EXTREME INTEGRAL POLYNOMIALS ON
A COMPLEX BANACH SPACE

SEÁN DINEEN*

Abstract

We obtain upper and lower set-theoretic inclusion estimates for the set of extreme points of the unit balls of $\mathscr{P}_I(^nE)$ and $\mathscr{P}_N(^nE)$, the spaces of n-homogeneous integral and nuclear polynomials, respectively, on a complex Banach space E. For certain collections of Banach spaces we fully characterise these extreme points. Our results show a difference between the real and complex space cases.

1. Introduction

Geometric properties of spaces of polynomials, e.g. smoothness, extreme points, exposed points, norm attaining polynomials, etc. have been investigated by a number of authors in recent years. We refer to [1], [3], [7], [8], [13], [12], [18], [20]. In particular, Ryan-Turett [18] and Boyd-Ryan [3] in their investigations examined the extreme points of the unit ball of the space of integral polynomials defined on a *real* Banach space. In this paper we study the extreme points of the unit ball of the space of integral polynomials defined on a *complex* Banach space. Throughout this paper E will, unless otherwise stated, denote a Banach space over the complex numbers C. We let B_E or B denote the open unit ball of E. We refer to [11] for basic facts on polynomials on Banach spaces and to [9], [10] for the geometry of Banach spaces.

2. Extreme Points

If E is a Banach space over C we let

$$\mathrm{Ext}_R(E) = \{ x \in E : \|x\| = 1, \|x + \lambda y\| \leq 1 \text{ for } -1 \leq \lambda \leq 1 \text{ implies } y = 0 \}$$

and

$$\mathrm{Ext}_C(E) = \{ x \in E : \|x\| = 1, \|x + \lambda y\| \leq 1 \text{ for } |\lambda| \leq 1 \text{ implies } y = 0 \}.$$

Thus, $\text{Ext}_R(E)$ is the set of all (real) extreme points of the unit ball of E and $\text{Ext}_C(E)$ is the set of complex extreme points of the same set. Clearly $\text{Ext}_R(E) \subset \text{Ext}_C(E)$. The inclusion may be proper since $\text{Ext}_R(L^1(0, 1))$ is empty while $\text{Ext}_C(L^1(0, 1))$ consists of all unit vectors.

A Banach space is strictly convex (or rotund) (respectively strictly c-convex) if

$$\text{Ext}_R(E) = \{\, x \in E : \|x\| = 1 \,\}$$

(respectively $\text{Ext}_C(E) = \{\, x \in E : \|x\| = 1 \,\}$). As noted above the Banach space $L^1(0, 1)$ is strictly c-convex but is not strictly convex.

The role of extreme points in functional analysis, convexity theory, linear programming and optimisation theory is well documented. Complex extreme points were introduced by Thorpe and Whitley [21] in order to prove a strong maximum modulus principle for Banach-valued holomorphic functions. Since their introduction they have proved useful in the study of the Shilov boundary, complex geodesics, invariant metrics, bounded symmetric domains and JB*-triple systems. We refer to [2], [4], [5], [14], [15], [22], [23] for details.

3. Integral and Nuclear Polynomials

If E is a complex Banach space $\mathscr{P}(^nE)$ denotes the space of continuous n-homogeneous polynomials on E endowed with the norm $\|\cdot\| := \|\cdot\|_B$ of uniform convergence over the unit ball B of E. A polynomial $P \in \mathscr{P}(^nE)$ is said to be *integral* if there exists a regular Borel measure μ on the unit ball of E', $B_{E'}$ endowed with the weak* topology, such that

$$(1) \qquad\qquad P(x) = \int_{B_{E'}} \phi(x)^n \, d\mu(\phi)$$

for all x in E. We let $\mathscr{P}_I(^nE)$ denote the space of all n-homogeneous integral polynomials on E and we endow this space with the norm $\|\cdot\|_I := \inf \|\mu\|$ where the infimum is taken over all μ satisfying (1).

An n-homogeneous polynomial P on E is *nuclear* if there exists a bounded sequence $(\phi_j)_j \subset E'$ and $(\lambda_j)_j \in l_1$ such that

$$(2) \qquad\qquad P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n$$

for all x in E. The space of all nuclear n-homogeneous polynomials on E, $\mathscr{P}_N(^nE)$, is a Banach space when $\|P\|_N$ is defined to be the infimum of $\sum_{j=1}^{\infty} |\lambda_j| \, \|\phi_j\|^n$ taken over all representations of P satisfying (2).

