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FC$^-$-ELEMENTS IN TOTALLY DISCONNECTED GROUPS AND AUTOMORPHISMS OF INFINITE GRAPHS

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Abstract

An element in a topological group is called an FC$^-$-element if its conjugacy class has compact closure. The FC$^-$-elements form a normal subgroup. In this note it is shown that in a compactly generated totally disconnected locally compact group this normal subgroup is closed. This result answers a question of Ghahramani, Runde and Willis. The proof uses a result of Trofimov about automorphism groups of graphs and a graph theoretical interpretation of the condition that the group is compactly generated.

Introduction

This note has two purposes: to give a partial answer to a question of Ghahramani, Runde and Willis [2, Question 8.4(i)], and to continue the work in [4], [6], [7], [11] and [12] on how one can relate ideas and concepts in the theory of topological groups to ideas and concepts from permutation group theory. As explained in [2] the new information regarding [2, Question 8.4(i)] has consequences in the theory of derivations on group algebras.

The concept in the limelight here is that of a topological group being compactly generated. A topological group G is *compactly generated* if there is a compact subset that generates G. It will be shown that for a totally disconnected locally compact group G it is equivalent that G is compactly generated, and that for every compact open subgroup U there is a finitely generated group H that acts transitively on the coset space G/U (Lemma 2). It follows that a totally disconnected locally compact group is compactly generated if and only if it acts transitively on some connected locally finite graph such that the stabiliser of a vertex is both compact and open (Corollary 1). These results support Palmer's view [5, p. 685] that "compactly generated" is a type of a very weak connectedness condition.

These ideas are then used to give a partial answer to the question of Ghahramani, Runde and Willis. In a topological group G an element g is said to be an

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FC$^-$-*element* if the conjugacy class of g has compact closure (some authors use the term *bounded element*). The FC$^-$-elements form a subgroup $B(G)$ of G. The question asked in [2] is whether the closure, $B(G)^-$, of $B(G)$ is an FC$^-$-group, i.e. whether all elements in $B(G)^-$ are FC$^-$-elements in $B(G)^-$. It follows from [8, Corollaire 1] that if G is a connected locally compact group then the subgroup $B(G)$ of FC$^-$-elements is closed. For totally disconnected groups the situation is not so clear. An example of a totally disconnected locally compact group, not compactly generated, where $B(G)$ is not closed and $B(G)^-$ is not an FC$^-$-group can be found in [8, Proposition 3]. The main result of this note is

THEOREM 2. *Let G be a totally disconnected locally compact group. Assume furthermore that G is compactly generated. Then the subgroup $B(G)$ of* FC$^-$-*elements in G is closed in G.*

This result fills a gap in our knowledge about when the subgroup of FC$^-$-elements must be closed and also gives a positive answer to the question of Ghahramani, Runde and Willis in the case of compactly generated totally disconnected locally compact groups. The proof presented here of Theorem 2 takes the scenic route: the question is reduced to the case of automorphism groups of locally finite connected graphs, where one can apply a result of Trofimov [11] (cf. [12, Theorem 3]). The ideas behind the reduction to the automorphism groups of graphs are explained in detail. A possible route to a direct proof is described in a final remark.

1. Preliminaries

Let G be a group acting on a set Ω. The action of G on Ω will be written on the right so that the image of a point $\alpha \in \Omega$ under an element $g \in G$ is written as αg. For a point $\alpha \in \Omega$ the *stabiliser* in G of α is the subgroup

$$G_\alpha = \{g \in G \mid \alpha g = \alpha\},$$

and if Δ is a subset of Ω the *pointwise stabiliser* in G of Δ is the subgroup

$$G_{(\Delta)} = \{g \in G \mid \delta g = \delta \text{ for every } \delta \text{ in } \Delta\}.$$

