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INDICES, CONVEXITY AND CONCAVITY OF CALDERÓN-LOZANOVSKII SPACES

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Abstract

In this article we discuss lattice convexity and concavity of Calderón-Lozanovskii space E_φ, generated by a quasi-Banach space E and an increasing Orlicz function φ. We give estimations of convexity and concavity indices of E_φ in terms of Matuszewska-Orlicz indices of φ as well as convexity and concavity indices of E. In the case when E_φ is a rearrangement invariant space we also provide some estimations of its Boyd indices. As corollaries we obtain some necessary and sufficient conditions for normability of E_φ, and conditions on its nontrivial type and cotype in the case when E_φ is a Banach space. We apply these results to Orlicz-Lorentz spaces receiving estimations, and in some cases the exact values of their convexity, concavity and Boyd indices.

0. Introduction

The Calderón-Lozanovskii spaces E_φ, where E is a Banach space and φ is a convex Young function, have been recently studied in several articles mostly in order to characterize their geometric properties like (local) uniform rotundity or monotonicity conditions (e.g. [2], [6], [7]). Here we extend our studies to the Calderón-Lozanovskii spaces E_φ, generated by a quasi-Banach space E and an increasing Orlicz function φ. This more general setting seems to be a natural environment for the main purpose of this article which is an investigation of the lattice convexity and concavity of E_φ.

The paper is divided into five parts. The first one, called preliminaries, contains all necessary definitions and recalls some auxiliary results. In the second part we discuss some basic properties of the Calderón-Lozanovskii spaces E_φ such as the Fatou property, its completeness and we state some results on comparison between these spaces generated by different functions φ and ψ. The third part consists of the main results of the paper, which are estimations of convexity and concavity indices and some corollaries on nontrivial type and cotype of E_φ. The Boyd indices of E_φ, in the case when E_φ is a rearrangement

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invariant space, are studied in the fourth part. Finally, we apply these results to Orlicz-Lorentz spaces, that are treated separately in the fifth section. In certain type of Orlicz-Lorentz spaces some results along this line have been also obtained in [15] and [23].

1. Preliminaries

We start with some notions and definitions which we will need further in the paper. In the following N, R, R_+ and $\overline{\mathsf{R}}_+$ stand for the sets of natural numbers, reals, nonnegative reals and interval $[0, \infty]$, respectively. Given a vector space X the functional $x \mapsto \|x\|$ is called a *quasi-norm* if the following three conditions are satisfied: $\|x\| = 0$ iff $x = 0$; $\|ax\| = |a|\|x\|$, $x \in X$, $a \in \mathsf{R}$; there exists $C \geq 1$ such that $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$, $x_1, x_2 \in X$. We will say that $\mathrm{X} = (X, \| \ \|)$ is a *quasi-Banach space* if it is complete. For $0 < p \leq 1$, $x \mapsto \|x\|$ is called a *p-norm* if it satisfies the first two conditions of the quasi-norm and the condition that for any $x_1, x_2 \in X$, $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$. Recall that the Aoki-Rolewicz theorem (cf. [9]) states that for any quasi-normed space there exists an equivalent p-norm for some $0 < p \leq 1$. We say that a quasi-Banach space X is *p-normable*, $0 < p \leq 1$, if there exists in X a p-norm equivalent to the quasi-norm in X. In the case when $p = 1$ we simply say that the space is *normable*. A quasi-Banach space $(X, \| \ \|)$ which in addition is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ is called a *quasi-Banach lattice*. Following Kalton in [8], a quasi-Banach lattice X or its quasi-norm $\| \ \|$ is said to be *p-convex* (order), $0 < p < \infty$, respectively *q-concave* (order), $0 < q < \infty$, if there is a constant $K > 0$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

respectively,

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}} \leq K \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} \right\|$$

for every choice of vectors $x_1, \ldots, x_n \in X$. A quasi-Banach lattice X is said to satisfy an *upper p-estimate*, $0 < p < \infty$, respectively a *lower q-estimate*, $0 < q < \infty$, if the definition of p-convexity, respectively q-concavity, holds true for any choice of disjointly supported elements $x_1, \ldots, x_n$ in X.

It is known that given $0 < p < \infty$, if X is p-convex (resp. p-concave), then X is r-convex (resp. r-concave) for $0 < r < p$ (resp. $r > p$) ([3],[8]). We also observe that for $0 < p \leq 1$, p-convexity implies p-normability and this in turn yields an upper p-estimate. The opposite implication is not satisfied since the weak space $L_{p,\infty}(0,1)$, $0 < p < 1$, is p-normable but not p-convex (cf. [8]). For $p = 1$, 1-convexity is equivalent to normability.

Given a quasi-Banach lattice X we define two types of *convexity and concavity indices* as follows:

$$
\begin{aligned}
p_c(X) &= \sup\{p > 0 : X \text{ is } p\text{-convex}\},\\
q_c(X) &= \inf\{q > 0 : X \text{ is } q\text{-concave}\},\\
p_d(X) &= \sup\{p > 0 : X \text{ satisfies an upper } p\text{-estimate}\},\\
q_d(X) &= \inf\{q > 0 : X \text{ satisfies a lower } q\text{-estimate}\}.
\end{aligned}
$$

The indices $p_d(X)$ and $q_d(X)$ were introduced by T. Shimogaki in 1965 for order complete Banach lattices, by J. J. Grobler in 1975 for Banach function spaces and in 1977 by P. Dodds for general Banach lattices (cf. [28] for suitable references). Obviously $p_c(X) \le p_d(X) \le q_d(X) \le q_c(X)$, and by the Aoki-Rolewicz theorem, $p_d(X) > 0$. It is also well known that for Banach lattices, $p_c(X) = p_d(X)$ and $q_c(X) = q_d(X)$ ([17]). For quasi-Banach lattices Kalton proved (Th. 2.2 in [8]) that $p_c(X) = p_d(X)$ iff X is L-convex, i.e., there exists $0 < \epsilon < 1$ so that if $y \in X$ with $\|y\| = 1$ and $0 \le x_i \le y$, $i = 1, \dots, n$, satisfy $(x_1 + \dots + x_n)/n \ge (1 - \epsilon)y$, then $\max_{1 \le i \le n} \|x_i\| \ge \epsilon$. He also gave an example of a quasi-Banach space X which is not L-convex, that is $0 = p_c(X) < p_d(X)$.

Given $0 < p < \infty$ and a quasi-Banach lattice X let $X^{(p)}$ denote the p-convexification of X. Recall that $X^{(p)} = \{x : |x|^p \in X\}$ and $\|x\|_{X^{(p)}} = \||x|^p\|^{1/p}$ is a quasi-norm in $X^{(p)}$. Observe that $X^{(p)}$ is 1-convex (resp. 1-concave) iff X is $1/p$-convex (resp. $1/p$-concave).

By L^0 we denote the space of all (equivalence classes of) Lebesgue-measurable functions f from I to R, where either $I = (0,1]$ or $I = (0,\infty)$ or $I = \mathsf{N}$. In the latter case L^0 is the space of all real valued sequences defined on a discrete measure space $(\mathsf{N}, 2^{\mathsf{N}})$ with a counting measure. A *quasi-normed function space* $E = (E, \|\ \|_E)$ is a quasi-normed sublattice of L^0 such that

(i) If $f \in L^0$, $g \in E$ and $|f| \le |g|$ a.e., then $f \in E$ and $\|f\|_E \le \|g\|_E$.

(ii) There exists $f \in E$ such that $f(t) \neq 0$ for all $t \in I$.

If $E = (E, \|\ \|_E)$ is complete then it is called a *quasi-Banach function space*. We say that an element $f \in E$ is *order continuous*, if for any sequence (f_n) in E such that $|f_n| \to 0$ a.e. and $|f_n| \le f$ a.e., there holds $\|f_n\|_E \to 0$. Let E_a denote the subspace of all order continuous elements in E. Then E is called *order continuous* if $E = E_a$. We say that $(E, \|\ \|_E)$ has the *Fatou property*, if whenever $0 \le f_n \in E$ for $n \in \mathsf{N}$, $f \in L^0$, $f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_E < \infty$, then $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$.

