MATH. SCAND. 92 (2003), 93–102

CARTAN SUBALGEBRAS AND BIMODULE DECOMPOSITIONS OF II_1 FACTORS

SORIN POPA and DIMITRI SHLYAKHTENKO

Abstract

Let $A \subset M$ be a MASA in a II_1 factor M. We describe the von Neumann subalgebra of M generated by A and its normalizer $\mathcal{N}(A)$ as the set $N_q^w(A)$ consisting of those elements $m \in M$ for which the bimodule $\overline{AmA}$ is discrete. We prove that two MASAs A and B are conjugate by a unitary $u \in N_q^w(A)$ iff A is discrete over B and B is discrete over A in the sense defined by Feldman and Moore [5]. As a consequence, we show that A is a Cartan subalgebra of M iff for any MASA $B \subset M$, $B = uAu^*$ for some $u \in M$ exactly when A is discrete over B and B is discrete over A.

1. Introduction

Let M be a type II_1 von Neumann algebra with a trace τ. If $A \subset M$ is a maximal abelian subalgebra (MASA), then A is called a Cartan subalgebra (see [2], [6]), if its normalizer $\mathcal{N}(A) = \{u \in U(M) : uAu^* = A\}$ generates M. Feldman and Moore [5] characterized pairs $A \subset M$, where A is a Cartan subalgebra, as those coming from r-discrete transitive measure groupoids with a finite measure space X as base (and a certain cocycle). Given such a groupoid, the algebra A is $L^\infty(X)$, and the group of bisections of the groupoid embeds into the unitary group of M as the normalizer of A. In this paper we make use of an alternate characterization of a Cartan subalgebra in M: namely, A is a Cartan subalgebra if the Hilbert space $L^2(M)$, viewed as an A, A-bimodule, is in a certain way discrete.

Recall that if $A = L^\infty(X, \nu_X)$, $B = L^\infty(Y, \nu_Y)$ are diffuse commutative von Neumann algebras, and μ is a measure on $X \times Y$, so that its push-forwards by the coordinate projections onto X and Y are absolutely continuous with respect to ν_X and ν_Y, then $L^2(X \times Y, \mu)$ carries a pair of commuting normal representations of A and B given by

$$(a \cdot f)(s,t) = a(s)f(s,t), \qquad (f \cdot b)(s,t) = b(t)f(s,t).$$

for $f \in L^2(X \times Y, \mu)$ and $a \in A, b \in B$. In this way, $L^2(X \times Y, \mu)$ is an A, B-bimodule. It can be shown that any (abstract) A, B-bimodule containing

Received November 21, 2000; in revised form June 22, 2001.

a vector ξ for which $A\xi B$ is dense, is isomorphic to $L^2(X \times Y, \mu)$ for some measure μ.

Denote by $C(A, B)$ the set of all A, B-bimodules. Then the operation of relative tensor product of bimodules gives a multiplication $\otimes_B : C(A, B) \times C(B, C) \to C(A, C)$. Inside $C(A, B)$, there is a remarkable subset $C_d(A, B)$ consisting of *discrete* modules, i.e., modules which are direct sums of $L^2(X \times Y, \mu)$ for which μ can be disintegrated as $\mu(s, t) = \mu_t(s)\nu_Y(t)$ with $\mu_t(s)$ atomic for almost all t. One has $C_d(A, B) \otimes_B C_d(B, C) \subset C_d(A, C)$. An example of a module in $C_d(A, B)$ is the bimodule $L^2(X \times Y, \mu)$ for which μ is supported on the graph of an isomorphism $\alpha : X \to Y$. In this case the bimodule contains a vector ξ for which $a \cdot \xi = \xi\alpha(a)$ and $A\xi B$ is dense.

If H is any A, B bimodule, one can construct a submodule $H_d \subset H$ consisting of those elements $\xi \in H$ for which $\overline{A \cdot \xi \cdot B} \in C_d(A, B)$. H_d is the maximal subbimodule of H which lies in $C_d(A, B)$.

