. Assume that $g(X)$ is not a group. Then $g(X)$ is isomorphic to N_0, and it follows that H, being an ideal of $g(X)$, is isomorphic to N_p for some $p \in \mathsf{N}_0$. The case of $p = 0$ is excluded since N_0 is not archimedean.

It remains to be shown that $S = S+S$. We have seen that S is a Z-semigroup. Being R_+-separative, S is $*$-separative. Being semiperfect, S is adapted. Now the desired conclusion follows from Theorem 4.7.

Suppose S is a semiperfect countable R_+-separative locally C-finite semigroup. Let H be an archimedean component of S. By the preceding Theorem, H is isomorphic to $\{0\}$, Z, or N_p for some $p \in \mathsf{N}$. If H is isomorphic to N_p for some $p \in \mathsf{N}$, is it possible that $p \geq 2$? Suppose $p = 2$. Let X be the least face of S containing H. Then X, being a face of the semiperfect semigroup S, is semiperfect. Obviously X is countable, R_+-separative, and locally C-finite. Moreover, H is the greatest element of $\mathscr{J}(X)$ with respect to the canonical partial ordering. In other words, we may assume that H is the greatest element of $\mathscr{J}(S)$ with respect to the canonical partial ordering. Let $(S_i)_{i\in I}$ be the family of all archimedean components of S, with the index set I made into a semilattice (isomorphic to $\mathscr{J}(S)$) by the requirement that $S_i + S_j \subset S_{i+j}$ for all $i, j \in I$. Then I has a greatest element k, and $S_k = \mathsf{N}_2$. For definiteness, assume that for $i < k$ the semigroup S_i is isomorphic to $\{0\}$, Z, or N.

THEOREM 6.2. *We may assume that S is a subsemigroup of the product semigroup $I \times \mathsf{N}_0$.*

PROOF. Let g be the mapping $g_{S_k}\colon S = X_{S_k} \to G_{S_k} = S_k - S_k = \mathsf{N}_2 - \mathsf{N}_2 = \mathsf{Z}$. Since $g(S) + S_k = S + S_k \subset S_k$ then $g(S) \subset \mathsf{N}_0$. Define $f\colon S \to I \times \mathsf{N}_0$ by $f(s) = \big(i, g(s)\big)$ for $s \in S_i$. We leave it as an easy exercise to verify that f is a homomorphism. Being a homomorphic image of the semiperfect semigroup S, the semigroup $T = f(S)$ is semiperfect. Since S is countable, so is T. The archimedean components of T are the sets $T_i = f(S_i)$ $(i \in I)$, which are isomorphic to subsemigroups of N_0, hence C-finite. Thus T is locally C-finite. Since g is the identity on S_k then the archimedean component T_k of T is isomorphic to N_2. Thus T satisfies all the assumptions on S and in addition is a subsemigroup of $I \times \mathsf{N}_0$.

In the following, we assume that I is a countable semilattice with greatest element k and that S is a subsemigroup of $I \times \mathsf{N}_0$. For $i \in I$ we define $S_i = \{\, n \mid (i, n) \in S \,\}$. We assume that $S_k = \mathsf{N}_2$ while if $i < k$ then S_i is $\{0\}$ or N. Assume that S is semiperfect.

THEOREM 6.3. *The semilattice I is infinite.*

PROOF. Suppose I is finite. Then S is C-finite. In fact, for every finite subset U of S the set $C(U)$ is contained in the set given by [7], equation (8). Thus

S is a semiperfect countable R_+-separative C-finite semigroup. Hence each archimedean component of S is isomorphic to $\{0\}$, Z, or N, contradicting the fact that S has an archimedean component isomorphic to N_2.

For every set A, denote by 2^A the set of all subsets of A.

THEOREM 6.4. *We may assume that I is a subsemilattice of the semilattice* $(2^{\mathsf{N}}, \cap)$.

PROOF. Choose an enumeration $(i_1, i_2, \ldots)$ of the set $I \setminus \{k\}$, i.e., a sequence in which each element of that set occurs exactly once. Define a mapping $f\colon I \to 2^{\mathsf{N}}$ by $f(i) = \{\, n \mid i \leq i_n \,\}$. Then f is a homomorphism of I into $(2^{\mathsf{N}}, \cap)$. Indeed, for $i, j \in I$, since $i + j$ is the least upper bound on the set $\{i, j\}$ then $f(i + j)$ is the set of those n such that both $i \leq i_n$ and $j \leq i_n$, i.e., $f(i + j) = f(i) \cap f(j)$. Define a homomorphism $F\colon I \times \mathsf{N}_0 \to 2^{\mathsf{N}} \times \mathsf{N}_0$ by $F(i, n) = \big(f(i), n\big)$. Now replace S by $F(S)$.

We discontinue the discussion at this point in order that the paper will not become too long.

REMARK. After the rest of this paper was written, we discovered that there is a semiperfect countable R_+-separative locally C-finite semigroup S which has an archimedean component isomorphic to N_2. The semigroup S is constructed as follows: Choose a sequence $(A_p)_{p=1}^{\infty}$ of pairwise disjoint infinite subsets of N with union N. Let I be the semilattice

$$\left(\{\mathsf{N}\} \cup \left\{ \bigcup_{p \in P} A_p \;\middle|\; P \in 2^{(\mathsf{N})} \right\} \cup 2^{(\mathsf{N})}, \cap\right),$$

define $J = I \setminus \{\emptyset\}$, and let S be the semigroup obtained by adjoining a neutral element to the subsemigroup $(J \times \mathsf{N}) \cup (\{\emptyset\} \times \mathsf{N}_2)$ of $I \times \mathsf{N}$. The proof is rather complicated.

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