A quasi-Banach function space E is said to be *rearangement invariant* (or r.i.) if for every $f \in L^0$ and $g \in E$ with $\mu_f = \mu_g$, we have $f \in E$ and $\|f\|_E = \|g\|_E$. Recall that μ_f denotes the distribution function of f, i.e., $\mu_f(\lambda) = |\{t \in I : |f(t)| > \lambda\}|$, $\lambda \geq 0$, where $|\ |$ is the Lebesgue measure on $(0, 1]$ or $(0, \infty)$ or a counting measure in the discrete case. The decreasing rearrangement f^* of f is defined by $f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}$, $t \in I$.

Given a r.i. space E, let Φ be its fundamental function, that is $\Phi(0) = 0$ and if $I = (0, 1]$ or $I = (0, \infty)$ then $\Phi(t) = \|\chi_{(0,t)}\|_E$, $t \in I$. Obviously Φ is increasing. One can also show, by following the proof in the Banach space case (see e.g. Th. 4.7 and 4.5 in [16]), that $\Phi(u)/u^r$ is decreasing, where $r > 0$ is the constant such that E is r-normable. This implies among others that Φ is continuous on $(0, \infty)$ and right-continuous at 0 iff $E = E_a$, whenever E is defined on $(0, 1]$ or $(0, \infty)$. For a r.i. quasi-Banach space E over $(I, |\ |)$ the *lower* and *upper Boyd* indices are defined analogously as for r.i. Banach spaces that is

$$p(E) = \sup\{p > 0 : \text{there exists } C > 0,\ \|D_a\| \leq Ca^{-\frac{1}{p}} \text{ for all } 0 < a < 1\},$$
$$q(E) = \inf\{q > 0 : \text{there exists } C > 0,\ \|D_a\| \leq Ca^{-\frac{1}{q}} \text{ for all } a > 1\},$$

where $D_a : E \to E$ is a dilation operator defined on $I = (0, \infty)$ as $D_a f(t) = f(at)$ and on $I = (0, 1]$ as $D_a f(t) = f(at)$ for $0 \leq t \leq \min(a^{-1}, 1)$ and $D_a f(t) = 0$ for $\min(a^{-1}, 1) < t \leq 1$ ([17]). In the case of the discrete measure we define the Boyd indices similarly replacing the dilation of functions by dilation of sequences defined for $f = (x_1, x_2, \ldots)$ and $n \in \mathbb{N}$ as

$$d_n f = n^{-1}\left(\sum_{i=1}^{n} x_i, \sum_{i=n+1}^{2n} x_i, \ldots\right) \quad \text{or} \quad d_{1/n} f = (\overbrace{x_1, \ldots, x_1}^{n}, \overbrace{x_2, \ldots, x_2}^{n}, x_3, \ldots).$$

For a r.i. quasi-Banach function space E, $p_c(E) \leq p(E) \leq q(E) \leq q_c(E)$ (cf. [17], p. 132).

Given an arbitrary function $F : J \to \mathbb{R}_+$, where J is an interval in $\mathbb{R}_+$, we define the *lower* and *upper Matuszewska-Orlicz indices* as follows:

$$\alpha(F) = \sup\{p \in \mathbb{R} : F(au) \leq Ca^p F(u) \text{ for some } C > 0 \text{ and all } u \in J, 0 < a \leq 1, au \in J\},$$

$$\beta(F) = \inf\{q \in \mathbb{R} : F(au) \leq Ca^q F(u) \text{ for some } C > 0 \text{ and all } u \in J, a \geq 1, au \in J\}.$$

If $J = \mathbb{R}_+$ and $F : \mathbb{R}_+ \to \mathbb{R}_+$ then $\alpha(F)$ and $\beta(F)$ will be often denoted by $\alpha^a(F)$ and $\beta^a(F)$. For $F : \mathbb{R}_+ \to \mathbb{R}_+$ we shall also consider the indices for "large arguments"

$$\alpha^{\infty}(F) = \sup\{p \in \mathsf{R} : F(au) \leq Ca^{p} F(u) \text{ for some } C > 0, u_0 \geq 0 \text{ and all } u \geq u_0, 0 < a \leq 1\},$$

$$\beta^{\infty}(F) = \inf\{q \in \mathsf{R} : F(au) \leq Ca^{q} F(u) \text{ for some } C > 0, u_0 \geq 0 \text{ and all } u \geq u_0, a \geq 1\},$$

and for "small arguments"

$$\alpha^{0}(F) = \sup\{p \in \mathsf{R} : F(au) \leq Ca^{p} F(u) \text{ for some } C > 0, u_0 > 0 \text{ and all } 0 \leq u \leq u_0, 0 < a \leq 1\},$$

$$\beta^{0}(F) = \inf\{q \in \mathsf{R} : F(au) \leq Ca^{q} F(u) \text{ for some } C > 0, u_0 > 0 \text{ and all } 0 \leq u \leq u_0, a \geq 1\}.$$

Let $\tilde{F}(u) = 1/F(1/u)$ assuming that $1/\infty = 0$ and $1/0 = \infty$. Some auxiliary relations between indices and operations on functions are listed in the following proposition.

PROPOSITION 1.1 ([20], [21]). *Let $F, G : \mathsf{R}_+ \to \mathsf{R}_+$ be strictly increasing unbounded functions. Then the following equalities are satisfied:*

(i) $\alpha^{a}(F) = \alpha^{a}(\tilde{F})$, $\beta^{a}(F) = \beta^{a}(\tilde{F})$, $\alpha^{\infty}(F) = \alpha^{0}(\tilde{F})$, $\beta^{\infty}(F) = \beta^{0}(\tilde{F})$.

(ii) $\alpha^{j}(F^{-1}) = 1/\beta^{j}(F)$ *for* $j = \infty, 0, a$.

(iii) $\alpha^{j}(F \circ G) \geq \alpha^{j}(F)\alpha^{j}(G)$, $\beta^{j}(F \circ G) \leq \beta^{j}(F)\beta^{j}(G)$ *for* $j = \infty, 0, a$.

The equalities hold if either F or G is a power function.

A mapping $\varphi : \mathsf{R}_+ \to \mathsf{R}_+$ is said to be an *Orlicz function* if $\varphi(0) = 0$, φ is continuous, strictly increasing and $\lim_{u\to\infty} \varphi(u) = \infty$. For any Orlicz function φ and any quasi-Banach function lattice $(E, \| \ \|_E)$, we define the Calderón-Lozanovskii space E_φ by

$$E_\varphi = \{f \in L^0 : \varphi \circ (\lambda|f|) \in E \ \text{ for some } \ \lambda > 0\},$$

where $\varphi \circ |f|(t) = \varphi(|f(t)|)$ for any $t \in I$. For every $f \in E_\varphi$ the following functional is finite

$$\|f\| := \|f\|_\varphi = \inf\{\lambda > 0 : \rho_\varphi(f/\lambda) \leq 1\},$$

where

$$\rho(f) := \rho_\varphi(f) = \begin{cases} \|\varphi \circ |f|\|_E, & \text{if } \varphi \circ |f| \in E \\ \infty, & \text{otherwise.} \end{cases}$$

If E is a Banach function space with the Fatou property and φ is a convex Orlicz function, then $(E_\varphi, \| \|_\varphi)$ is a Banach function space ([2], [6]) and then E_φ is a special case of a general Calderón-Lozanovskii construction $\Psi(E, F)$, where E is a Banach function space and $F = L^\infty$ (cf. [19]). If $E = L^1$ then E_φ is an Orlicz space. If $\varphi(u) = u^p$ with $p \geq 1$, then E_φ is a p-convexification $E^{(p)}$ of E and by analogy E_φ is called a φ-convexification of E whenever φ is convex. We also observe that if E is an r.i. space then E_φ is also rearrangement invariant.

In the process of studying the properties of E_φ, we extract three classes of quasi-Banach spaces E:

(1) $L^\infty \subset E$,

(2) $E \subset L^\infty$,

(3) neither $L^\infty \subset E$ nor $E \subset L^\infty$.