Denote by $C_d(A) \subset C_d(A, A) \subset C(A, A)$ the set of such bimodules $H \in C_d(A, A)$ for which $\bar{H} \in C_d(A, A)$ as well (here $\bar{H}$ is H with the opposite Hilbert space structure and the right and left actions of A switched). Equivalently, $C_d(A)$ is the maximal subset C in $C(A, A)$ satisfying $C_d(B, A) \otimes_A C \otimes_A C_d(A, C) \subset C_d(B, C)$.

Returning now to the situation that $A \subset M$ is a MASA, $L^2(M)$ is a bimodule over A, since elements of A act on $L^2(M)$ by right and left multiplication. Every element $x \in M$ defines a subbimodule $\overline{AxA} \subset L^2(M)$. Denoting by $N_q^w(A)$ the set of all elements x for which $\overline{AxA} \in C_d(A)$ we obtain a certain subset of M. Because $C_d(A)$ is closed under tensor products, $N_q^w(A)$ is a subalgebra (in fact, a von Neumann subalgebra) of M. Every element in the normalizer of A is in N_q^w (in fact, its associated bimodule is the bimodule constructed out of an automorphism α of A above). We show that $N_q^w(A)$ is exactly the von Neumann subalgebra of M generated by A and its normalizer. Hence A is a Cartan subalgebra iff $N_q^w(A) = M$, i.e., $L^2(M, \tau) \in C_d(A)$.

One can similarly consider two MASAs in M and the subset $N_q^w(A, B)$ of $x \in M$ for which $\overline{AxB} \in C_d(A, B)$ and $\overline{Bx^*A} \in C_d(B, A)$. It turns out that the condition that $1 \in N_q^w(A, B)$ is equivalent to the condition that A is discrete over B in the sense of Feldman and Moore [5].

Feldman and Moore proved that two Cartan subagebras A and B are conjugate by a unitary $u \in M$ iff B is discrete over A and A is discrete over B. We show that this characterization of conjugacy characterizes Cartan subalgebras: A is a Cartan subalgebra of M iff uAu^* is discrete over A for all unitaries $u \in M$ (i.e., Cartan subalgebras are precisely the subalgebras of M for which the Feldman-Moor criterion of inner conjugacy applies). More generally, we prove that A is discrete over B and B is discrete over A iff A and B are conjugate by a unitary from $N_q^w(A)$.

2. Quasi-normalizer of a MASA

Let X be a measure space. By a local Borel map from X to X we mean a triple (ϕ, D, R), where $D, R \subset X$ are Borel subsets, and $\phi : D \to R$ is a Borel map. We further say that (ϕ, D, R) is a local isomorphism, if ϕ is a measure-preserving Borel isomorphism of D with R. For a function $f \in L^\infty(X)$, we write $\phi(f)$ for the function $\chi_R \cdot \phi(f \cdot \chi_D)$.

Whenever A, B are commutative finite W^*-algebras with fixed finite traces and $\eta : A \to B$ is a completely-positive map, we can identify $A \cong L^\infty(X, \mu)$ and $B \cong L^\infty(Y, \nu)$ so that the fixed traces on A and B correspond to integration with respect to μ and ν, respectively. With this identification, there exists a measure $\hat{\eta}$ on $X \times Y$, so that:

(1) The push-forwards of $\hat{\eta}$ onto X and Y via projections maps are absolutely-continuous with respect to μ and ν, respectively;

(2) For all $f \in A$ and $g \in B$,

$$\int g(y)\eta(f)(y)\, d\nu(y) = \iint f(x)g(y)\, d\hat{\eta}(x, y).$$

The measure $\hat{\eta}$ can be disintegrated along the y direction: there is a measurable family of measures $\hat{\eta}_y$ on X, for which

$$\iint f(x, y)\hat{\eta}(x, y) = \int \left(\int f(x, y)\, d\hat{\eta}_y(x) \right) dy.$$

DEFINITION 2.1. (cf. [5]) A completely-positive map $\eta : A \to B$ is called *discrete*, if the measures η_y are atomic for almost all $y \in Y$.

