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A CLASS OF RATIONAL SURFACES IN P^4

CRISTIAN VOICA*

Abstract

In this paper, we obtain a complete classification of all rational surfaces embedded in P^4 so that all their exceptional curves are lines. These surfaces are exactely the rational surfaces shown by I.Bauer to project isomorphicaly from P^5 from one of their points, although no a priori reason is known why such a surface should be projectable in this way.

1. Introduction

As is well known, every surface embeds into P^5. A remarkable theorem due to Ellingsrud-Peskine (see [7]) tells us that surfaces in P^4 have very special properties: those not of general type belong to finitely many families. This gives a strong motivation for the (both classical and contemporary) efforts to get a complete list of the special ones. In particular understanding all rational surfaces in P^4 is a very exciting problem. Hartshorne was the first to conjecture that the degree of rational surfaces in P^4 is bounded. It is belived that the actual bound should be 12.

How can we obtain rational surfaces in P^4?

Severi [17] proved, a hundred years ago, that the natural way, namely projections from an exterior point, is not appliable in this case. More specifically, he proved that except the Veronese surface, there is no other non-degenerate surface in P^5 (meaning that the given surface is not contained in a hyperplane) which can be projected isomorphically to P^4. There exists a second natural way to embed a rational surface in P^4: projection from a point on the surface. This means to embed in P^4 the blow-up of a given rational surface in P^5 using the linear system of hyperplane sections passing through a point. In fact, we can start with a rational surface $S \subset \mathsf{P}^n$ and we can try to project this surface into P^4 from some of its points. Unfortunately, we only know that such a projection can be an isomorphism (in the above sense) if $\deg(S) \geq 2\pi + 2$, where π is the sectional genus of S (see [11] for more details). Since after some projections we obtain $\deg(S) < 2\pi + 2$, it is not clear that we can continue. In

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this direction, Bauer [5] proved that there exist only six (families of) rational, non-degenerate surfaces in P^5 which can be projected isomorphically in P^4 from one of their points.

In the present paper, we find all rational, non-degenerate surfaces $S \subset \mathsf{P}^4$ embedded such that all their exceptional curves are lines. More specifically, we find all possible minimal rational surfaces $S_{\min}$ and linear systems L on $S_{\min}$ such that there exist r simple points $P_1, P_2, \ldots, P_r \in S_{\min}$ such that the blown-up surface $\widehat{S_{\min}}(P_1, \ldots, P_r)$ can be embedded in P^4 using the linear system $\left|f^*(L) - \sum_{i=1}^r E_i\right|$, where f is the blow-up morphism.

Our main result is the following:

THEOREM 1.1. *If S is a rational surface embedded in P^4 via the linear system $|H| = \left|f^*(L) - \sum_{i=1}^r E_i\right|$, where $f : S \to S_{\min}$ is the birational morphism on a minimal rational surface, L is a very ample divisor on $S_{\min}$ and $E_1, E_2, \ldots, E_r$ are the exceptional curves, then $(S, |H|)$ can be:*

$$(\widehat{\mathsf{P}^2}(P_1), |f^*(2L) - E_1|),$$

$$\left(\widehat{\mathsf{P}^2}(P_1, \ldots, P_5), \left|f^*(3L) - \sum_{i=1}^{5} E_i\right|\right),$$

$$\left(\widehat{\mathsf{P}^2}(P_1, \ldots, P_{10}), \left|f^*(4L) - \sum_{i=1}^{10} E_i\right|\right),$$

$$\left(\widehat{\mathsf{F}_0}(P_1, \ldots, P_4), \left|f^*(2C_1 + 2C_2) - \sum_{i=1}^{4} E_i\right|\right),$$

$$\left(\widehat{\mathsf{F}_0}(P_1, \ldots, P_7), \left|f^*(3C_1 + 2C_2) - \sum_{i=1}^{7} E_i\right|\right)$$

or

$$\left(\widehat{\mathsf{F}_2}(P_1, \ldots, P_7), \left|f^*(2C_0 + 5F) - \sum_{i=1}^{7} E_i\right|\right).$$

Even if we find the same surfaces as in [5], our result can not be reduced to Bauer's theorem, because we do not know a priori that all our surfaces can be projected from P^5. The obstruction is that, for a given surface S and for a divisor D on S, $|f^*(D) - E|$ can be a very ample linear system on $\widehat{S}(P)$, while P is a base point of $|D|$. In fact, Bauer [5] proved that a smooth rational surface $S \subset \mathsf{P}^4$ is projection of a smooth surface $Y \subset \mathsf{P}^5$ with center on Y if and only if S contains an exceptional line and $h^1(S, O_S(1)) = 0$. For our surfaces there is no a priori reason ensuring that this condition is fulfilled.