These classes determine conditions imposed on φ. In general, the first class is associated with the behaviour of φ for large arguments, the second class with small arguments, and the third one with all arguments. Therefore the Matuszewska-Orlicz indices marked with "∞" will usually appear in case (1) of E, those with "0" will occur in case (2) and the indices with "a" will be of use for class (3) of E. Notice that if E is a r.i. space over $((0, 1], | |)$ then $L^\infty \subset E$ and if E is over $(\mathsf{N}, 2^{\mathsf{N}})$ then $E \subset L^\infty$.

Since in the sequel we frequently use the terms "all arguments", "large arguments" and "small arguments" we will abbreviate them as "a.a.", "l.a." and "s.a.", respectively.

Recall that φ satisfies condition Δ_2 for l.a., s.a. or a.a. whenever there exist $K > 0$ and $u_0 \geq 0$ such that $\varphi(2u) \leq K\varphi(u)$ for all $u \geq u_0$, for all $0 \leq u \leq u_0$ with $u_0 > 0$, or for all $u \geq 0$, respectively. It is well known that φ satisfies condition Δ_2 for l.a., s.a. or a.a. iff $\beta^j(\varphi) < \infty$ for $j = \infty$, $j = 0$ or $j = a$, respectively ([20], [21]). The Orlicz functions φ and ψ are said to be *equivalent* for a.a. (resp. l.a., s.a.) if there exist positive constants C_i, K_i, $i = 1, 2$, such that $C_1\varphi(K_1 u) \leq \psi(u) \leq C_2\varphi(K_2 u)$ for every $u \geq 0$ (resp. $u \geq u_0$, $0 \leq u \leq u_0$ with $u_0 > 0$).

For equivalent functions the suitable Matuszewska-Orlicz indices are equal ([22], [20], [21]), where "a", "∞" or "0" indices are associated with equivalence for all, large or small arguments, respectively.

An arbitrary function $F : \mathsf{R}_+ \to \mathsf{R}_+$ is said to be *pseudo-increasing* for a.a. (resp. l.a., s.a.) whenever there exist $C > 0$, $u_0 \geq 0$ such that $F(u) \leq CF(v)$ for all $0 \leq u < v$ (resp. $u_0 \leq u < v$, $0 \leq u < v \leq u_0$). F is said to be *pseudo-decreasing* if the suitable reverse inequality is satisfied. The following result is well known.

THEOREM 1.2 ([22]). *Let φ be an Orlicz function.*

(i) *If $\varphi(u)/u$ is pseudo-increasing (for a.a., l.a. or s.a.), then there exists a convex Orlicz function equivalent to φ (for a.a., l.a. or s.a. respectively).*

(ii) *If $\varphi(u)/u$ is pseudo-decreasing (for a.a., l.a. or s.a.), then there exists a concave Orlicz function equivalent to φ (for a.a., l.a. or s.a. respectively).*

2. Properties of Calderón-Lozanovskii spaces

We start this section with two lemmas that have their analogies in Banach spaces.

LEMMA 2.1. *A quasi-normed function space $(E, \| \ \|_E)$ with the Fatou property is complete.*

PROOF. Let $(f_n) \subset E$ be a Cauchy sequence. By the Aoki-Rolewicz theorem we assume that $(E, \| \ \|_E)$ is a p-norm for some $0 < p \leq 1$. Following the proof of Theorem 1 on p. 96 in [14] we can show that there exists a subsequence (f_{n_k}) and $f \in L^0$ such that $f_{n_k} \to f$ a.e.. Assuming that $f_n \to f$ a.e. and applying the Fatou property we have that $f \in E$ and for all $n \in \mathsf{N}$, $\|f - f_n\| \leq \liminf_{m\to\infty} \|f_m - f_n\|_E$, which completes the proof.

LEMMA 2.2. *Let $(E, \| \ \|_E)$ be a quasi-Banach space with the Fatou property. Then the following properties are satisfied:*

(i) *For all $f \in E_\varphi$, $\rho(f) \leq 1$ if and only if $\|f\| \leq 1$.*

(ii) *The space $(E_\varphi, \| \ \|)$ has the Fatou property.*

PROOF. Since $(E, \| \ \|_E)$ satisfies the Fatou property, the function $h(\lambda) = \rho(\lambda f)$, $f \in L^0$, is left-continuous on $(0, \infty)$. This fact immediately implies (i).

In order to show (ii), let $0 \leq f_n \in E_\varphi$, $f \in L^0$, $f_n \uparrow f$ a.e. and $M = \sup_{n\in\mathsf{N}} \|f_n\|_E < \infty$. Assuming $f_n \neq 0$ a.e., $\rho(f_n/\|f_n\|) \leq 1$ for all $n \in \mathsf{N}$ by left-continuity of $h(\lambda)$. By the Fatou property of E, $\varphi(|f|/M) \in E$ and

$$\rho(f/M) = \|\varphi(|f|/M)\|_E \leq \liminf_{n\to\infty} \|\varphi(|f_n|/\|f_n\|)\|_E \leq 1.$$

Hence $f \in E_\varphi$ and $\|f\| = \sup_n \|f_n\|$, which completes the proof.

We will need further the following result comparing different E_φ spaces.

THEOREM 2.3. *Let E be a quasi-Banach function space and φ and ψ be Orlicz functions with $\alpha^j(\varphi) > 0$ and $\alpha^j(\psi) > 0$ for $j = \infty$, $j = 0$ and $j = a$ whenever E is in class* (1), (2) *or* (3), *respectively. Then*

$$E_\psi \subset E_\varphi \quad \textit{and} \quad \|f\|_\varphi \leq K\|f\|_\psi$$

for all $f \in E_\varphi$ *and some* $K > 0$*, if there exist* K_i*,* $i = 1, 2$*, and* $u_0 \geq 0$ *such that*

$$\varphi(K_1 u) \leq K_2 \psi(u), \tag{2.1}$$

for all $u \geq u_0$ *in case* (1) *of* E*, for* $0 \leq u \leq u_0$ *with* $u_0 > 0$ *when* E *is in class* (2)*, and for all* $u \geq 0$ *if* E *is in class* (3).

PROOF. Assume that E is in class (1) and that the inequality (2.1) holds for large arguments and let $\alpha^\infty(\varphi) > 0$. Let $C > 1$ be the constant in a triangle inequality of $\| \ \|_E$. We choose the constants K_i, $i = 1, 2$, in (1.1) such that

$$C\|\varphi(K_1 u_0)\|_E \leq 1/2,$$

and $K_2 > 1$. Letting $f \in E_\psi$ with $\|f\|_\psi \leq 1$, $\|\psi(|f|)\|_E \leq 1$ and

$$\varphi(K_1|f(t)|) \leq \varphi(K_1 u_0) + K_2 \psi(|f(t)|)$$

for all $t \in I$. Hence

$$\|\varphi(K_1|f|)\|_E \leq C\|\varphi(K_1 u_0)\|_E + CK_2 = M,$$

where $M > 1$. By the assumption $\alpha^\infty(\varphi) > 0$, for some $p > 0$ and all $t \in I$,

$$\varphi(K_1/(2MC)^{1/p}|f(t)|) \leq \varphi(K_1 u_0) + (2M)^{-1}\varphi(K_1|f(t)|).$$

Thus for $K^{-1} = K_1/(2MC)^{1/p}$,

$$\rho_\varphi(K^{-1}f) \leq C\|\varphi(K_1 u_0)\|_E + (2MC)^{-1}\|\varphi(K_1|f|)\|_E \leq 1,$$

which implies clearly that $\|f\|_\varphi \leq K$. Thus $\|f\|_\varphi \leq K\|f\|_\psi$ and $E_\psi \subset E_\varphi$. The proofs of the other two cases are similar so we omit the details.

We are able to provide partial converse of the above comparison result. Below we present a sample of such result in the case when $L^\infty \subset E$ and the measure is nonatomic.

THEOREM 2.4. *Let* E *be a quasi-Banach function space on* $(0, 1]$ *or* $(0, \infty)$ *with the Fatou property. Assume that* $L^\infty \subset E$ *and that* $E_a \neq \{0\}$*. Given Orlicz functions* φ *and* ψ*, a necessary condition for the inclusion* $E_\varphi \subset E_\psi$ *is the inequality* (2.1).