Notice that if $\phi : C \to B$ is an isomorphism, then $\eta \circ \phi$ is discrete if and only if η is discrete.

Let M be a type II_1 factor and $A \subset M$ be a MASA. Let $x \in M$ be an element. Then x defines a completely positive map $\eta_x : A \to A$ by:

$$\eta_x(a) = E_A(x^* a x).$$

Identifying A with $L^\infty[0, 1]$, η_x determines a finite positive measure $\mu^x = \hat{\eta}_x$ on $[0, 1]^2$, by:

$$\iint a(t)b(s)\, d\mu^x(t, s) = \tau(a\eta_x(b)) = \tau(ax^*bx), \quad \forall a, b \in A = L^\infty[0, 1].$$

If x is self-adjoint, the measure μ^x is symmetric; more precisely, if $T : [0, 1]^2 \to [0, 1]^2$ is given by $T(t, s) = (s, t)$, then $T_* \mu^x = \mu^x$. The push-forwards of μ^x by the coordinate projections from $[0, 1]^2$ onto the first copy and second copies of [0, 1] are absolutely continuous with respect to Lebesgue measure. For each $t \in [0,1]$, μ^x can be disintegrated along the t-axis: there exists a family of measures μ_t^x, so that

$$\iint f(s,t)d\mu^x(s,t) = \iint f(s,t)d\mu_t^x(s)dt.$$

LEMMA 2.2. *The Hilbert space* $H(x) = \overline{\{a_1xa_2 : a_1, a_2 \in A\}} \subset L^2(M)$ *can be isometrically identified with* $L^2([0,1]^2, \mu^x)$*, in such a way that* x *is identified with the constant function* 1 *on* $[0,1]^2$*, and the element* a_1xa_2 *is identified with the function* $a_1(s)a_2(t) \in L^2([0,1]^2, \mu^x)$*. In the case that* x *is self-adjoint, the restriction of the Tomita conjugation operator* J *to* $H(x)$ *is given by* $J(f(x,y)) = \overline{f(y,x)}$.

NOTATION 2.3. We consider the following sets of elements in M:

(1) $\mathcal{N}(A) = \{u \in M \text{ unitary} : uAu^* = A\}$, the normalizer of A;

(2) $\mathcal{GN}(A) = \{v \in M \text{ partial isometry} : vAv^* \subset A, v^*Av \subset A\}$, the full group of the normalizer of A;

(3) $N_1(A) = \{x \in M : \text{ there is a local isomorphism } \phi : A \to A, \text{ s.t. } \phi(a)x = xa, \forall a \in A\}$;

(4) $N_q(A) = \{x \in M : \mu_t^x, \mu_t^{x^*}$ are both atomic with a finite number of atoms for all most all $t\}$, the quasi-normalizer of A;

(5) $N_q^w(A) = \{x \in M : \mu^x, \mu^{x^*}$ are both discrete$\}$, the weak quasi-normalizer of A.

Note that $N_q(A)$ and $N_q^w(A)$ are $*$-subalgebras of M. Indeed, if μ^x, μ^{x^*} and μ^y, μ^{y^*} are discrete, the support of the measure μ_t^{xy} is contained in the set $\{s : \exists t' \text{ s.t. } \mu_t^x(\{t'\}) \cdot \mu_{t'}^y(\{s\}) \neq 0\}$, which is finite if $x, y \in N_q(A)$ and countable if $x, y \in N_q^w(A)$.