PROPOSITION 3.1. *Let E denote a complex Banach space.*

(a) *The space $\mathscr{P}_I({}^nE)$ is isometrically isomorphic to $\left(\widehat{\bigotimes}_{n,s,\epsilon} E\right)'$.*

(b) $\mathscr{P}_N({}^nE) \subset \mathscr{P}_I({}^nE) \subset \mathscr{P}({}^nE)$ *and* $\|\cdot\| \le \|\cdot\|_I \le \|\cdot\|_N$.

(c) *If $\phi \in E'$ then $\phi^n \in \mathscr{P}_N({}^nE)$ and*

$$\|\phi\|^n = \|\phi^n\|_I = \|\phi^n\|_N$$

(d) *If E' has the approximation property then $\mathscr{P}_N({}^nE)$ is isometrically isomorphic to $\widehat{\bigotimes}_{n,s,\pi} E'$ and $\mathscr{P}_N({}^nE)' = \mathscr{P}({}^nE')$ isometrically.*

(e) *If $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{n,s,\epsilon} E$ (in particular if E' has the Radon-Nikodým Property) then $\mathscr{P}_I({}^nE)$ and $\mathscr{P}_N({}^nE)$ are isometrically isomorphic.*

(f) $\text{Ext}_R(\mathscr{P}_I({}^nE)) \subset \{\phi^n : \phi \in E', \|\phi\| = 1\}$.

(g) $\text{Ext}_R(\mathscr{P}_I({}^nE)) \subset \text{Ext}_R(\mathscr{P}_N({}^nE))$.

PROOF. Parts (a), (b), (c) and (d) are well known. Parts (e) and (f) are due independently to C. Boyd-R. A. Ryan [3] and D. Carando-V. Dimant [6] while (g) can be deduced from the proof of [3, Theorem 2].

REMARKS. (i) By (b) and (g) we have

$$\mathscr{P}_N({}^nE) \subset \mathscr{P}_I({}^nE) \qquad \text{and} \qquad \text{Ext}_R(\mathscr{P}_I({}^nE)) \subset \text{Ext}_R(\mathscr{P}_N({}^nE)).$$

This contrasting information is quite useful since $\text{Ext}_R(\mathscr{P}_I({}^nE))$ has good abstract properties while calculations are easier with elements of $\text{Ext}_R(\mathscr{P}_N({}^nE))$.

(ii) In general we do not know if $\mathscr{P}_N({}^nE)$ is a dual Banach space but, by (a) and (g), $\text{Ext}_R(\mathscr{P}_N({}^nE))$ is non-empty.

For *real* Banach space we also have the following result which is in contrast to the results we obtain in the next section for *complex* Banach spaces.

PROPOSITION 3.2 ([3], [18]). *If E is a reflexive Banach space over R and $n > 1$ then*
$$\text{Ext}_R(\mathscr{P}_I({}^nE)) = \{\pm\phi^n : \phi \in E', \|\phi\| = 1\}.$$

4. Extreme Polynomials

PROPOSITION 4.1. *If E is a complex Banach space and $n \ge 1$ then*

$$\{\phi \in E' : \phi^n \in \text{Ext}_R(\mathscr{P}_N({}^nE))\} \subset \text{Ext}_C(E').$$

PROOF. Suppose $\|\phi\| = 1$ and $\phi \notin \text{Ext}_C(E')$. Then there exists $\psi \in E'$, $\psi \neq 0$, such that $\|\phi + \lambda\psi\| \leq 1$ for all $\lambda \in \mathsf{C}$, $|\lambda| \leq 1$. If $\omega = e^{\frac{2\pi i}{n+1}}$ then

$$\sum_{j=0}^{n} (\phi + \omega^j \psi)^n = \sum_{k=0}^{n} \binom{n}{k} \psi^k \phi^{n-k} \left(\sum_{j=0}^{n} \omega^{kj} \right).$$

If $k \neq 0$ then $\sum_{j=0}^{n} \omega^{kj} = 0$. Hence

$$\frac{1}{n+1} \sum_{j=0}^{n} (\phi + \omega^j \psi)^n = \phi^n.$$

Since $\|\phi\| = 1$ and $\|\phi + \omega^j \psi\| \leq 1$ for all j, Proposition 3.1(c) implies $\|\phi^n\|_N = 1$ and, $\|(\phi + \omega^j \psi)^n\|_N \leq 1$ for all j. Hence $\phi^n \notin \text{Ext}_R(\mathscr{P}_N(^nE))$. This completes the proof.