PROOF. Without loss of generality we assume that $E_a = E$. Let $0 < r \leq 1$ be the number such that E is r-normable. At first observe that $\nu(A) = \|\chi_A\|_E$ defined on a σ-algebra of Lebesgue measurable sets in I is a submeasure in the sense of Definition 1 in [5]. Indeed, $\lim_n \nu(A_n) = 0$ for every sequence

(A_n) with $A_n \searrow \emptyset$, in view of order continuity of E. Moreover, for every set A and $\epsilon > 0$ if $\delta = ((\nu(A)+\epsilon)^r - \nu^r(A))^{1/r}$, then for every B with $\nu(B) \le \delta$,

$$\nu(A \cup B) \le (\|\chi_A\|^r + \|\chi_B\|^r)^{1/r} \le (\nu^r(A) + \delta^r)^{1/r} = \nu(A) + \epsilon.$$

Similarly, in view of $\nu^r(A \setminus B) \ge \nu^r(A) - \nu^r(A \cap B)$, it yields that if $\delta = (\nu^r(A) - (\nu(A) - \epsilon)^r)^{1/r}$ and $\nu(B) \le \delta$ then

$$\nu(A) - \nu(A \setminus B) \le \nu(A) - (\nu^r(A) - \delta^r)^{1/r} = \epsilon.$$

As a consequence, by Theorem 10 in [5], we conclude that the submeasure $\nu(A) = \|\chi_A\|_E$ has the Darboux property.

After this preparation, assume that inequality (2.1) is not satisfied for large arguments. Thus, for every $n, m \in \mathsf{N}$ there exists $u_{nm} > 0$ such that $u_{nm} \to \infty$ as $n, m \to \infty$ and

$$\varphi\big(2^{-(n+m)} u_{nm}\big) \ge 2^{2(n+m)} \psi(u_{nm})$$

for all $n, m \in \mathsf{N}$. By the Darboux property of ν, there exist measurable disjoint sets A_{nm} such that

$$\|\chi_{A_{nm}}\|_E = \|\chi_I\|_E \big/ \big(2^{n+m} \psi(u_{nm})\big),$$

for sufficiently large $n, m \in \mathsf{N}$. Define for $t \in I$,

$$f(t) = \begin{cases} u_{nm}, & \text{if } t \in A_{nm} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \rho_\psi^r(f) &= \Big\| \sum_{n=1}^\infty \sum_{m=1}^\infty \psi(u_{nm}) \chi_{A_{nm}} \Big\|_E^r \\ &\le \sum_{n=1}^\infty \sum_{m=1}^\infty \psi^r(u_{nm}) \|\chi_{A_{nm}}\|_E^r = \|\chi_I\|_E^r \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{2^{(n+m)r}} < \infty, \end{aligned}$$

and so $f \in E_\psi$. However, on the other hand for any $\lambda > 0$ there exists $M \in \mathsf{N}$ such that $2^{-M} < \lambda$ and assuming that $\varphi(\lambda f) \in E$ we have for every $n, m > M$,

$$\begin{aligned} \infty > \rho_\varphi(\lambda f) = \|\varphi(\lambda f)\|_E &\ge \Big\| \sum_{n>M} \sum_{m>M} \varphi\big(2^{-(n+m)} u_{nm}\big) \chi_{A_{nm}} \Big\|_E \\ &\ge \varphi\big(2^{-(n+m)} u_{nm}\big) \|\chi_{A_{nm}}\|_E = \frac{\varphi\big(2^{-(n+m)} u_{nm}\big)}{2^{n+m} \psi(u_{nm})} \|\chi_I\|_E \ge 2^{n+m} \|\chi_I\|_E, \end{aligned}$$

which is a contradiction. Thus $\rho_\varphi(\lambda f) = \infty$ for every $\lambda > 0$, and so $f \notin E_\varphi$.

REMARK. If $E_a = \{0\}$ then the above result does not need to hold. Indeed, let $E = L^\infty$. Then $E_a = \{0\}$ and for any Orlicz functions φ and ψ, $E_\varphi = E_\psi = L^\infty$ and $\varphi^{-1}(1)\|f\|_\varphi = \|f\|_\infty = \psi^{-1}(1)\|f\|_\psi$ for every $f \in L^\infty$.

THEOREM 2.5. *Let E be a quasi-Banach space with the Fatou property and φ be an Orlicz function such that $\alpha^j(\varphi) > 0$ for $j = \infty$, $j = 0$ or $j = a$ whenever E is in class* (1), (2) *or* (3), *respectively. Then $\| \, \|_\varphi$ is a quasi-norm in E_φ and the space $(E_\varphi, \| \, \|_\varphi)$ is complete.*

PROOF. We shall show that $\| \, \|_\varphi$ is a quasi-norm under the assumption that $\alpha^a(\varphi) > 0$. For other indices the proof will be analogous. At first observe that if E is r-normable for some $0 < r \leq 1$ and when φ is convex then

$$\left\| \sum_{i=1}^n f_i \right\|_\varphi \leq \left(\sum_{i=1}^n \|f_i\|_\varphi^r \right)^{1/r}$$

for any $f_1, \ldots, f_n$ in E_φ. Indeed, for any $\epsilon > 0$, setting $a^r = \sum_{i=1}^n (\|f_i\|_\varphi + \epsilon)^r$, we obtain

$$\left\| \varphi\left(\left| \sum_{i=1}^n f_i \right| \Big/ a \right) \right\|_E \leq \left\| \sum_{i=1}^n \frac{\|f_i\|_\varphi + \epsilon}{a} \varphi\left(\frac{f_i}{\|f_i\|_\varphi + \epsilon} \right) \right\|_E$$
$$\leq \left(\sum_{i=1}^n \left\| \frac{\|f_i\|_\varphi + \epsilon}{a} \varphi\left(\frac{f_i}{\|f_i\|_\varphi + \epsilon} \right) \right\|_E^r \right)^{1/r} \leq 1.$$

Since $\alpha^a(\varphi) > 0$, there exists $0 < p \leq 1$ such that $\varphi(u^{1/p})/u$ is pseudo-increasing for a.a., and by Theorem 1.2 one can find an Orlicz function ψ equivalent to φ such that $\psi(u^{1/p})$ is convex. Thus, in view of Theorem 2.3, we can assume that $\varphi(u^{1/p})$ is convex. Observe also that $\|f\|_\varphi = \||f|^p\|_{\varphi(u^{1/p})}^{1/p}$. By combining the above, for any $f_1, \ldots, f_n \in E_\varphi$,

$$\left\| \sum_{i=1}^n f_i \right\|_\varphi \leq \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_\varphi = \left\| \sum_{i=1}^n |f_i|^p \right\|_{\varphi(u^{1/p})}^{1/p}$$
$$\leq \left(\sum_{i=1}^n \||f_i|^p\|_{\varphi(u^{1/p})}^r \right)^{1/rp} = \left(\sum_{i=1}^n \|f_i\|_\varphi^{pr} \right)^{1/rp},$$

which means that $\| \, \|_\varphi$ is a quasinorm. Finally we note that the space is complete in view of Lemmas 2.1 and 2.2, which ends the proof.

3. Indices of convexity and concavity of Calderón-Lozanovskii spaces

Before we prove our main result we will need the following lemma.

LEMMA 3.1. *Let φ be an Orlicz function and $0 < s < \infty$. Asume also that $\varphi(u^s)$ is convex (resp. concave). Then for any $f \in E_\varphi$, the following holds true:*

(i) *If $\rho(f) \leq 1$ then $\|f\| \geq \rho^s(f)$ (resp. $\|f\| \leq \rho^s(f)$).*

(ii) *If $\rho(f) \geq 1$ then $\|f\| \leq \rho^s(f)$ (resp. $\|f\| \geq \rho^s(f)$).*

PROOF. We shall prove only (i) assuming that $\varphi(u^s)$ is concave. Indeed, if $\rho(f) \leq 1$ then

$$\varphi((u/\rho(f))^s) \leq \varphi(u^s)/\rho(f),$$

for all $u \geq 0$, whence

$$\rho(f/\rho^s(f)) = \|\varphi((|f|^{1/s}/\rho(f))^s)\|_E \leq \|\varphi(|f|)/\rho(f)\|_E = 1.$$

Hence $\|f\| \leq \rho^s(f)$.