LEMMA 2.4. *Let* $x = x^* \in M$*, and let* f, g *be bounded* μ^x*-measurable functions on* $[0,1]^2$*, such that* $\text{supp}(f), \text{supp}(g) \subset \Gamma$*, where*

$$\Gamma = \bigcup_{j=1}^{N} \{ (s, \phi_j(s)) \,;\, x \in [0,1] \} \cup \{ (\phi_j(s), s) : s \in [0,1] \}$$

and ϕ_j *are local isomorphisms. Identify* $\overline{AxA}$ *with* $L^2([0,1]^2, \mu^x)$ *as in Lemma 2.2. Let* $f \cdot x$ *be the element in* $\overline{AxA}$*, corresponding via this identification to* $f \in L^2([0,1]^2, \mu^x)$.

Then:

(1) $f \cdot x \in M$.

(2) $\mu^{f\cdot x} = |f|^2\mu^x$.

(3) $g\cdot(f\cdot x) = (gf)\cdot x$ *(Chain rule).*

(4) *If f is symmetric (i.e., $f(s,t) = \overline{f(t,s)}$), then $f\cdot x$ is self-adjoint.*

PROOF. Let $a_i, b_i \in A$, $i = 1,\dots,n$ be functions in A. Then if $y = \sum a_i x b_i$, we have that $\mu^y = \left|\left(\sum a_i(s)b_i(t)\right)\right|^2\mu^x$. Note that

$$\left\|\sum a_i x b_i\right\| = \left\|(a_1 \quad \dots \quad a_n)\begin{pmatrix} x & \dots & 0\\ \vdots & \ddots & \vdots \\ 0 & \dots & x\end{pmatrix}\begin{pmatrix} b_1\\ \vdots\\ b_n\end{pmatrix}\right\|$$

$$\leq \|x\|\cdot\left\|\sum a_i^* a_i\right\|^{1/2}\cdot\left\|\sum b_i^* b_i\right\|^{1/2}.$$

Choose now $a_i^{(k)}, b_i^{(k)}$ in such a way that $\left\|\sum a_i^* a_i\right\|, \left\|\sum b_i^* b_i\right\| \leq 4N^2\|f\|^2$, and $\left|\sum a_i(s)b_i(t)\right| \to f$ in $L^2([0,1]^2,\mu^x)$. This is possible because of the assumptions on f: for sufficiently fine partitions $A_1^{(k)},\dots,A_p^{(k)}$, $k = 1,\dots,N$ and $i = pj + r$, $0 \leq r < N$, one can take $a_p^{(k)}$ to be the characteristic function of $A_j^{(r-1)}$ and $b_p^{(k)}$ be a constant times the characteristic function of $\phi_r(A_j^{(r)})$.

Then $y_k = \sum_i a_i^{(k)} x b_i^{(k)}$ converges in $L^2(M)$ to some vector $y \in L^2(M)$. Since $\|y_k\|$ is bounded, we get that $y \in M$. Define $f\cdot x$ to be equal to y. The claimed properties of $\cdot$ follow easily. To show that $f\cdot x$ is self-adjoint if f is symmetric, notice that in this case $a_i^{(k)}$ and $b_i^{(k)}$ can be chosen so that $\sum_i a_i^{(k)}(s)b_i^{(k)}(t)$ is symmetric. But then it follows that $Jy_k = y_k$, so that $Jy = y$, so that $f\cdot x = y$ is self-adjoint.