Note that there exist rational surfaces $S \subset \mathsf{P}^4$ containing an exceptional line and with $h^1(S, O_S(1)) \neq 0$, for example the surface of degree 8 and sectional genus $\pi = 6$ constructed by Okonek [13] as a blow-up of P^2 in 16 points, embedded by a linear system of the form $|H| = \left|6L - 2\sum_{i=1}^{4} x_i - \sum_{j=5}^{16} x_j\right|$.

2. Preliminaries

2.1. Notations

In this paper, *surface* means a projective, smooth, irreducible algebraic variety of dimension 2, defined over an algebraically closed field of characteristic zero. We will use standard notation as for instance those in [9]. For a surface S, we denote:

H a (smooth) hyperplane section of S;

K the canonical divisor of S;

$d = d(S)$ and $\pi = \pi(S)$ the degree and the sectional genus of S, respectively;

$\chi(O_S)$ the Euler characteristic of the structure sheaf of S.

Recall that $d = (H^2)$ and that $2\pi - 2 = d + (H.K)$ (see, e.g., [9, p. 361]).

2.2. Blow-ups

Let S be a surface and let P be a point on S. Denote by $\widehat{S} = \widehat{S}(P)$ the blow-up of S at P and by E the exceptional locus. Using Nakai-Moishezon criterion of ampleness (see, e.g., [9, p. 365]) we can prove:

LEMMA 2.1. *If $f^*(D) + aE$ is an ample divisor on $\widehat{S}$, then $a < 0$ and D is an ample divisor on S.*

2.3. Rational minimal surfaces

Let S be a rational surface. There exists a birational morphism $f : S \to S_{\min}$, where $S_{\min}$ is P^2 or one of the Hirzebruch surfaces F_n, $n \neq 1$ (see, e.g., [9, p. 419]). Every rational surface S is (isomorphic to) a blow-up of P^2 or F_n and $\chi(O_S) = 1$.

2.4. Divisors on F_n

Let $\mathsf{F}_n, n \geq 2$ be a rational minimal surface and let $g : \mathsf{F}_n \to \mathsf{P}^1$ be the canonical morphism. Denote by F and C a fiber of g and the unique section with negative self-intersection, respectively. $D = aC + bF$ is an ample divisor on F_n if and only if $a > 0$ and $b > an$ (see, e.g., [9, p. 380]). The same result is true for F_0, C being any section of g. The canonical divisor on F_n is $K = -2C - (n+2)F$ (see, e.g., [9, p. 373]).

2.5. Double-point formula

Let S be a surface in P^4 and let $d = d(S)$. Then

$$d^2 - 10d - 5(H.K) - 2(K^2) + 12\chi(O_S) = 0$$

(see, e.g., [9, p. 434]).

In particular, let S be a rational surface embedded in P^4 so that all its exceptional curves are lines. If

$$H = H_0 - \sum_{i=1}^{r} E_i \qquad \text{and} \qquad K = K_0 + \sum_{i=1}^{r} E_i,$$

where H_0 and K_0 are (the pull-backs of) an ample divisor and the canonical divisor of $S_{\min}$, then:

$$d = (H^2) = (H_0^2) - r,$$
$$(H.K) = (H_0.K_0) + r = (H_0.K_0) + (H_0^2) - d,$$
$$(K^2) = (K_0^2) - r = (K_0^2) - (H_0^2) + d$$

and the double-point formula becomes

$$(d-3)(d-4) = 3(H_0^2) + 5(H_0.K_0) + 2(K_0^2).$$

2.6. Linkage

Two surfaces S and S_1 in P^4 are said to be geometrically linked (m, m_1) if there exist hypersurfaces X and X_1 of degree m and m_1, respectively, such that $X \bigcap X_1 = S \bigcup S_1$. From the standard sequence of linkage ([14]):

$$0 \to O_{S_1}(K) \to O_{S \bigcup S_1}(m + m_1 - 5) \to O_S(m + m_1 - 5) \to 0,$$