THEOREM 3.2. *Let E and φ be as in Theorem 2.5. Then the following inequalities hold:*

(i) $p_d(E)\alpha^j(\varphi) \leq p_d(E_\varphi) \leq p_d(E)\beta^j(\varphi)$,

(ii) $q_d(E)\alpha^j(\varphi) \leq q_d(E_\varphi) \leq q_d(E)\beta^j(\varphi)$,

for $j = \infty$, $j = 0$ or $j = a$ whenever E is in class (1), (2) or (3), respectively.

PROOF. We shall prove only the right hand side inequalities in both (i) and (ii) for $j = \infty$. The remaining inequalities can be obtained analogously. Starting with (i) we assume that both $\beta^\infty(\varphi)$ and $p_d(E)$ are finite. For any $r > p_d(E)$, E does not satisfy an upper r-estimate, that is for every $n \in \mathbb{N}$ there exist disjoint nonnegative functions $f_1, \ldots, f_m$ in E such that

$$\left\|\sum_{i=1}^m f_i\right\|_E \geq 2^n \left(\sum_{i=1}^m \|f_i\|_E^r\right)^{1/r}$$

and $\sum_{i=1}^m \|f_i\|_E^r = 1$. Setting $g_i = \varphi^{-1}(f_i)$ it holds

$$\rho(g_i) = \|f_i\|_E \leq 1 \quad \text{and} \quad \rho\left(\sum_{i=1}^m g_i\right) = \left\|\sum_{i=1}^m f_i\right\|_E \geq 1.$$

Let $q > \beta^\infty(\varphi)r$. Then $q/r > \beta^\infty(\varphi)$ and so $\varphi(u^{r/q})/u$ is pseudo-decreasing for l.a.. Thus in view of Theorems 1.2 and 2.3, we assume without loss of generality that $\varphi(u^{r/q})$ is concave. Now by Lemma 3.1,

$$\|g_i\|^q \le \rho^r(g_i) \quad \text{and} \quad \Big\|\sum_{i=1}^m g_i\Big\|^q \ge \rho^r\Big(\sum_{i=1}^m g_i\Big).$$

Consequently

$$(2^n)^{r/q}\Big(\sum_{i=1}^m \|g_i\|^q\Big)^{1/q} \le (2^n)^{r/q}\Big(\sum_{i=1}^m \rho^r(g_i)\Big)^{1/q} = \Big(2^n\Big(\sum_{i=1}^m \|f_i\|_E^r\Big)^{1/r}\Big)^{r/q}$$
$$\le \Big\|\sum_{i=1}^m f_i\Big\|_E^{r/q} = \rho^{r/q}\Big(\sum_{i=1}^m g_i\Big) \le \Big\|\sum_{i=1}^m g_i\Big\|,$$

which proves that E_φ does not satisfy an upper q-estimate. We conclude that $p_d(E_\varphi) \le p_d(E)\beta^\infty(\varphi)$.

In order to show (ii) we shall prove at first that E_φ satisfies a lower q-estimate whenever E satisfies a lower q/s-estimate and $\varphi(u^{1/s})$ is concave. Since $\alpha^\infty(\varphi) > 0$, $\varphi(u^r)/u$ is pseudo-increasing for l.a. and some $r > 0$. Thus in view of Theorem 1.2, we can assume that $\varphi(u^r)$ is convex.

Now let $\{f_i\}_{i=1}^n \subset E_\varphi$ be a sequence of functions with disjoint supports. Setting $a^q = \sum_{i=1}^n \|f_i\|^q$, we shall show that $\rho\big(\sum f_i/a\big)$ is bounded below. Assuming that $\rho\big(\sum f_i/a\big) < \infty$ it follows

$$\rho\Big(\sum_{i=1}^n f_i/a\Big) = \Big\|\varphi\Big(\sum_{i=1}^n f_i/a\Big)\Big\|_E$$
$$= \Big\|\sum_{i=1}^n \varphi(f_i/a)\Big\|_E \ge K\Big(\sum_{i=1}^n \|\varphi(f_i/a)\|_E^{q/s}\Big)^{s/q}$$
$$= K\Big(\sum_{i=1}^n (\rho^{1/s}(f_i/a))^q\Big)^{s/q} \ge K\Big(\sum_{i=1}^n \|f_i/a\|^q\Big)^{s/q} = K,$$

where K is a constant in the lower q/s-estimate. Since we can always take $0 < K < 1$, by convexity of $\varphi(u^r)$ we obtain

$$\rho\Big(\sum_{i=1}^n f_i/K^r a\Big) = \Big\|\varphi\Big(\frac{1}{K}\Big(\sum_{i=1}^n f_i/a\Big)^{1/r}\Big)^r\Big\|_E$$
$$\ge (1/K)\Big\|\varphi\Big(\sum_{i=1}^n f_i/a\Big)\Big\|_E = \rho\Big(\sum_{i=1}^n f_i/a\Big)/K \ge 1,$$

which simply means that E_φ satisfies a lower q-estimate.

If $q > q_d(E)\beta^\infty(\varphi)$ then there exist r, s such that $r > q_d(E)$ and $s > \beta^\infty(\varphi)$ and $rs = q$. Thus $q_d(E) < q/s$. Since $\beta^\infty(\varphi) < s$, the function $\varphi(u^{1/s})/u$ is pseudo-decreasing for l.a., and by Theorem 1.2 there exists an Orlicz function ψ equivalent to φ for l.a. and such that $\psi(u^{1/s})$ is concave. By the first part of the proof, E_ψ and hence E_φ satisfies a lower q-estimate. Thus we have showed the right hand side of inequality (ii) and so the proof of the theorem is complete.

PROPOSITION 3.3. *Let E be an L-convex quasi-Banach function space with the Fatou property and let φ be an Orlicz function such that* $0 < \alpha^j(\varphi) \leq \beta^j(\varphi) < \infty$ *for* $j = \infty, 0, a$ *if E is in class* (1), (2), (3), *respectively. Then E_φ is L-convex.*

PROOF. Suppose at first that φ is convex and let $1/r > \beta^j(\varphi)$. Then in view of Theorems 1.2 and 2.3, we assume without loss of generality that $\varphi(u^r)$ is concave. Let now $0 \leq f_i \leq h$, $\|h\| = 1$ and

$$(f_1 + \ldots + f_n)/n \geq (1-\delta)h,$$

where $\delta > 0$ is such that $(1-\delta)^{1/r} = 1 - \epsilon$, and $0 < \epsilon < 1$ is the constant from L-convexity of E. Then $\varphi(h) \geq \varphi(f_i)$, $\|\varphi(h)\|_E = 1$ and

$$\begin{aligned}(\varphi(f_1) + \ldots + \varphi(f_n))/n &\geq \varphi((f_1 + \ldots + f_n)/n) \\ &\geq \varphi((1-\delta)h) \geq (1-\delta)^{1/r}\varphi(h) = (1-\epsilon)\varphi(h).\end{aligned}$$

The L-convexity of E then implies that $\max \|\varphi(f_i)\|_E \geq \epsilon$. Thus $\max \|\varphi(f_i/\epsilon)\|_E \geq \max \|(1/\epsilon)\varphi(f_i)\|_E \geq 1$ and so $\max \|f_i\| \geq \epsilon$. It follows that E_φ is L-convex whenever φ is convex.

By the assumption that the lower index of φ is positive and in view of Theorems 1.2 and 2.3, we assume that $\varphi(u^{1/p})$ is convex for some $0 < p < \infty$. Now applying the first part, $E_{\varphi(u^{1/p})}$ is L-convex and so s-convex for some $0 < s \leq 1$ by Th. 2.2 in [8]. Therefore, for any functions $f_1, \ldots, f_n$ in E_φ,

$$\begin{aligned}\left\|\left(\sum_{i=1}^n |f_i|^{ps}\right)^{1/ps}\right\|_\varphi &= \left\|\left(\sum_{i=1}^n |f_i|^{ps}\right)^{1/s}\right\|_{\varphi(u^{1/p})}^{1/p} \\ &\leq \left(\left(\sum_{i=1}^n \||f_i|^p\|_{\varphi(u^{1/p})}^s\right)^{1/s}\right)^{1/p} = \left(\sum_{i=1}^n \|f_i\|_\varphi^{ps}\right)^{1/ps}.\end{aligned}$$

Thus E_φ is ps-convex and so it is L-convex.