If E is a complex Banach space we say that x and y in E are equivalent if there exists $\alpha \in \mathsf{C}$, $|\alpha| = 1$, such that $x = \alpha y$. If $A \subset E$ we denote by $\diamond[A]$ the cardinality of the set of equivalence classes in A. For example it is well known that $\diamond[\text{Ext}_R(\ell_1^n)] = n$. For complex extreme points we obtain a different type of result.

COROLLARY 4.2. *If E is a finite dimensional complex Banach space of dimension ≥ 2 then*

$$\diamond[\text{Ext}_C(E)] = \infty.$$

PROOF. By Proposition 4.1 and the Krein-Milman Theorem

$$\dim(\mathscr{P}_I(^nE')) \leq \diamond[\text{Ext}_C(E)]$$

for any positive integer n. Since the monomials $z^j w^{n-j}$, $j = 1, 2, \ldots, n$ are linearly independent, $\dim(\mathscr{P}_I(^nE')) = \dim(\mathscr{P}(^nE'))$ is at least n. Hence $\diamond[\text{Ext}_C(E)] \geq n$ for all n. This completes the proof.

If E is a Banach space we say that $A \subset E'$ is *E-transitive* if for all θ, ϕ in A there exists an isometry of E onto itself, T, such that $\phi \circ T = \theta$, that is $^tT(\phi) = \theta$ where ^tT denotes the transpose of T.

COROLLARY 4.3. *If the set of complex extreme points of the unit ball of E' is E-transitive then*

$$\text{Ext}_R(\mathscr{P}_I(^nE)) = \{ \phi^n : \phi \in E', \phi \in \text{Ext}_C(E') \}$$

and, in particular,

$$\text{Ext}_R(E') = \text{Ext}_C(E').$$

PROOF. By Proposition 3.1(a), (f) and (g) and Proposition 4.1, $\text{Ext}_R(\mathscr{P}_I(^nE))$ contains an element ϕ^n where $\phi \in \text{Ext}_C(E')$. Let $\psi \in \text{Ext}_C(E')$. By E-transitivity of the set of complex extreme points there exists an isometry T of E such that $\psi \circ T = \phi$. If $P \in \mathscr{P}_I(^nE)$ and $\|\psi^n \pm P\|_I \leq 1$ then $\|\psi^n \circ T \pm P \circ T\|_I \leq 1$. Since $\psi^n \circ T = (\psi \circ T)^n = \phi^n$ and $\phi^n \in \text{Ext}_R(\mathscr{P}_I(^nE))$ this implies $P \circ T = 0$. Hence $P = 0$ and $\psi^n \in \text{Ext}_R(\mathscr{P}_I(^nE))$. An application of Proposition 4.1 completes the proof.

EXAMPLE 4.4. Let $\mathscr{A}$ denote a $\mathscr{W}^*$-algebra , i.e. a $\mathscr{C}^*$-algebra which is also a dual Banach space. We can suppose, without loss of generality, that $\mathscr{A}$ is a $\mathscr{C}^*$-subalgebra of $\mathscr{B}(\mathscr{H})$, the space of bounded linear operators on the Hilbert space $\mathscr{H}$. By [15] the real and complex extreme points of the unit ball of a $\mathscr{C}^*$-algebra coincide.

Let

$$\mathscr{A}^{\perp} = \{\, w \in \mathscr{N}(\mathscr{H}) : \text{trace}(vw) = 0 \text{ for all } v \in \mathscr{A} \,\}.$$

where $\mathscr{N}(\mathscr{H})$ is the space of trace class operators on $\mathscr{H}$. By [17, Theorem 4.2.9], $\mathscr{A}_* := \mathscr{N}(\mathscr{H})/\mathscr{A}^{\perp}$ is the unique isometric predual of $\mathscr{A}$. The $(\mathscr{A}, \mathscr{A}_*)$ duality is given by

$$(u, w + \mathscr{A}^{\perp}) = \text{trace}(uw) = \sum_{x \in E} \langle uw(x), x \rangle$$

where $u \in \mathscr{A}, w + \mathscr{A}^{\perp} \in \mathscr{A}_*$ and E is an orthonormal basis for $\mathscr{H}$. Let $\mathscr{U}$ denote the set of unitary elements in $\mathscr{A}$, i.e.