LEMMA 2.5. *Assume that $x = x^* \in N_q^w(M)$ and $\epsilon > 0$. Then there exists $y \in N_q(M)$, so that $\|x - y\|_2 \leq \epsilon$.*

PROOF. By [5, Theorem 1] (see also [1, Lemma 3 (a)]), there exists local isomorphisms $\sigma_i : A_i \to B_i$, $A_i, B_i \subset X$, so that the support of the measure μ^x is contained in the union of graphs $\Gamma_{\sigma_j} = \{(x,\sigma_j(x)) : x \in A_j\}$, and the graphs are disjoint. By Lemma 2.4, denoting by f_j the characteristic function of Γ_{σ_j}, we find elements $x_j = f_j\cdot x \in M$, so that $\mu^{x_j} = f_j\cdot\mu^x$. It follows that $x_j \in L^2(M)$ are perpendicular, and $x = \sum_j x_j$. Moreover, each $x_j \in N_q(M)$. Now, given $\epsilon > 0$, there exists N so that if we set $y = \sum_{j=1}^N x_j$, then $\|x - y\|_2 \leq \epsilon$. Since N_q is an algebra, $y \in N_q$.

PROPOSITION 2.6. *$N_q^w(A)$ is a von Neumann subalgebra of M.*

PROOF. Let $x_n \in N_q^w(A)$ be a sequence of elements, converging $*$-strongly to an element $x \in M$ and such that $\|x_j\| \leq \|x\|$. By Lemma 2.5 we may assume that $x_n \in N_q$. We must show that $x \in N_q^w(M)$. If not, then let $X \subset [0,1]^2$ be the set of atoms of μ_t^x, $t \in [0,1]$, and we have that $\mu^x(X) = \|x\|_2 - \delta$ for some $\delta > 0$. Hence we have that for any f satisfying the hypothesis of Lemma 2.4 and valued in $\{0,1\}$, $\|x - f \cdot x\|_2^2 \geq \delta$. On the other hand, we clearly have for all such f that $\|f \cdot x_n - f \cdot x\|_2^2 = \|f \cdot (x_n - x)\|_2^2 \leq \|x_n - x\|_2^2$, since f is valued in $\{0,1\}$. Now choose x_n so that $\|x_n - x\|_2^2 < \delta^2/4$; then there is an f for which $f \cdot x_n = x_n$. Hence $\|f \cdot x_n - f \cdot x\|_2^2 \leq \delta^2/4$, and it follows that

$$\|x - f \cdot x\|_2 \leq \|x - x_n\|_2 + \|f \cdot x_n - f \cdot x\|_2 < \delta,$$

which is a contradiction.

THEOREM 2.7. *Let $A \subset M$ be a MASA. Then the sets $\mathscr{N}(A)$, $\mathscr{G}\mathscr{N}(A)$, $N_1(A)$, $N_q(A)$ and $N_q^w(A)$ generate the same von Neumann subalgebras in M.*

PROOF. Note that A is contained in all of the sets listed in the statement. Clearly $\mathscr{N}(A) \subset \mathscr{G}\mathscr{N}(A)$; also, $N_q(A) \subset N_q^w(A)$. If $x \in N_1(A)$, then for a certain local isomorphism $\phi : [0,1] \to [0,1]$, μ_t^x and $\mu_t^{x^*}$ are supported on $\{\phi(t), \phi^{-1}(t)\}$ if t is in the domain of ϕ, and zero otherwise. Hence $W^*(N_1(A)) \subset W^*(N_q(A))$.

By Lemma 2.5 and Proposition 2.6, we have that $W^*(N_q(A)) = N_q^w(A)$.

By a result of H. Dye (cf. [3], [4]), we have that $\mathscr{G}\mathscr{N}(A) = \mathscr{N}(A)A$, so that $W^*(\mathscr{G}\mathscr{N}(A)) = W^*(\mathscr{N}(A))$.