The next Corollary is an immediate consequence of Theorem 3.2, Proposition 3.3 and Th. 2.2 in [8].

COROLLARY 3.4. *Let E and φ be as in Proposition* 3.3. *Then* $p_d(E) = p_c(E)$, $p_d(E_\varphi) = p_c(E_\varphi)$ *and in consequence*

$$p_c(E)\alpha^j(\varphi) \le p_c(E_\varphi) \le p_c(E)\beta^j(\varphi)$$

for $j = \infty$, $j = 0$ *or* $j = a$ *whenever E is in class* (1), (2) *or* (3), *respectively.*

COROLLARY 3.5. *Let E and φ be as in Proposition* 3.3, *and assume that* $0 < p \le 1$. *Then E_φ is p-normable whenever* $p_c(E)\alpha^j(\varphi) > p$, *and it is not p-normable when* $p_c(E)\beta^j(\varphi) < p$. *In particular if* $p_c(E)\alpha^j(\varphi) > 1$ *then E_φ is normable, and if* $p_c(E)\beta^j(\varphi) < 1$ *then E_φ is not normable.*

PROOF. If $p_c(E)\alpha^j(\varphi) > p$ then $p_c(E_\varphi) > p$ by Corollary 3.4, and so E_φ is p-convex which yields p-normability. If $p > p_c(E)\beta^j(\varphi)$ then $p > p_c(E_\varphi) = p_d(E_\varphi)$, and then E_φ does not have an upper p-estimate and hence it is not p-normable.

In the case when E is a Banach function space and φ is a convex Orlicz function, so E_φ is a Banach lattice, we can restate Theorem 2.5 in terms of indices $p_c(E_\varphi)$ and $q_c(E_\varphi)$. We can also infer some corollaries about type and cotype of E_φ. We refer to [17] for definition of type and cotype of Banach spaces and their relations with convexity and concavity in Banach lattices.

COROLLARY 3.6. *Let E be a Banach function space with the Fatou property and φ be a convex Orlicz function. Then the following inequalities hold:*

(i) $p_c(E)\alpha^j(\varphi) \le p_c(E_\varphi) \le p_c(E)\beta^j(\varphi)$,

(ii) $q_c(E)\alpha^j(\varphi) \le q_c(E_\varphi) \le q_c(E)\beta^j(\varphi)$,

for $j = \infty$, $j = 0$ *or* $j = a$ *whenever E is in class* (1), (2) *or* (3), *respectively.*

In the next two corollaries we will say that φ satisfies condition Δ_2 whenever φ satisfies Δ_2 for l.a., s.a. or a.a. for E in class (1), (2) or (3), respectively.

COROLLARY 3.7. *Let E be a Banach function space with the Fatou property and φ be a convex Orlicz function. Then E_φ has a finite cotype if and only if E has a finite cotype and φ satisfies condition* Δ_2.

PROOF. If E has a finite cotype then $q_c(E) < \infty$, and if φ satisfies condition Δ_2 then $\beta^j(\varphi) < \infty$ for suitable j. Now, by (ii) of Corollary 3.6, $q_c(E_\varphi) < \infty$ which yields that cotype of E_φ is finite ([17], p. 100). Conversely, if E_φ has finite cotype then again by (ii) of Corollary 3.6, E must also have a finite cotype. If φ does not satisfy condition Δ_2 then by Corollary 5 in [6], E_φ is not order coninuous and thus, it contains an order isomorphic copy of l^∞ ([14], [17]), and so E_φ has no finite cotype.

COROLLARY 3.8. *Let E and φ be as in Corollary 3.7. If φ satisfies condition Δ_2 and E has a finite cotype and either E has a nontrivial type or $\alpha^j(\varphi) > 1$ then E_φ has a nontrivial type. If E_φ has a nontrivial type, then φ satisfies condition Δ_2 and E has a finite cotype.*

PROOF. It follows from the estimation of $p_c(E_\varphi)$ in (i) of Corollary 3.6, the relations between type and convexity ([17], p. 100) and the well known fact that a Banach space with a nontrivial type must possess a nontrivial cotype.

REMARK. The full converse of the first part of the above corollary does not hold. The following example of E and φ (cf. [2]) shows that $p_c(E) = 1 = \alpha^a(\varphi)$, E has finite cotype, φ satisfies condition Δ_2 for a.a., and yet type of E_φ is 2. Let

$$\varphi(u) = \begin{cases} u, & \text{if } 0 \le u \le 1 \\ u^2, & \text{if } u > 1, \end{cases} \qquad \psi(u) = \begin{cases} u^2, & \text{if } 0 \le u \le 1 \\ 2u - 1, & \text{if } u > 1, \end{cases}$$

and let E be an Orlicz space L_ψ on $(0, \infty)$. It is easy to check that $\alpha^a(\varphi) = 1 = \alpha^a(\psi)$ and that both φ and ψ satisfy condition Δ_2 for a.a.. Thus $\beta^a(\psi) < \infty$. Moreover, it is well known that $p_c(L_\psi) = \alpha^a(\psi)$ and $q_c(L_\psi) = \beta^a(\psi)$ (cf. [17], p. 139). Hence $p_c(L_\psi) = 1$ and $q_c(L_\psi) < \infty$. The latter means that the cotype of L_ψ is finite. However $E_\varphi = L_{\psi\circ\varphi} = L^2$, and so the type of E_φ is 2.

4. Boyd indices of Calderón-Lozanovskii spaces

Below there are given some estimations of the Boyd indices of the Calderón-Lozanovskii space E_φ in the case when E is a r.i. space.

THEOREM 4.1. *Given a r.i. quasi-Banach function space E with the Fatou property and an Orlicz function φ the following inequalities are satisfied:*

(i) $p(E)\alpha^j(\varphi) \le p(E_\varphi) \le q(E_\varphi) \le q(E)\beta^j(\varphi)$ *for* $j = \infty$, $j = 0$ *or* $j = a$ *when E is in class* (1), (2) *or* (3), *respectively.*

(ii) *If* $I = (0, 1]$ *or* $I = (0, \infty)$ *then* $p(E_\varphi) \le 1/\beta(\tilde{\varphi}^{-1} \circ \Phi)$, $q(E_\varphi) \ge 1/\alpha(\tilde{\varphi}^{-1} \circ \Phi)$, *where Φ is a fundamental function of E.*

PROOF. (i) We will carry out the proof only for lower index, in the case when $L^\infty \subset E$. Assume that $\|1\|_E = 1$, $p(E) > 0$ and $\alpha^\infty(\varphi) > 0$. Let $0 < p < \alpha^\infty(\varphi)$ and $0 < r < p(E)$. For any $f \in E_\varphi$ with $\|f\| \le 1$ it holds $\|\varphi(|f|)\|_E \le 1$. Thus $\varphi(|f|) \in E$ and clearly $\varphi(|D_a f|) \in E$ for any $0 < a < 1$. By the definition of $p(E)$ we have the estimation

$$\|\varphi(|D_a f|)\|_E \le K a^{-1/r} \|\varphi(|f|)\|_E \le K a^{-1/r},$$

for every $0 < a < 1$ and some $K > 1$. Since $p < \alpha^\infty(\varphi)$, in view of Theorems 1.2 and 2.3 we assume without loss of generality that $\varphi(u^{1/p})$ is convex. Thus for any $0 < a < 1$ and all $u \in I$

$$K^{-1}a^{1/r}\varphi(|D_a f(u)|) \geq \varphi(K^{-1/p}a^{1/rp}|D_a f(u)|)$$

and so

$$\|\varphi(K^{-1/p}a^{1/rp}|D_a f|)\|_E \leq K^{-1}a^{1/r}\|\varphi(|D_a f|)\|_E \leq 1.$$

Hence $\|D_a f\| \leq K^{1/p}a^{-1/rp}$ for every f with $\|f\| \leq 1$ and thus $p(E_\varphi) \geq p(E)\alpha^\infty(\varphi)$.