$$\mathscr{U} = \{\, u \in \mathscr{A} : uu^* = u^*u = 1_{\mathscr{H}} \,\}.$$

Clearly $uw \in \mathscr{A}^{\perp}$ for u unitary in $\mathscr{A}$ and w in $\mathscr{A}^{\perp}$. Hence the mapping

$$u_t : \mathscr{A}_* \hookrightarrow \mathscr{A}_*, u_t(v + \mathscr{A}^{\perp}) := uv + \mathscr{A}^{\perp}$$

is well defined and easily seen to be an isometry of $\mathscr{A}_*$. If v_1 and v_2 are unitary elements in $\mathscr{A}$ then $v_2^* v_1$ is also unitary and for all $w + \mathscr{A}^{\perp} \in \mathscr{A}_*$ we have

$$(v_2 \circ (v_2^* v_1)_t, w + \mathscr{A}^{\perp}) = (v_2, v_2^* v_1 w + \mathscr{A}^{\perp}) = \sum_{x \in E} \langle v_2 v_2^* v_1 w(x), x \rangle$$

$$= \sum_{x \in E} \langle v_1 w(x), x \rangle = (v_1, w + \mathscr{A}^{\perp}).$$

Hence

$$v_1 = v_2 \circ (v_2^* v_1)_t$$

and $\mathscr{U}$ is $\mathscr{A}_*$-transitive. Results in [16] (see also Example 4.6) imply that

$$\mathrm{Ext}_C(\mathscr{A}) = \mathrm{Ext}_R(\mathscr{A}) = \mathscr{U}$$

i.e. the real (and complex) extreme points coincide with the set of unitary elements when $\mathscr{A}$ is one of the following :

(a) a commutative $\mathscr{W}^*$-algebra,

(b) $\mathscr{B}(\mathscr{H})$, $\mathscr{H}$ finite dimensional,

(c) a type II_1 factor.

Hence, by Corollary 4.3, if $\mathscr{A}$ is any of the above $\mathscr{W}^*$-algebras then

$$\mathrm{Ext}_R(\mathscr{P}_I(^n\mathscr{A}_*)) = \{\, \phi^n : \phi \in \mathrm{Ext}_C(\mathscr{A})\,\}.$$

If $\dim(\mathscr{A}) \geq 2$ then $\mathrm{Ext}_C(\mathscr{A})$ is a proper subset of $\{x : x \in \mathscr{A}, \|x\| = 1\}$. (See for instance [5, Theorem 1.9]). For example if $\mathscr{A} = \ell_\infty^m$ then its predual is ℓ_1^m and $\mathrm{Ext}_R(\mathscr{P}_I(^n\ell_1^m))$ consists of all ϕ^n where ϕ lies in the distinguished boundary of the unit polydisc in $\mathbf{C}^m$.

The above method does not extend to all $\mathscr{W}^*$-algebras. By [16, Corollary 2]

$$\mathrm{Ext}_C(\mathscr{B}(\mathscr{H})) = \{\, u \in \mathscr{B}(\mathscr{H}) : uu^* = I_{\mathscr{H}} \text{ or } u^*u = I_{\mathscr{H}}\,\}.$$

If $\mathscr{H}$ is infinite dimensional then the forward shift, S, is a non-unitary extreme point of the unit ball of $\mathscr{B}(\mathscr{H})$. By [16] isometries of $\mathscr{B}(\mathscr{H})$ map extreme points to extreme points and unitaries to unitaries. Hence no isometry of $\mathscr{B}(\mathscr{H})$ maps S to $I_{\mathscr{H}}$ and $\mathrm{Ext}_C(\mathscr{B}(\mathscr{H})) = \mathrm{Ext}_R(\mathscr{B}(\mathscr{H}))$ is not $\mathscr{B}(\mathscr{H})_*$-transitive.

PROPOSITION 4.5. *If E is a finite dimensional complex Banach space then*

$$\{\, \phi^n : \phi \in \mathrm{Ext}_R(E')\,\} \subset \mathrm{Ext}_R(\mathscr{P}_I(^nE))$$

for all n.