Summarizing, we have:

$$W^*(\mathscr{N}(A)) = W^*(\mathscr{G}\mathscr{N}(A)) \subset W^*(N_1(A)) \subset W^*(N_q(A)) = N_q^w(A).$$

Next, we prove that $N_q(A) \subset W^*(N_1(A))$. Assume that $x = x^* \in N_q(A)$. As in Lemma 2.4, by finding suitable functions f_i, we can write $x = \sum f_i \cdot x$, $x_i = f_i \cdot x \in N_q(A)$, so that μ^{x_i} is supported on the set $\{(s, \phi(s)\} \cup \{(\phi(s), s)\}$ for some local isomorphism ϕ (depending on i). It is therefore sufficient to consider those x, for which μ^x is supported on such a set. Letting g be the characteristic function of $\{(s, \phi(s) : s \in [0,1]\}$ and h be the characteristic function of $\{(\phi(s), s) : s \in [0,1]\}$, we get that $x = g \cdot x + h \cdot x - hg \cdot x$. Now, $y_1 = g \cdot x$ satisfies $y_1 a = \phi(a) y_1$ for all $a \in A$, hence $y_1 \in N_1(A)$. Similarly, $y_2 = h \cdot x$ is in $N_1(A)$. Lastly, $y_3 = hg \cdot x$ satisfies $y_3 a = \chi_X a y_3$ for all $a \in A$, where X is the projection of the support of hg onto the t axis; it follows that $y_3 \in N_1(A)$. Thus $W^*(N_q(A)) \subset W^*(N_1(A))$.

Lastly, we prove that $N_1(A) \subset W^*(\mathscr{G}\mathscr{N}(A))$. Assume that $x \in N_1(A)$. There exists a local isomorphism $\phi : A \to A$, so that $xa = \phi(a)x$, for all $a \in A$. Let $x = v(x^*x)^{1/2}$ be the polar decomposition of x; let D and R be the domain and range of ϕ. Then $(x^*x)^{1/2}\chi_D = (x^*x)^{1/2}$. Moreover, for

$a \in A$, we have $x^*xa = x^*\phi^{-1}(a)x = ax^*x$, so that $[a, (x^*x)^{1/2}] = 0$. Since A is a MASA, this implies that $(x^*x)^{1/2} \in A$. Since $A \subset W^*(\mathcal{GN}(A))$ and $v \in \mathcal{GN}(A)$, $x \in W^*(\mathcal{GN}(A))$.

The same proof works to show the following:

THEOREM 2.8. *For an arbitrary diffuse unital abelian subalgebra $A \subset M$, we have $W^*(N_1(A)) = W^*(N_q(A)) = N_q^w(A)$.*

3. Conjugacy of MASAs

Let $A, B \subset M$ be diffuse commutative subalgebras. Let $\eta : B \to A$ be the restriction to B of the conditional expectation from M onto A. As a completely positive map, E defines a measure $\hat{\eta}$ on $[0, 1]^2$ by

$$\iint f(s)g(t)\, d\hat{\eta}(s,t) = \tau(fE(g)) = \tau(fg), \quad f \in A, g \in B.$$

Recall [5, Part II, Definition 5.3] that B is called discrete over A if $E : A \to B$ is discrete as a completely positive map. That is to say, in the disintegration $\hat{\eta}(s,t) = \hat{\eta}_t(s)\, dt$ the measures $\hat{\eta}_t$ are atomic for almost all t.

Let $x \in M$. Define the completely-positive maps $\lambda_x : A \to B$ and $\rho_x : B \to A$ by

$$\lambda_x(a) = E_B(xax^*), \quad \rho_x(b) = E_A(x^*bx), \qquad a \in A, b \in B.$$

DEFINITION 3.1. The relative quasi-normalizer $N_q^w(A, B)$ is defined to be the set of all $x \in M$, for which both λ_x and ρ_x are discrete.

Note that $N_q^w(A, B) = N_q^w(B, A)^*$.

THEOREM 3.2. *(compare [5]) Let $A, B \subset M$ be two MASAs in M. The following are equivalent:*

(1) *A is discrete over B and B is discrete over A;*

(2) *$A = uBu^*$ for some $u \in N_q^w(A)$;*

(3) *$N_q^w(A, B) = N_q^w(A)$;*

(4) *$A \subset N_q^w(A, B)$;*

(5) *$1 \in N_q^w(A, B)$.*

PROOF. We prove (1) ⇔ (2), (3) ⇒ (4) ⇒ (5) ⇒ (1) and (2) ⇒ (3).