(ii) Let's show only the first inequality. By the assumption of symmetry of E, for every measurable $A \subset I$ with $|A| < \infty$, $\chi_A \in E_\varphi$. Moreover, $\|\chi_A\| = \tilde{\varphi}^{-1} \circ \Phi(|A|)$. Now, for any $0 < p < p(E_\varphi)$ there exists $C > 0$ such that

$$\|\chi_{(0,a^{-1}|A|)}\| \leq Ca^{-1/p}\|\chi_{(0,|A|)}\|$$

for every $0 < a < 1$, every measurable A with $|A| < \infty$ and such that $a^{-1}|A| \leq 1$ in case of $I = (0, 1]$. By the definition of the fundamental function we obtain for $r = 1/p$,

$$\tilde{\varphi}^{-1} \circ \Phi(at) \leq Ca^r\tilde{\varphi}^{-1} \circ \Phi(t)$$

for every $a > 1$ and every $t \geq 0$ if $I = (0, \infty)$, and for every $a > 1$ and $t \geq 0$ with $at \leq 1$ in the case when $I = (0, 1]$. It follows that $p(E_\varphi) \leq 1/\beta(\tilde{\varphi}^{-1} \circ \Phi)$.

5. Orlicz-Lorentz spaces

Now we apply the results from the previous part to Orlicz-Lorentz spaces. Although we will consider only spaces defined on $I = (0, 1]$ or $I = (0, \infty)$, the discrete case may be handled analogously. Let $w : (0, \infty) \to (0, \infty)$ be a measurable function such that

$$S(t) := \int_0^t w < \infty \quad \text{for all } t \in I,$$

and S satisfies condition Δ_2, that is $S(2t) \leq KS(t)$ for all $t \in \frac{1}{2}I$ and some $K > 0$, and

$$\int_0^\infty w = \infty \quad \text{in the case when } I = (0, \infty).$$

Such a function w will be called a *weight function*. If $\alpha(S) > 0$ then the weight w is often called *regular*. The Lorentz space $\Lambda_{1,w}$ is the set of $f \in L^0$ defined on (0, 1] or $(0, \infty)$ and such that

$$\|f\|_w = \int_I f^* w = \int_I f^*(s)w(s)\,ds < \infty,$$

where $\| \ \|_w$ is a quasi-norm and $(\Lambda_{1,w}, \| \ \|_w)$ is a r.i. quasi-Banach function space with the Fatou property and its fundamental function equal to S ([11]). It is clear that $L^\infty \subset \Lambda_{1,w}$ whenever $I = (0, 1]$ and that neither $L^\infty \subset \Lambda_{1,w}$ nor $L^\infty \supset \Lambda_{1,w}$ whenever $I = (0, \infty)$. Thus $\Lambda_{1,w}$ is in class (1) when $I = (0, 1]$ and in class (3) when $I = (0, \infty)$.

Given an Orlicz function φ, let $\Lambda_{\varphi,w} := (\Lambda_{1,w})_\varphi$. The space $\Lambda_{\varphi,w}$ is then called the *Orlicz-Lorentz space* (cf. [10], [20]). In the case when $\varphi(u) = u^p$, $0 < p < \infty$, $\Lambda_{\varphi,w}$ is denoted by $\Lambda_{p,w}$. If $\alpha^j(\varphi) > 0$ for $j = a$ or $j = \infty$ depending on whether $I = (0, 1]$ or $I = (0, \infty)$, then by Theorem 2.5, $\Lambda_{\varphi,w}$ is a r.i. quasi-Banach function space with the quasi-norm $\|f\| := \|f\|_{\Lambda_{\varphi,w}} = \inf\{\epsilon > 0 : \int_I \varphi(f^*/\epsilon)w \le 1\}$.

The Boyd indices of $\Lambda_{p,w}$ under these general assumptions that $0 < p < \infty$ and w is an arbitrary weight, can be calculated analogously as in the case when $\Lambda_{p,w}$ is a Banach space, that is when w is decreasing and $p \ge 1$ ([16]). Its convexity and concavity indices are also known (see [11] and [25] in case of arbitrary weight w and $0 < p < \infty$ and also [26] in case when w is decreasing and $p \ge 1$). Below we summarize all these results.

THEOREM 5.1. *Given* $0 < p < \infty$ *and a weight function* w*, the following holds true:*

(i) $p(\Lambda_{p,w}) = p\alpha(S^{-1})$, $q(\Lambda_{p,w}) = p\,\beta(S^{-1})$.

(ii) $p_d(\Lambda_{p,w}) = p_c(\Lambda_{p,w}) = p\min(\alpha(S^{-1}), 1)$,
$q_d(\Lambda_{p,w}) = q_c(\Lambda_{p,w}) = p\ \max(\beta(S^{-1}), 1)$.

Note that $\Lambda_{p,w}$ is L-convex. Indeed, by condition Δ_2 of S, $\alpha(S^{-1}) = 1/\beta(S) > 0$, which means that $\Lambda_{p,w}$ has some positive convexity and so it is L-convex.

In view of Theorems 3.2 and 5.1, Corollary 3.4 and the above characterization for $p = 1$, we can state immediately the following theorem.

THEOREM 5.2. *Let* φ *be an Orlicz function and* $j = a$ *or* $j = \infty$ *for* $I = (0, \infty)$ *or* $I = (0, 1]$*, respectively. Assume that* $0 < \alpha^j(\varphi) \le \beta^j(\varphi) < \infty$*. Then the following inequalities are satisfied:*

(i) $\min(\alpha(S^{-1}), 1)\alpha^j(\varphi) \le p_c(\Lambda_{\varphi,w}) = p_d(\Lambda_{\varphi,w}) \le \min(\alpha(S^{-1}), 1)\beta^j(\varphi)$,
$\alpha(S^{-1})\alpha^j(\varphi) \le p(\Lambda_{\varphi,w}) \le 1/\beta(\tilde{\varphi}^{-1} \circ S)$;

(ii) $\max(\beta(S^{-1}), 1)\alpha^j(\varphi) \leq q_d(\Lambda_{\varphi,w}) \leq \max(\beta(S^{-1}), 1)\beta^j(\varphi)$,
$\beta(S^{-1})\beta^j(\varphi) \geq q(\Lambda_{\varphi,w}) \geq 1/\alpha(\tilde{\varphi}^{-1} \circ S)$.

REMARK. In view of the above estimations, the Boyd indices of $\Lambda_{\varphi,w}$ may be exactly calculated in the case when either φ or S is a power function (for w decreasing and φ convex see Th. 6.3 in [20]). Indeed, if e.g. $I = (0, \infty)$ and $S(t) = t^p$, $p > 0$, then in view of Proposition 1.1,

$$\beta(\tilde{\varphi}^{-1} \circ S) = \beta(\tilde{\varphi}^{-1})\beta(S) = \beta(S)/\alpha(\tilde{\varphi}) = \beta(S)/\alpha(\varphi) = 1/\alpha(\varphi)\alpha(S^{-1}).$$

Thus $p(\Lambda_{\varphi,w}) = \alpha(S^{-1})\alpha(\varphi)$. With the same S and $I = (0, 1]$, $S : (0, 1] \to (0, 1]$. Applying the same techniques as in the proof of Proposition 1.1, we can show that $\beta(\tilde{\varphi}^{-1} \circ S) = \beta^0(\tilde{\varphi}^{-1})\beta(S)$ and that $\beta(S) = 1/\alpha(S^{-1})$. Hence $p(\Lambda_{\varphi,w}) = \alpha(S^{-1})\alpha^\infty(\varphi)$. In other cases we obtain similar results. Notice also that the estimations of Boyd indices of $\Lambda_{\varphi,w}$ given in the above theorem can be obtained from the estimations given in Theorem 3.1 in [23] as well.

By applying Theorem 5.2(ii) we can easily state an appropriate result on p-normability of $\Lambda_{\varphi,w}$. Since the question on normability is always of some importance let's state this result explicitely.