PROOF. Let $\phi \in \mathrm{Ext}_R(E')$. Suppose $\phi^n \notin \mathrm{Ext}_R(\mathscr{P}_I(^nE))$. Since E is finite dimensional Proposition 3.1(e) implies that $\phi^n \notin \mathrm{Ext}_R(\mathscr{P}_N(^nE))$.

Hence there exist P and Q in $\mathscr{P}_N(^nE)$, $P \neq Q$, and $\lambda \in \mathbf{R}$, $0 < \lambda < 1$, such that

$$\phi^n = \lambda P + (1 - \lambda)Q$$

and

$$\|P\| = \|Q\| = 1.$$

Since E is finite dimensional so also is $\mathscr{P}_N(^nE)$. Hence the unit ball of $\mathscr{P}_N(^nE)$ is the convex hull of its extreme points and thus, by Proposition 3.1(f), there exists $(\phi_i)_{i=1}^{k+l} \subset E'$, $\phi_i \neq 0$, such that $\lambda P = \sum_{i=1}^{k} \phi_i^n$, $\|\lambda P\|_N = \sum_{i=1}^{k} \|\phi_i^n\|$, $(1 - \lambda)Q = \sum_{i=k+1}^{k+l} \phi_i^n$ and $\|(1 - \lambda)Q\|_N = \sum_{i=k+1}^{k+l} \|\phi_i^n\|$.

Hence

$$\|\phi^n\|_N = \|\phi^n\| = 1 = \sum_{i=1}^{k+l} \|\phi_i^n\|.$$

Choose $x_0 \in E$, $\|x_0\| = 1$, such that $\phi(x_0) = 1$. On differentiating $\phi^n = \sum_{i=1}^{k+l} \phi_i^n$ at x_0 we obtain

$$\phi = \phi^{n-1}(x_0)\phi = \sum_{i=1}^{k+l} \phi_i^{n-1}(x_0)\phi_i.$$

Hence

$$1 = \|\phi\| \leq \sum_{i=1}^{k+l} |\phi_i^{n-1}(x_0)| \|\phi_i\| \leq \sum_{i=1}^{k+l} \|\phi_i\|^n = 1.$$

and we can choose for each i, $\beta_i \in \mathsf{C}$, with $|\beta_i| = 1$ such that

$$\phi = \sum_{i=1}^{k+l} |\phi_i^{n-1}(x_0)| \cdot \|\phi_i\| \cdot \frac{\beta_i \phi_i}{\|\phi_i\|}.$$

Since $\phi \in \mathrm{Ext}_R(E')$ this implies that for each i, $\phi_i = \alpha_i \phi$ for some $\alpha_i \in \mathsf{C}$. Hence $P = \phi^n = Q$. This contradicts our hypothesis and shows that $\phi^n \in \mathrm{Ext}_R(\mathscr{P}_N(^nE)) = \mathrm{Ext}_R(\mathscr{P}_I(^nE))$ and completes the proof.

EXAMPLE 4.6.   If $\mathscr{J}$ is a JB*-algebra then, by [4, Lemma 4.1] and [15, Theorem 11],
$$\mathrm{Ext}_R(\mathscr{J}) = \mathrm{Ext}_C(\mathscr{J}).$$

Hence Propositions 4.1 and 4.5 imply

$$\{\phi^n : \phi \in \mathrm{Ext}_R(\mathscr{J})\} = \mathrm{Ext}_R(\mathscr{P}_I(^n\mathscr{J}'))$$

for any finite dimensional JB*-algebra $\mathscr{J}$. This result gives an independent proof of the result in Example 4.4 for finite dimensional $\mathscr{C}^*$-algebras.

EXAMPLE 4.7.   If E' is a strictly convex finite dimensional Banach space then
$$\mathrm{Ext}_R(\mathscr{P}_I(^nE)) = \{\phi^n : \phi \in E', \|\phi\| = 1\}.$$

This is similar to the result for real Banach spaces quoted above (Proposition 3.2).

To extend Proposition 4.5 to infinite dimensional spaces we require *weak*-exposed* points. For convenience we introduce these in complex form.