We first prove that (1) implies (2); the proof is based on [5].

Consider $N = M_{2\times 2}(M)$. Let

$$D = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \subset N$$

be a commutative subalgebra, and let

$$u^* = u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N.$$

Let

$$d = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in D.$$

Then

$$\eta_u(d) = E_D(udu^*) = E_D \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} E_A(b) & 0 \\ 0 & E_B(a) \end{pmatrix}.$$

Since E_A and E_B are both discrete, it follows that $\eta_u = \eta_{u^*}$ is discrete. Hence $u \in N_q^w(D)$. Identify $A \cong L^\infty(X, \mu)$ and $B \cong L^\infty(Y, \nu)$, $D \cong L^\infty(X \sqcup Y, \mu \sqcup \nu)$. Then μ^u is supported inside the set $\{(s,t) \in (X \sqcup Y)^2 : s \in X, y \in Y \text{ or } x \in Y, y \in X\}$. Since u is a unitary, for a. e. $x \in X$, there is a $y \in Y$, so that $\mu_x^u(\{y\}) \neq 0$. Since μ_x^u is symmetric, it follows that there exists an measure-preserving isomorphism $\phi : X \to Y$, so that for each $x \in X$, $\mu_x(\{\phi(x)\}) = \mu_{\phi(x)}(\{x\}) \neq 0$. Let $f \in L^2((X \sqcup Y)^2, \mu^u)$ be the function given by

$$f(s,t) = \begin{cases} 0 & \text{if } s \neq \phi(t) \text{ and } t \neq \phi(s), \\ 1 & \text{if } s = \phi(t) \text{ or } t = \phi(s). \end{cases}$$

Let $y = f \cdot u$, and v be the polar part in the polar decomposition of y. Then v has the form

$$\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}$$

for some $w \in M$, and $[wAw^*, B] = \{0\}$. Since A and B are MASAs, this implies that $wAw^* = B$. Since $\eta_w(a) = E_A(waw^*) = (E_A|_B)(waw^*)$, it follows that η_w is discrete; since $\eta_{w^*}(a) = E_A(w^*aw) = w(E_B|_A)(a)w^*$, also η_{w^*} is discrete. Hence $w \in N_q^w(A)$.

Next, we prove that (2) implies (1). Indeed, if $w \in N_q^w(A)$, η_w and η_{w^*} are discrete. If $B = wAw^*$, it follows that $E_A|_B(b) = \eta_w(w^*bw)$ and $E_B|_A(a) = w\eta_w(a)w^*$ are both discrete.

Clearly, (3) implies (4), since $A \subset N_q^w(A)$.

Clearly, (4) implies (5), since $1 \in A$.

We now prove that (5) implies (1). If (5) holds, then $1 \in A \subset N_q(A, B)$, and hence $B \ni b \mapsto E_A(b)$ and $A \ni a \mapsto E_B(a)$ are both discrete, hence (1).

We next prove that (2) implies (3). If (2) holds, then for $x \in N_q^w(A)$ we have for $a \in A$,

$$\lambda_x(a) = E_B(xax^*) = uE_A(u^*xax^*u)u^* = u\eta_{u^*x}(a)u^*,$$

which is discrete, since $u^* \in N_q^w(A)$, $x \in N_q^w(A)$ and hence $u^*x \in N_q^w(A)$. Similarly, for $b \in B$,

$$\rho_x(b) = E_A(x^*bx) = E_A(x^*uu^*buu^*x) = \eta_{xu^*}(u^*bu),$$

which is discrete since $u^*x \in N_q^w(A)$.