For a point α in Ω the orbits of the stabiliser G_α are called *suborbits* of G. The orbits of G on Ω^2 are called *orbitals*. When G acts transitively on Ω there is a simple one-to-one correspondence between the orbits of G_α and the orbitals: the suborbit βG_α corresponds to the orbital $(\alpha, \beta)G$. An *orbital graph* $\Gamma = (V\Gamma, E\Gamma)$ is formed by letting the set of vertices $V\Gamma$ be equal to Ω and letting the set of edges $E\Gamma$ be a union of some set of orbitals. The edges in an orbital graph are ordered pairs of vertices so an orbital graph is a directed graph. We say that a vertex β is a *neighbour* of another vertex α if either (α, β) or (β, α) is an edge. An orbital graph is said to be connected if for any two vertices α and β there is a path $\alpha_0, \alpha_1, \dots, \alpha_n$ with $\alpha = \alpha_0$ and $\beta = \alpha_n$ such that α_i and α_{i+1} are neighbours for every $i = 0, \dots, n-1$. It is easy to see that G acts on an orbital graph as a group of automorphisms, because if $g \in G$ and (α, β) is an edge in an orbital graph, then $(\alpha g, \beta g)$ is in the same orbital and thus also an edge. When all the suborbits of G are finite and the edge set of an orbital graph is a union of finitely many orbitals then this orbital graph is locally finite (i.e. each vertex in it has only finitely many neighbours).

It is possible to use the action of G on Ω to define a topology, *the permutation topology*, on G. This topology is defined by taking as a neighbourhood basis of the identity element the family of all subgroups of the form $G_{(\Phi)}$ where Φ is a finite subset of Ω. The subgroups $G_{(\Phi)}$ are both open and closed in G. If G acts faithfully on Ω then G, with this topology, is totally disconnected. Conversely we can start with a topological group G and an open subgroup U of G. If it is assumed that G is totally disconnected and locally compact then one can choose U to be both compact and open (see [3, Theorem 7.7]). Let Ω be the coset space G/U. Because U is open, the permutation topology defined by the action of G on Ω is a subtopology of the original topology on G.

2. Compactly generated groups

The following lemma belongs to the large class of mathematical results that are termed as "folklore" and it is implicit in several arguments in the literature but it is difficult to locate an explicit reference.

LEMMA 1. *Let G be a group acting transitively on a set Ω. Assume that all suborbits of G are finite. Then the following are equivalent:*

(i) *G contains a finitely generated transitive subgroup;*

(ii) *there are orbitals $\Delta_1, \dots, \Delta_n$ such that the orbital graph $(\Omega, \Delta_1 \cup \dots \cup \Delta_n)$ is connected.*

The implication (i) $\Rightarrow$ (ii) *is valid without any assumptions about the finiteness of suborbits.*

PROOF. (i) $\Rightarrow$ (ii): Assume that H is a finitely generated transitive subgroup and assume also that we have chosen a finite generating set $\{h_1, \dots, h_n\}$ that is closed under taking inverses. Pick a point $\alpha \in \Omega$. Let Δ_i denote the orbital $(\alpha, \alpha h_i)G$. We will prove that the orbital graph $\Gamma = (\Omega, \Delta_1 \cup \dots \cup \Delta_n)$ is connected. We must show that if β is a vertex in Γ then there is a path in Γ from α to β. Say $\beta = \alpha g$ where $g = h_{i_1} h_{i_2} \dots h_{i_m} \in H$. A path from α to β

consists of the sequence of vertices:

$$\alpha, \alpha h_{i_m}, \alpha h_{i_{m-1}} h_{i_m}, \ldots, \alpha h_{i_2} \ldots h_{i_{m-1}} h_{i_m}, \alpha h_{i_1} h_{i_2} \ldots h_{i_{m-1}} h_{i_m} = \beta.$$

One sees that this is a path in Γ because $(\alpha h_{i_k} \ldots h_{i_{m-1}} h_{i_m}, \alpha h_{i_{k-1}} h_{i_k} \ldots h_{i_{m-1}} h_{i_m})$ is equal to $(\alpha, \alpha h_{i_{k-1}}) h_{i_k} \ldots h_{i_{m-1}} h_{i_m} \in \Delta_{i_{k-1}}$, and is therefore an edge in the orbital graph Γ.

(ii) $\Rightarrow$ (i): Now assume that (ii) holds. Then we have a locally finite connected graph Γ with vertex set equal to Ω and the action of G on Ω gives an action of G as a group of automorphisms on Γ. Choose a fixed vertex α in Γ. Enumerate the neighbours of α as $\beta_1, \ldots, \beta_n$. For each i let h_i be an element in G such that $\alpha h_i = \beta_i$. The claim is that $H = \langle h_1, \ldots, h_n \rangle$ is transitive. Clearly every vertex adjacent to α is in the orbit αH. If $\beta = \alpha h$ for some $h \in H$ then the set $\{\beta h^{-1} h_i h\}_{i=1}^n$ contains precisely all the neighbours of β. So, if β is in the orbit αH then every vertex adjacent to β is also in αH. Since Γ is connected this implies that $\alpha H = V\Gamma = \Omega$, i.e. H is transitive.