DEFINITION 4.8.   Let E denote a Banach over C. A linear functional $\phi \in E'$ is a *weak*-exposed point* of the unit ball of E' if $\|\phi\| = 1$ and there exists

$x \in E$, $\|x\| = 1$, such that $\Re(\phi(x)) = 1$ and $\Re(\psi(x)) < 1$ for all $\psi \in E'$, $\|\psi\| \leq 1$ and $\psi \neq \phi$. We say that x weak*-exposes ϕ. We let $\mathrm{Exp}_{\omega^*}(E')$ denote the set of all weak*-exposed points of the unit ball of E'.

If $\phi : E \longrightarrow \mathsf{C}$ is complex linear then $\tilde{\phi} := \Re(\phi)$ is a real linear mapping. Moreover, $\|\phi\| = \|\tilde{\phi}\|$ and $\phi(x) = \tilde{\phi}(x) - i\tilde{\phi}(ix)$ for all x in E. Hence if ϕ_1 and ϕ_2 are complex linear then $\phi_1 = \phi_2$ if and only if $\Re(\phi_1) = \Re(\phi_2)$.

If x weak*-exposes ϕ and $\phi = \frac{\phi_1+\phi_2}{2}$ with $\|\phi_1\| = \|\phi_2\| = 1$ then, $|\phi_1(x)| \leq 1$ and $|\phi_2(x)| \leq 1$ imply $\phi_1(x) = \phi_2(x) = 1$. Hence $\phi_1 = \phi_2 = \phi$ and $\mathrm{Exp}_{\omega^*}(E') \subset \mathrm{Ext}_R(E')$.

For our next result, which was motivated by [3, Proposition 5], we require the following Lemma [3, Lemma 4].

LEMMA 4.9. *Let E be a normed space and let ϕ be a unit vector in E'. Suppose that for each finite dimensional subspace F of E there exists a subspace G of E, $F \subset G$, such that $\phi|_G$ is an extreme point of the unit ball of G'. Then ϕ is an extreme point of the unit ball of E'.*

PROPOSITION 4.10. *If E is a complex Banach space then*

$$\{\, \phi^n : \phi \in \mathrm{Exp}_{\omega^*}(E') \,\} \subset \mathrm{Ext}_R(\mathscr{P}_I(^nE)).$$

PROOF. Let $\phi \in \mathrm{Exp}_{\omega^*}(E')$ and suppose x weak*-exposes ϕ. Given a finite dimensional subspace X of $\bigotimes_{n,s,\epsilon} E$ choose a finite dimensional subspace F of E such that $x \in F$ and $X \subset \bigotimes_{n,s,\epsilon} F$. Since $x \in F$, $\|\phi|_F\| = 1$. Let $\psi \in F'$, $\|\psi\| = 1$ and $\psi \neq \phi|_F$. By the Hahn-Banach Theorem there exists $\tilde{\psi} \in E'$ such that $\tilde{\psi}|_F = \psi$ and $\|\tilde{\psi}\| = \|\psi\| = 1$. Since $\tilde{\psi} \neq \phi$ we have $\Re(\tilde{\psi}(x)) = \Re(\psi(x)) < 1$ and $\phi|_F \in \mathrm{Exp}_{\omega^*}(F') \subset \mathrm{Ext}_R(F')$.

By Proposition 4.5

$$(\phi|_F)^n = \phi^n|_F \in \mathrm{Ext}_R(\mathscr{P}_I(^nF)) = \mathrm{Ext}_R\Big(\big(\bigotimes_{n,s,\epsilon} F\big)'\Big).$$

By Proposition 3.1(a) and Lemma 4.9,

$$\phi^n \in \mathrm{Ext}_R\Big(\big(\bigotimes_{n,s,\epsilon} E\big)'\Big) = \mathrm{Ext}_R(\mathscr{P}_I(^nE)).$$

This completes the proof.

EXAMPLE 4.11. If E is a complex Banach space let

$D(E) = \{x \in E : \|x\| = 1$ and $f(\cdot) := \|\cdot\|$ is real Gâteaux differentiable at $x\}$.