we obtain that

$$\chi(O_S) = \chi(O_{X \bigcap X_1}) - \chi(O_{S_1}(m + m_1 - 5)).$$

2.7. Relations between numerical invariants

THEOREM 2.2 (Roth, Gruson-Peskine, [16], [8]). *Let S be a surface in P^4, not contained in a hypersurface of degree less than s, where $s(s-1) < d$. Then*

$$\pi \le 1 + \frac{d}{2}\left(\frac{d}{s} + s - 4\right) - \frac{t(s-t)(s-1)}{2s},$$

where $d + t \equiv 0$ (mod s), $0 \leq t < s$. Furthermore, equality occurs if and only if S is projectively Cohen-Macaulay and linked to a degenerate surface of degree t by hypersurfaces of degrees s and $\frac{d+t}{s}$ respectively.

2.8. Surfaces contained in hypersurfaces of P^4

THEOREM 2.3 (Aure, [3]). *The surface S is contained in a hyperquadric of* P^4 *if and only if*

$$\pi = 1 + \left[\frac{d(d-4)}{4}\right].$$

In this case, S is either the complete intersection of the hyperquadric with another hypersurface, or S is linked to a plane in the complete intersection of the hyperquadric with another hypersurface.

THEOREM 2.4 (Aure, Koelblen, [3], [12]). *If the surface S is contained in a hypercubic of* P^4*, then either S is projectively Cohen-Macaulay and linked to a surface of degree ≤ 3 in the complete intersection of the hypercubic with another hypersurface, or S is linked to a Veronese surface or to an elliptic scroll of degree 5 in the complete intersection of the hypercubic with another hypersurface.*

Using this explicit description, one can prove:

COROLLARY 2.5 (Popescu, [15]). *Every surface of degree ≥ 9 contained in a cubic hypersurface of* P^4*, is of general type.*

3. Some results on surfaces in P^4

In this section we give a list of rational surfaces in P^4 which can be ruled out directely. In all these cases we can decide using the already known classification of surfaces in P^4 (see, e.g., [10], [4], [1], [2], [13]). We prefer to give some simpler arguments in order to make the paper self-contained.

In this section, *surface* means a *non-degenerate surface in* P^4, other than the Veronese surface.

LEMMA 3.1. *Let $S \subset \mathsf{P}^4$ be a rational surface of degree d contained in a hyperquadric of* P^4*. Then $d \leq 5$.*

PROOF. Suppose S as above. Using Theorem 2.3 and the results of 2.6, we can compute

$$\chi(O_S) = 1 - h^0\left(\mathsf{P}^4, O_{\mathsf{P}^4}\left(\frac{d}{2} - 5\right)\right) + h^0\left(\mathsf{P}^4, O_{\mathsf{P}^4}\left(\frac{d}{2} - 3\right)\right),$$

for d even and

$$\chi(O_S) = 1 - h^0\left(\mathsf{P}^4, O_{\mathsf{P}^4}\left(\frac{d+1}{2} - 5\right)\right) + h^0\left(\mathsf{P}^4, O_{\mathsf{P}^4}\left(\frac{d+1}{2} - 3\right)\right) - \frac{(d-1)(d-3)}{8},$$

for d odd.

Since $\chi(O_S) = 1$, an easy computation gives $d \leq 5$.

LEMMA 3.2. *Let $S \subset \mathsf{P}^4$ be a rational surface with degree d and sectional genus π. Then:*

(1) $\pi \leq 1 + \left[\frac{d(d-4)}{4}\right]$ *for any d and π;*

(2) *if $d = 6$, then $\pi \leq 3$;*

(3) *if $d \geq 7$, then $\pi \leq 1 + \left[\frac{d(d-3)}{6}\right]$;*

(4) *if $d \geq 9$, then $\pi \leq \left[\frac{d(d-3)}{6}\right]$;*

(5) *if $d \geq 13$, then $\pi \leq \frac{d^2}{8}$ if d is divisible by 4, or $\pi \leq \frac{d^2}{8} - 1$ in the contrary case.*

PROOF. First of all we apply Corollary 2.5 and Lemma 3.1 to observe that if S is a non-degenerate rational surface in P^4, then S is not contained in a hyperquadric if $d \geq 6$ and S is not contained in a hypercubic if $d \geq 9$. Then the sectional genus π verifies the inequalities of Theorem 2.2 for $s = 2$ and any d, for $s = 3$ if $d \geq 7$ and for $s = 4$ if $d \geq 13$. We have only to prove that the equality case of some of these inequalities can not be fulfilled for a rational surface.