LEMMA 2. *Let G be a compactly generated totally disconnected locally compact group, and U a compact open subgroup of G. Set $\Omega = G/U$. Then G has a finitely generated subgroup H such that H acts transitively on Ω.*

Conversely, if U is a compact open subgroup of a totally disconnected locally compact group G and G contains a finitely generated subgroup H that acts transitively on $\Omega = G/U$ then G is compactly generated.

PROOF. Assume G is compactly generated and S is some compact generating set of G. The family of open sets $\{Ug\}_{g \in G}$ gives an open covering of S and as S is compact there is a finite subcovering $Ug_1, \ldots, Ug_m$ of S. Hence $G = \langle U, g_1, \ldots, g_m \rangle$. If we take some element $g \in G$ then $U \cap (g^{-1}Ug)$ is an open subgroup of the compact group U and therefore $|U : U \cap (g^{-1}Ug)| < \infty$. When we look at the action of G on the coset space $\Omega = G/U$ then the subgroup $g^{-1}Ug$ is precisely the subgroup in G of all the elements in G that fix the coset Ug. That $|U : U \cap (g^{-1}Ug)| < \infty$ implies that the orbit of the coset Ug under the subgroup U is finite, i.e. the subset UgU of G can be written as a union of finitely many right cosets of U. Hence there are for each i elements $u_1, \ldots, u_{n_i} \in U$ such that $Ug_iU = \cup_{j=1}^{n_i} Ug_iu_j$. Now let $\{h_1, \ldots, h_n\}$ be the set of all elements g_iu_j arising in this fashion. The point of this is that now $h_k u$, with $u \in U$, can be written as $u'h_{k'}$ for some k' and some $u' \in U$. That is so, because $h_k u$ is contained in Ug_iU for some i but Ug_iU can be written as a union of sets of the type Uh_j. Note that $G = \langle U, h_1, \ldots, h_n \rangle$.

Let α denote the coset $U \in G/U = \Omega$. Take some point β in Ω, say $\beta = Ub$ with $b \in G$. Write $b = u_1 h_{i_1} u_2 h_{i_2} \ldots u_k h_{i_k} u_{k+1}$ where $u_i \in U$. Because $h_{i_{j-1}} u_j = u'_j h_{i'_{j-1}}$ for some $u'_j \in U$ and some index i'_{j-1} one can assume that $b = uh_{i_1}h_{i_2}\dots h_{i_k}$. Then, with $h = h_{i_1}h_{i_2}\dots h_{i_k} \in H$, we see that $\alpha h = \beta$. One concludes that H acts transitively on Ω.

For the second part of the lemma one only needs to note that if $\{h_1, \dots, h_n\}$ is a finite generating set for H then G is generated by the compact set $U \cup \{h_1, \dots, h_n\}$.

Combining the two lemmas above we get the following corollary.

COROLLARY 1. *Let G be a totally disconnected locally compact group.*

If G is compactly generated then there is a locally finite connected graph Γ *such that:*

(i) *G acts as a group of automorphisms on* Γ *and is transitive on* $V\Gamma$*;*

(ii) *for every vertex* α *in* Γ *the subgroup*

$$G_\alpha = \{g \in G \mid \alpha g = \alpha\}$$

is compact and open in G;

(iii) *if* $\mathrm{Aut}(\Gamma)$ *is given the permutation topology then the homomorphism* $\pi : G \to \mathrm{Aut}(\Gamma)$ *given by the action of G on* Γ *is continuous, the kernel of this homomorphism is compact and the image of* π *is closed in* $\mathrm{Aut}(\Gamma)$.

Conversely, if G acts as a group of automorphisms on a locally finite connected graph Γ *such that G is transitive on the vertex set of* Γ *and the stabilisers of the vertices in* Γ *are compact and open, then G is compactly generated.*

PROOF. Pick a compact open subgroup U of G. By Lemma 2 we know that G has a finitely generated subgroup that acts transitively on $\Omega = G/U$. Now we use the implication (i) $\Rightarrow$ (ii) in Lemma 1 to see that it is possible to find finitely many orbitals such that the resulting orbital graph Γ is connected. As pointed out before all the suborbits in the action of G on Ω are finite and therefore the orbital graph Γ is locally finite.