Then, see [19] and [10],

$$\mathrm{Exp}_{\omega^*}(E') = \{\, \phi \in E' : \phi = f'(x) \text{ for some } x \in D(E) \,\}.$$

For many classical Banach spaces, e.g. ℓ_p, $1 < p < \infty$, this result can be used to show that

$$\{\, \phi : \phi \in E', \|\phi\| = 1 \,\} = \mathrm{Exp}_{\omega^*}(E')$$

and hence, by Proposition 4.10, that

$$\mathrm{Ext}_R(\mathscr{P}_I(^nE)) = \{\, \phi^n : \phi \in E', \|\phi\| = 1 \,\}.$$

If $E = \ell_1$ then $x = (x_n)_n \in D(\ell_1)$ if and only if $x_n \neq 0$ for all n. Hence $\phi := (y_n)_n \in \mathrm{Exp}_{\omega^*}(\ell_\infty)$ if and only if $|y_n| = 1$ for all n and $\mathrm{Exp}_{\omega^*}(\ell_\infty) = \mathrm{Ext}_R(\ell_\infty)$. Since $\mathrm{Ext}_R(\ell_\infty) = \mathrm{Ext}_C(\ell_\infty)$ this implies

$$\{\, \phi^n : \phi \in \mathrm{Ext}_R(\ell_\infty) \,\} = \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_1))$$

and we recover a special case of Example 4.4 since ℓ_∞ is a $\mathscr{W}^*$-algebra.

EXAMPLE 4.12.   Let ℓ_∞^m denote $\mathbf{C}^m$ endowed with the supremum norm. By Proposition 3.1(f)

$$\mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m)) \subset \{\, \phi^n : \phi \in \ell_1^m, \|\phi\| = 1 \,\}$$

for $n \geq 2$. If $\phi := (w_1, \ldots, w_m) \in \ell_1^m$, $\theta := (\theta_1, \ldots, \theta_m) \in \mathbf{R}^m$ and σ is a permutation of $\{1, \ldots, m\}$ let

$$\phi_{\theta,\sigma} := (e^{i\theta_1} w_{\sigma(1)}, \ldots, e^{i\theta_m} w_{\sigma(m)}).$$

It is easily seen that $\phi_{\theta,\sigma}^n \in \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$ if and only if $\phi^n \in \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$. Hence to show

$$(3) \qquad \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m)) = \{\, \phi^n : \phi \in \ell_1^m, \|\phi\| = 1 \,\}.$$

it suffices to show that $\alpha^n \in \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$ where $\alpha := (\alpha_1, \ldots, \alpha_m)$ satisfies $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$ and $\sum_{i=1}^m \alpha_i = 1$. Let l denote the smallest positive integer such that $\alpha_l > 0$. If $l = m$ then $\alpha \in \mathrm{Ext}_R(\ell_1^m)$ and Proposition 4.5 implies that $\alpha^n \in \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$. We may thus suppose, from now on, that $l < m$.

By Proposition 3.1(d), $(\mathscr{P}_I(^n\ell_\infty^m))' = \mathscr{P}(^n\ell_1^m)$ with duality $\langle \phi^n, P \rangle = P(\phi)$ for $P \in \mathscr{P}(^n\ell_1^m)$ and $\phi \in (\ell_\infty^m)' = \ell_1^m$. By the Krein-Milman Theorem $|P|$, $P \in \mathscr{P}(^n\ell_1^m)$, achieves its maximum over the unit ball of ℓ_1^m at a point ϕ where $\phi^n \in \mathrm{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$.

Now consider the 2-homogeneous polynomial on ℓ_1^m

$$P_\alpha(z_1, \ldots, z_m) := \sum_{l \leq i < j \leq m} z_i z_j + \sum_{k \geq l} \frac{\alpha_k - \alpha_l}{2\alpha_k} z_k^2.$$

Since

$$|P_\alpha(z_1, \ldots, z_m)| = |P_\alpha(0, \ldots, 0, z_l, \ldots, z_m)|$$

$$\leq \sum_{l \leq i < j \leq m} |z_i||z_j| + \sum_{k \geq l} \frac{\alpha_k - \alpha_l}{2\alpha_k} |z_k|^2$$

$$= P_\alpha(0, \ldots, 0, |z_l|, \ldots, |z_m|)$$

$$\leq \|P_\alpha\| \cdot \left(\sum_{k \geq l} |z_k| \right)^2$$

it follows that $|P_\alpha|$ achieves its maximum, $\|P_\alpha\| := \|P_\alpha\|_{B_{\ell_1^m}}$, at some point which has the form $e^{i\theta}(0, \ldots, 0, x_l, \ldots, x_m)$ where $x_i \geq 0$ for $i \geq l$, $\sum_{i=l}^m x_i = 1$ and $\theta \in \mathbf{R}$.