COROLLARY 5.3. *Let φ and j be the same as in Theorem* 5.2.
If $\min(\alpha(S^{-1}), 1)\alpha^j(\varphi) > 1$, *then $\Lambda_{\varphi,w}$ is normable.*
If $\min(\alpha(S^{-1}), 1)\beta^j(\varphi) < 1$, *then $\Lambda_{\varphi,w}$ is not normable.*

It is known that the Lorentz space $\Lambda_{p,w}$, $0 < p < \infty$, is normable whenever the Hardy operator

$$Hf(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$$

is bounded on $\Lambda_{p,w}$ ([25], [27]). We obtain the similar result in the case of Orlicz-Lorentz space.

COROLLARY 5.4. *Let φ and j be the same as in Theorem* 5.2.
If $\alpha(S^{-1})\alpha^j(\varphi) > 1$, then the Hardy operator H is bounded on $\Lambda_{\varphi,w}$. Consequently, if $\min(\alpha(S^{-1}), 1)\alpha^j(\varphi) > 1$ *then*

$$|||f||| := \|Hf\|_{\Lambda_{\varphi,w}}$$

is a norm in $\Lambda_{\varphi,w}$ equivalent to $\|f\|_{\Lambda_{\varphi,w}}$.

PROOF. The boundedness of H follows from Theorem 5.2(i) and the Mongomery-Smith result ([24]), stating that the Hardy operator H is bounded on a r.i. quasi-Banach function space iff its lower Boyd index is strictly bigger than one.

Recall that $L_\varphi(w)$ denotes the space of all $f \in L^0$ such that $\|f\|_{L_\varphi(w)} < \infty$, where $\|f\|_{L_\varphi(w)} = \inf\{\epsilon > 0 : \int_I \varphi(|f|/\epsilon)w \le 1\}$. If φ is convex then $\| \ \|_{L_\varphi(w)}$ is a norm. Now, if $\min(\alpha(S^{-1}), 1)\alpha^j(\varphi) > 1$ then $\alpha^j(\varphi) > 1$, and in view of Theorems 1.2 and 2.3 we assume that φ is convex. For every $f \in \Lambda_{\varphi,w}$, the functional $\|Hf\|_{\Lambda_{\varphi,w}}$ is finite and equivalent to $\|f\|_{\Lambda_{\varphi,w}}$ by boundedness of H. Moreover, $\|Hf\|_{\Lambda_{\varphi,w}} = \|Hf\|_{L_\varphi(w)}$ since $(Hf)^* = Hf$. By subadditivity of H, $\|Hf\|_{L_\varphi(w)}$ satisfies the triangle inequality and so $|||f||| = \|Hf\|_{\Lambda_{\varphi,w}}$ is a norm in $\Lambda_{\varphi,w}$ equivalent to $\|f\|_{\Lambda_{\varphi,w}}$.

We conclude the paper with a corollary on type and cotype of $\Lambda_{\varphi,w}$. Recall that $\varphi_*(u) = \sup_{v\ge 0}\{uv - \varphi(v)\}$, $u \ge 0$, is a complementary function to φ.

THEOREM 5.5 ([15]). *Let φ be a convex Orlicz function and w be a decreasing weight function.*

(i) *$\Lambda_{\varphi,w}$ has a finite cotype if and only if $\alpha(S) > 0$ and φ satisfies condition Δ_2 for a.a. (resp. l.a.) when $I = (0, \infty)$ (resp. $I = (0, 1]$).*

(ii) *$\Lambda_{\varphi,w}$ has a nontrivial type if and only if $\alpha(S) > 0$ and both φ and its complementary function φ_* satisfy condition Δ_2 for a.a. (resp. l.a.) when $I = (0, \infty)$ (resp. $I = (0, 1]$).*

PROOF. (i) By Theorem 5.2(ii), $\Lambda_{1,w}$ has finite cotype iff $\alpha(S) > 0$. Now it is enough to apply Corollary 3.7.

(ii) Note that φ_* satisfies condition Δ_2 for a.a. or l.a. iff $\alpha^j(\varphi) > 1$ for $j = a$ or $j = \infty$, respectively. Thus the sufficiency holds in view of (i) and Corollary 3.8. Conversely, if the type of E_φ is nontrivial then the cotype is finite and so $\alpha(S) > 0$ and φ satisfies a suitable condition Δ_2. It has been proved in [13] that if φ_* does not satisfy condition Δ_2, then $\Lambda_{\varphi,w}$ contains an isomorphic copy of l^1, and so it has a trivial type ([4], [17]).

REFERENCES

1. Calderón, A. P., *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1965), 133–190.
2. Cerdà, J., Hudzik, H. and Mastyło, M., *On the geometry of some Calderón-Lozanovskii interpolation spaces*, Indag. Math., N.S. 6 (1995), 35–49.
3. Cuartero, B. and Triana, M. A., *(p, q)-convexity in quasi-Banach lattices and applications*, Studia Math. 84 (1986), 113–124.
4. Diestel, J., Jarchov, H. and Tonge, A., *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43 (1995).
5. Dobrakov, I., *On Submeasures I*, Dissertationes Math. 112 (1974).
6. Hudzik, H., Kamińska, A. and M. Mastyło, *Geometric properties of Calderón-Lozanovskii spaces*, Houston J. Math. 22 (1996), 639–663.
7. Hudzik, H., Kamińska, A. and Mastyło, M., *Monotonicity and rotundity in Banach lattices*, Rocky Mountain J. Math. 30 (2000), 933–950.

8. Kalton, N. J., *Convexity conditions for non-locally convex lattices*, Glasgow Math. J. 25 (1984), 141–152.
9. Kalton, N. J., Peck, N. T. and Roberts, J. W., *An F-Space Sampler*, London Math. Soc. Lecture Note Ser. 89 (1984).
10. Kamińska, A., *Some remarks on Orlicz-Lorentz spaces*, Math. Nachr. 147 (1990), 29–38.
11. Kamińska, A. and Maligranda, L., *Order convexity and concavity in Lorentz spaces with arbitrary weight*, Technical Report, Luleå University of Technology 4 (1999), 1–21.
12. Kamińska, A., Maligranda, L. and Persson, L. E., *Convexity, concavity, type and cotype of Lorentz spaces*, Indag. Math. N.S. 9 (1998), 367–382.
13. Kamińska, A., Lin, P. K. and Sun, H., *Uniformly normal structure of Orlicz-Lorentz spaces*, Lecture Notes in Pure and Applied Math. 175 (1995), 229–238.
14. Kantorovich, L. V. and Akilov, G. P., *Functional Analysis*, Pergamon Press and "Nauka", Second Edition, 1982.
15. Katirtzoglou, E., *Geometric properties of Banach spaces*, Doctoral Dissertation, The University of Memphis 1996.
16. Krein, S. G., Petunin, Ju. I. and Semenov, E. M., *Interpolation of Linear Operators*, Amer. Math. Soc. Providence 1982.
17. Lindenstrauss, J. and Tzafriri, L., *Classical Banach Spaces II*, Springer-Verlag, 1979.
18. Lorentz, G. G., *Some new functional spaces*, Ann. of Math. 51 (1950), 27–55.
19. Lozanovskii, G. Ya., *On Banach lattices II*, Siberian Math. J. 12 (1971), 397–401.
20. Maligranda, L., *Indices and Interpolation*, Dissertationes Math. 234 (1985).
21. Maligranda, L., *Orlicz Spaces and Interpolation*, Seminars in Math. 5, University of Campinas, Campinas SP, Brazil 1989.
22. Matuszewska, W. and Orlicz, W., *On certain properties of φ-functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 439–443.
23. Montgomery-Smith, S. J., *Boyd indices of Orlicz-Lorentz spaces*, Lecture Notes in Pure and Appl. Math. 172 (1995), 321–334.
24. Montgomery-Smith, S. J., *The Hardy operator and Boyd indices*, Lecture Notes in Pure and Appl. Math. 175 (1995), 359–364.
25. Raynaud, Y., *On Lorentz-Sharpley spaces*, Israel Math. Conference Proc. 5 (1992), 207–228.
26. Reisner, S., *A factorization theorem in Banach lattices and its application to Lorentz spaces*, Ann. Inst. Fourier (Grenoble) 31 (1981), 239–255.
27. Sawyer, E., *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.
28. Zaanen, A. C., *Riesz Spaces II*, North-Holland Math. Library, 30 (1983).

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