If $d = 6$, we obtain from Theorem 2.2 that $\pi \leq 4$. In the equality case, S must be contained in a hyperquadric, which contradicts Lemma 3.1.

If $d \geq 9$, the sectional genus π verifies the inequality of Theorem 2.2 for $s = 3$. In the equality case, S must be contained in a hypercubic, which contradicts Corollary 2.5.

If $d \geq 13$, the sectional genus π verifies the inequality of Theorem 2.2 for $s = 4$.

In the equality case, we deduce from Theorem 2.2 that S is linked to a degenerate surface S_1 of degree t by hypersurfaces X and Y of degrees 4 and $\frac{d+t}{4}$ respectively, where $0 \leq t \leq 3$ and $d + t \equiv 0 \pmod 4$. Observe that, if $t \neq 0$, then $K_{S_1} \sim (t-4)H_{S_1}$. Using the standard sequence of linkage, we compute

$$\chi(O_S) = 1 - h^0(\mathsf{P}^4, O_{\mathsf{P}^4}(m-4)) + h^0(\mathsf{P}^4, O_{\mathsf{P}^4}(m)) - \frac{1}{2}mt(m-t+4) - \varepsilon,$$

where $m = \frac{d+t}{4} - 1$ and ε is 0 if $t = 0$ and 1 otherwise. It is easy to see that, if $d \geq 13$, then $\chi(O_S) > 1$, which is a contradiction.

LEMMA 3.3. *There are no rational surfaces $S \subset \mathsf{P}^4$ with $d = 9$ and $\pi = 9$.*

PROOF. Suppose such surfaces exist and let C be a smooth hyperplane section. Observe that the divisors $2H|_C$ and $3H|_C$ are non-special on C. Using the exact sequence:

$$0 \to O_S(-H) \to O_S \to O_C \to 0,$$

we deduce that

$$h^0(S, O_S(3H)) \leq h^0(S, O_S(H)) + h^0(C, O_C(2H)) + h^0(C, O_C(3H)) \leq 34.$$

From the exact sequence:

$$0 \to I_S(3) \to O_{\mathsf{P}^4}(3) \to O_S(3) \to 0,$$

we obtain that

$$h^0(\mathsf{P}^4, I_S(3)) \geq 1,$$

so S is contained in a hypercubic. We apply Corollary 2.5 in order to obtain a contradiction.

LEMMA 3.4. *There are no rational surfaces $S \subset \mathsf{P}^4$ with $d = 4$ and $\pi = 0$.*

PROOF. Suppose that such surfaces exist and let C be a smooth hyperplane section of S. We use the exact sequence

$$0 \to O_S(1) \to O_S(2) \to O_C(2H|_C) \to 0$$

to prove that S is contained in a hyperquadric.

Observe that $2H|_C$ is non-special on C; then

$$h^0(S, O_S(2H)) \leq h^0(S, O_S(H)) + h^0(C, O_C(2H|_C)) = 14.$$

From the exact sequence

$$0 \to I_S(2) \to O_{\mathsf{P}^4}(2) \to O_S(2) \to 0$$

we obtain that $h^0(I_S(2)) \geq 1$. Since S is contained in a hyperquadric, we can apply Theorem 2.3 to obtain that $\pi = 1$.

4. The case: $S_{\min} = \mathsf{P}^2$

In this section we will determine all the families of rational surfaces which dominate P^2 and which are embedded in P^4 such that all their exceptional curves are lines.

It is known that there exist only a finite number of such families: this fact was claimed by Hartshorne before the general theorem of Ellingsrud and Peskine ([7]) appeared. In this section, we give an alternative proof of this claim.

Let $S = \widehat{\mathsf{P}^2}(P_1, \dots, P_r) \subset \mathsf{P}^4$ be a surface as above. We suppose that S is neither degenerate, nor the Veronese surface. Our assumptions on S say that the linear system of the hyperplane sections is of the form

$$|f^*(mL) - E_1 - \dots - E_r|,$$

where L is a line in P^2, $m > 0$ and $E_1, E_2, \dots, E_r$ are the exceptional curves of the blow-up f.