Applying the method of Lagrange multipliers to the problem of maximizing $|P_\alpha|$ on the set $\mathbf{T} := \left\{ (0, \ldots, 0, x_l, \ldots, x_m) : x_i > 0 \text{ and } \sum_{i=l}^m x_i = 1 \right\}$ yields the equations

$$\sum_{i > l} x_i = \lambda = 1 - x_l$$

and

$$\left(\sum_{i=l}^m x_i \right) - x_k + \frac{\alpha_k - \alpha_l}{2\alpha_k} 2x_k = \lambda$$

for $k > l$. Hence $\alpha_l x_k = \alpha_k x_l$ for $k \geq l$. Since $\sum_{i=l}^m x_i = \sum_{i=l}^m \alpha_i = 1$ this implies $\alpha_i = x_i$ for $i \geq l$.

Hence

$$\sup\{ |P_\alpha(x)| : x \in \mathbf{T} \} = \sum_{l \leq i < j \leq m} \alpha_i \alpha_j + \sum_{k \geq l} \frac{\alpha_k - \alpha_l}{2\alpha_k} \alpha_k^2$$

$$= \frac{1}{2} \left(\sum_{i=l}^m \alpha_i \right)^2 - \frac{1}{2} \alpha_l \cdot \sum_{k=l}^m \alpha_k$$

$$= \frac{1}{2} (1 - \alpha_l).$$

If $l^* \geq l$, $\{l^*\} \subset S \subset \{l^*, \ldots, m\}$ let

$$\mathbf{S} = \left\{ x \in \mathbf{R}^m : x = (0, \ldots, 0, x_{l^*}, \ldots, x_m), \right.$$
$$\left. x_i > 0 \text{ if } i \in S, x_i = 0 \text{ if } i \notin S \text{ and } \sum_{i=l}^{m} x_i = 1 \right\}.$$

An analysis similar to the above shows that

$$\sup\{ |P_\alpha(x)| : x \in \mathbf{S} \} = \frac{\left(\sum_{i \in \mathbf{S}} \alpha_i\right) - \alpha_l}{2\left(\sum_{i \in \mathbf{S}} \alpha_i\right)} = \frac{1}{2}\left(1 - \frac{\alpha_l}{\sum_{i \in \mathbf{S}} \alpha_i}\right)$$

Since $\alpha_i > 0$ when $i \geq l$ this shows that $\|P_\alpha\| = |P_\alpha(z_1, \ldots, z_m)|$, $\sum_{i=1}^{m} |z_i| = 1$, if and only if $(z_1, \ldots, z_m) = e^{i\theta}\alpha$ for some $\theta \in \mathbf{R}$. Hence $\alpha^2 \in \text{Ext}_R(\mathscr{P}_I(^2\ell_\infty^m))$ and we have established (3) when $n = 2$.

If $\beta := (1, 1, \ldots, 1) \in \ell_\infty^m$ then $\|\beta\| = 1$ and $\langle \beta, (z_1, \ldots, z_m) \rangle = \sum_{i=1}^{m} z_i$ for $(z_1, \ldots, z_m) \in \ell_1^m$. Hence $|\beta|$ achieves its maximum over the unit ball of ℓ_1^m *precisely* at *all* points of the form $e^{i\theta}(x_1, \ldots, x_m)$ where $x_i \geq 0$, $\sum_{i=1}^{m} x_i = 1$ and $\theta \in \mathbf{R}$.

If $n > 2$ then $Q := P_\alpha \cdot \beta^{n-2}$ is an n-homogeneous polynomial on ℓ_1^m and $|Q|$ achieves its maximum over the unit ball of ℓ_1^m precisely at those points which have the form $e^{i\theta}\alpha$ where $\theta \in \mathbf{R}$ is arbitrary. Hence $\alpha^n \in \text{Ext}_R(\mathscr{P}_I(^n\ell_\infty^m))$ and we have established (3) for all $n \geq 2$ and all m.

Note that the function

$$f((y_n)_n) := 1 + P_\alpha\left(\left(e^{i\theta_n} y_{\sigma(n)}\right)_{n=1}^{m}\right)$$

satisfies $|f(y)| < |f(x)|$ for all $y \neq x$ in the closed unit ball of ℓ_1. This may be used to show that all finitely supported unit vectors are peak points of different algebras of holomorphic functions (we refer to [2] for details).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE DUBLIN
BELFIELD, DUBLIN 4
IRELAND