COROLLARY 3.3. *If A is discrete over B and B is discrete over A, then $N_q^w(B) = N_q^w(A)$; in particular, $A \subset N_q^w(B)$ and $B \subset N_q^w(A)$.*

PROOF. By Theorem 3.2, $B = uAu^*$ for some $u \in N_q^w(A)$. Hence $N_q^w(B) = uN_q^w(A)u^* = N_q^w(A)$, since $N_q^w(A)$ is an algebra.

COROLLARY 3.4. *If $1 \in N_q^w(A, B)$, then $A \subset N_q^w(A, B)$ and also $N_q^w(A, B) = N_q^w(B, A) = N_q^w(A) = N_q^w(B)$.*

PROOF. If $1 \in N_q^w(A, B)$, then $A \subset N_q^w(A, B)$ by Theorem 3.2. If $A \subset N_q^w(A, B)$, then A is discrete over B and B is discrete over A, by Theorem 3.2. Hence $N_q^w(B, A) = N_q^w(B)$ by the same theorem. Lastly, by Corollary 3.3, $N_q^w(A) = N_q^w(B)$.

4. Cartan subalgebras

THEOREM 4.1. *Let $A \subset M$ be a MASA. Then the following conditions are equivalent:*

(1) *A is a Cartan subalgebra of M.*
(2) *The weak quasi-normalizer $N_q^w(A)$ is equal to all of M.*
(3) *The quasi-normalizer $N_q(A)$ is dense in M.*
(4) *For a self-adjoint set of unitaries u_k in M, which are strongly dense in the unitary group $U(M)$ of M, the algebras $u_kAu_k^*$ are discrete over A.*
(5) *For any unitary $u \in M$, uAu^* is discrete over A.*

PROOF. Statement (1) is equivalent to saying that $W^*(\mathcal{N}(A), A) = M$. Hence (1), (2) and (3) are equivalent, by Theorem 2.7. The condition that uAu^* is discrete over A and A is discrete over uAu^* is equivalent to the condition that $u \in N_q^w(A)$, by Theorem 3.2. Note also that uAu^* is discrete over A iff A is discrete over u^*Au. Hence (2) and (5) are equivalent. Lastly, (5) implies (4), while (4) implies that $\{u_k\} \subset N_q^w(A)$, which because u_k are strongly dense, implies (2).

COROLLARY 4.2. *Let A and B be two MASAs in M. Then the following conditions are equivalent:*

(1) *A and B are both Cartan subalgebras of M and are conjugate by a unitary in M;*

(2) $M = N_q^w(A, B)$.

PROOF. If (1) holds, then by Theorem 4.1, $N_q^w(A) = M = N_q^w(B)$. If A and B are conjugate, then by Theorem 3.2, $N_q^w(A, B) = N_q^w(A)$, so that (1) implies (2).

If (2) holds, then $A \subset N_q^w(A, B) = M$, so that A and B are conjugate by Theorem 3.2. Furthermore, since by Corollary 3.4, $M = N_q^w(A, B) = N_q^w(A) = N_q^w(B)$, A and B are both Cartan by Theorem 4.1. Thus (2) implies (1).

REFERENCES

1. Connes, A., Feldman, J., and Weiss, B., *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems 1 (1981), 431–450.
2. Dixmier, J., *Sous anneux abéliens maximaux dans les facteurs de type fini*, Ann. of Math. 59 (1954), 279–286.
3. Dye, H. A., *On groups of measure preserving transformation. I*, Amer. J. Math. 81 (1959), 119–159.
4. Dye, H. A., *On groups of measure preserving transformations. II*, Amer. J. Math. 85 (1963), 551–576.
5. Feldman, J., and Moore, C. C., *Ergodic equivalence relations, cohomology, and von Neumann algebras I, II*, Trans. Amer. Math. Soc. 234 (1977), 289–359.
6. Singer, I. M., *Automorphisms of finite factors*, Amer. J. Math. 77 (1955), 117–133.

DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA 90095
USA
E-mail: popa@math.ucla.edu, shlyakht@math.ucla.edu