In our case the sectional genus of S is

$$\pi = \frac{(m-1)(m-2)}{2}$$

and the double-point formula (2.5) becomes:

$$(d-3)(d-4) = 3(m-2)(m-3).$$

First of all observe that $d \leq m\sqrt{3}$ if $d \geq 4$: in the contrary case we obtain that

$$(d-3)(d-4) = 3(m-2)(m-3) < \big(d - 2\sqrt{3}\big)\big(d - 3\sqrt{3}\big),$$

and then $d \leq 3$.

Now suppose $d \geq 13$. In this case, we can use the inequality $\pi \leq \frac{d^2}{8}$ from Lemma 3.2 in order to obtain that

$$\begin{aligned} d^2 \geq 4(m-1)(m-2) &= 4(m-2)(m-3) + 8m - 16 \\ &\geq \frac{4}{3}.(d-3)(d-4) + \frac{8d\sqrt{3}}{3} - 16 \end{aligned}$$

and is easy to see that $d \leq 14$.

So it will be sufficient to assign to d the values from 3 to 14, to compute the corresponding values of m and to determine the numerical invariants of S. In addition, note that $d \equiv 0$ or 1 (mod 3). We obtain for d, m and π the following values:

1. $d = 3, \pi = 0, m = 2;$
2. $d = 4, \pi = 1, m = 3;$

3. $d = 6, \pi = 3, m = 4$;
4. $d = 3, \pi = 1, m = 3$;
5. $d = 4, \pi = 0, m = 2$;
6. $d = 13, \pi = 21, m = 8$.

We use now Lemma 3.2 and Lemma 3.4 to decide that the cases 4., 5. and 6. are not possible.

Using the classification in low degrees ([10]) or a direct argument, we can decide that all the three remaining cases represent rational surfaces in P^4. In fact, we have only to show that the linear systems

$$|2L - P_1|, \qquad \left|3L - \sum_{i=1}^{5} P_i\right| \qquad \text{and} \qquad \left|4L - \sum_{i=1}^{10} P_i\right|$$

are very ample on P^2. We conclude:

THEOREM 4.1 (Hartshorne, unpublished). *There exist only three families of non-degenerate rational surfaces $S \subset \mathsf{P}^4$, which dominate P^2 and are embedded so that all their exceptional curves are lines. Such a polarised surface $(S, |H|)$ may be:*

$$(\widehat{\mathsf{P}^2}(P_1), |f^*(2L) - E_1|)$$

$$\left(\widehat{\mathsf{P}^2}(P_1, \ldots, P_5), \left|f^*(3L) - \sum_{i=1}^{5} E_i\right|\right)$$

or

$$\left(\widehat{\mathsf{P}^2}(P_1, \ldots, P_{10}), \left|f^*(4L) - \sum_{i=1}^{10} E_i\right|\right).$$

5. The case: $\mathsf{S}_{\mathbf{min}} = F_n$, $n \neq 1$

Let S be a rational surface in P^4 which dominates one of the surfaces F_n, $n \neq 1$, embedded such that all its exceptional curves are lines. Let

$$|H| = |f^*(H_0) - E_1 - \ldots - E_r|$$

be the linear system of hyperplane sections on S, where H_0 is an ample divisor on F_n. Using 2.4 and 2.3 we decide that

$$H_0 \sim aC + bF,$$

where

$$(C^2) = -n, \quad (F^2) = 0, \quad (C.F) = 1, \qquad a > 0 \quad \text{and} \quad b > an.$$

Let $K_0 = -2C - (n+2)F$ be the canonical divisor on F_n. We denote

$$\alpha := 2a, \qquad \beta := 2b - na.$$

Observe that $\alpha > 0$, $\beta > 0$ and $2\beta > n\alpha$. Since

$$\begin{aligned}(H_0^2) &= -na^2 + 2ab = \frac{1}{2}\alpha\beta,\\ (H_0.K_0) &= an - 2a - 2b = -(\alpha+\beta),\\ (K_0^2) &= 8,\end{aligned}$$

the double-point formula becomes

$$(3\alpha - 10)(3\beta - 10) = 6(d-3)(d-4) + 4$$

and

$$4\pi = (\alpha - 2)(\beta - 2).$$

Suppose for the moment that $\beta = 1$; since $\pi \geq 0$ and α is odd, we obtain that $(d-3)(d-4) = 4$, which is a contradiction.

Since $\alpha \geq 2$ and $\beta \geq 2$ we deduce that

$$\begin{aligned}(3\alpha - 10)(3\beta - 10) &= 9(\alpha-2)(\beta-2) - 12((\alpha-2) + (\beta-2)) + 16\\ &\leq 36\pi - 48\sqrt{\pi} + 16\end{aligned}$$

Suppose now that $d \geq 13$. We use Lemma 3.2 and the above inequality to get that

$$6d^2 - 42d + 76 \leq \frac{9}{2}d^2 - 12d\sqrt{2} + 16$$

and then $d \leq 13$.

So it will be sufficient to assign to d the values from 3 to 13 and to determine α and β so that $(3\alpha - 10)(3\beta - 10) = 6(d-3)(d-4) + 4$. Note that $3\alpha - 10$ and $3\beta - 10$ are $\equiv 2 \pmod 3$ and that $3\alpha - 10$ is odd. We obtain the following types of numerical invariants:

I.

1. $d = 4, \pi = 1, r = 4, \alpha = 4, \beta = 4$;
2. $d = 5, \pi = 2, r = 7, \alpha = 4, \beta = 6$ or $\alpha = 6, \beta = 4$.

II.

1. $d = 3, \pi = 1$;
2. $d = 6, \pi = 4$;
3. $d = 7, \pi = 7$;
4. $d = 8, \pi = 11$;
5. $d = 9, \pi = 9$ or $\pi = 16$;
6. $d = 10, \pi = 12$ or $\pi = 22$;
7. $d = 12, \pi = 37$;
8. $d = 13, \pi = 21$.

III.

1. $d = 6, \pi = 3, \alpha = 6, \beta = 5$;
2. $d = 11, \pi = 14, \alpha = 10, \beta = 9$.

We can use Lemma 3.2 and Lemma 3.3 to eliminate the cases II. 1–8. We use the inequality $2\beta > n\alpha$ and the condition $n \neq 1$ to eliminate the cases III. 1. and 2.: in both cases we obtain that n must be zero, and then $b = \frac{\beta}{2}$ can not be an integer.

The case I. 1. gives $n = 0, a = b = 2$ and the case I. 2. gives $n = 0, a = 2, b = 3$ or $n = 0, a = 3, b = 2$ or $n = 2, a = 2, b = 5$. For the existence of such surfaces in P^4, we can use the classification in low degrees ([10]) or an ad-hoc argument. In fact, using monoidal transformations, we have to verify the very ampleness of linear systems of type

$$\left|3L - \sum_{i=1}^{5} P_i\right| \qquad \text{and} \qquad \left|4L - 2P_0 - \sum_{i=1}^{7} P_i\right|$$

on P^2, where, in the second case, P_1 is an ordinary point or is infinitely near P_0. We obtain:

Proposition 5.1. *There exist only three families of non-degenerate rational surfaces $S \subset \mathsf{P}^4$ which dominate F_n, $n \neq 1$ and are embedded so that all their exceptional curves are lines. Such a polarised surface $(S, |H|)$ may be*

$$\left(\widehat{\mathsf{F}_0}(P_1, \ldots, P_4), \left|f^*(2C_1 + 2C_2) - \sum_{i=1}^{4} E_i\right|\right),$$

$$\left(\widehat{\mathsf{F}_0}(P_1, \ldots, P_7), \left|f^*(3C_1 + 2C_2) - \sum_{i=1}^{7} E_i\right|\right),$$

or

$$\left(\widehat{\mathsf{F}_2}(P_1, \ldots, P_7), \left| f^*(2C + 5F) - \sum_{i=1}^{7} E_i \right|\right).$$

REMARK 5.2. In [6], Ellia proves that if $S \subset \mathsf{P}^4$ is a smooth, non-degenerate surface isomorphic to F_n blown-up at r points $y_1, \ldots y_r$ such that the points y_i lie in different fibers of $g : \mathsf{F}_n \to \mathsf{P}^1$ and no y_i lies on C for $n \geq 1$, then $\deg(S) \leq 12$. Under more general conditions, we prove that, in fact, $\deg(S)$ can be only 4 or 5.

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DEPARTMENT OF MATHEMATICS
BUCHAREST UNIVERSITY
14 ACADEMIEI STR.
70109, BUCHAREST
ROMANIA
E-mail: voica@gta.math.unibuc.ro