The stabiliser in G of a vertex α in Γ is conjugate to U and thus compact and open in G. A basis of neighbourhoods of the identity element in the permutation topology on $\mathrm{Aut}(\Gamma)$ is given by the pointwise stabilisers of finite sets of vertices. The pre-image of such a set in G is the intersection of finitely many conjugates of U and thus open in G. Therefore π is continuous. The kernel of π is closed and contained in the compact subgroup U. Hence the kernel is compact. The stabiliser in $\pi(G)$ of a point α is the image under π of the stabiliser in G of α. Thus the stabiliser of α in $\pi(G)$ is compact, and thus closed, in $\mathrm{Aut}(\Gamma)$.

Let H denote the stabiliser of a vertex α in $\mathrm{Aut}(\Gamma)$. Now $\pi(G) \cap H$ is the stabiliser of α in $\pi(G)$ and is closed. Since the intersection of the subgroup $\pi(G)$ with the open subgroup H is closed we can refer to standard results about topological groups ([1, Proposition 2.4 in Chapter III] or [3, 5.37]) to conclude that $\pi(G)$ is closed in $\mathrm{Aut}(\Gamma)$.

Let us now turn to the latter part of the Corollary where it is assumed that G is acting on a locally finite connected graph. From Lemma 1 we learn that G must contain a finitely generated subgroup $H = \langle h_1, \ldots, h_n\rangle$ that acts transitively on the vertex set of Γ. If U is the stabiliser of some vertex in Γ then G is generated by the compact set $U \cup \{h_1, \ldots, h_n\}$.

3. Application to FC$^-$-elements

Before addressing the question of Ghahramani, Runde and Willis, we first repeat two lemmas from the paper [12] by Woess. Here these auxiliary results are needed in a slightly more general setting than that of Woess's paper, but the same proofs work equally well.

LEMMA 3 ([12], Lemma 2). *Let G be a topological group acting (not necessarily faithfully) on a set Ω such that the stabiliser in G of any point α in Ω is compact and open in G. A subset A of G has compact closure in G if and only if the orbit αA is finite for every $\alpha \in \Omega$.*

PROOF. Suppose that A^-, the closure of A in G, is compact. Then for a point $\alpha \in \Omega$ there is a finite open covering of A^- by sets of the type $G_\alpha g$; that is to say, we can find $g_1, \ldots, g_n \in \mathrm{Aut}(\Gamma)$ such that $A^- \subseteq \cup_{i=1}^n G_\alpha g_i$. Then $\alpha A \subseteq \{\alpha g_1, \ldots, \alpha g_n\}$.

Now suppose that $\alpha A = \{\beta_1, \ldots, \beta_n\}$. Let g_i be an element in A such that $\alpha g_i = \beta_i$. Then $A \subseteq \cup_{i=1}^n G_\alpha g_i$. The latter set is compact, so the closure of A is compact.

For a connected graph Γ we let d denote the usual graph metric on the vertex set $V\Gamma$, i.e. $d(\alpha, \beta)$ is the least possible number of edges in a path from α to β. An automorphism g of Γ is said to be *bounded* if there is a constant M such that $d(\alpha, \alpha g) \leq M$ for every vertex α in Γ. It is not difficult to show that the bounded automorphisms of Γ form a subgroup $B(\Gamma)$ of the automorphism group.

LEMMA 4 ([12], Lemma 4). *Let G be a topological group acting transitively (but not necessarily faithfully) on a graph Γ such that the stabiliser in G of a vertex α in Γ is compact and open in G. An element $g \in G$ acts on Γ as a bounded automorphism if and only if g is an* FC$^-$*-element of G.*

PROOF. Suppose $g \in G$ acts as a bounded automorphism on Γ. Find a number M such that $d(\alpha g, \alpha) \leq M$ for every $\alpha \in V\Gamma$. For $h \in G$, write $g^h = h^{-1}gh$. It is clear that $d(\alpha g^h, \alpha) = d(\alpha h^{-1}g, \alpha h^{-1}) \leq M$ for every

$\alpha \in V\Gamma$. Set $g^G = \{g^h \mid h \in \mathrm{Aut}(X)\}$. We see that the set αg^G is finite and by Lemma 3 the conjugacy class g^G has compact closure.

Suppose now that the conjugacy class g^G has compact closure. Then, for every $\alpha \in V\Gamma$ the set αg^G is finite. Let M be a number such that for some vertex $\alpha \in V\Gamma$ we have $d(\alpha g^h, \alpha) \leq M$, for every $h \in G$. Take some $\beta \in V\Gamma$. Let $h \in G$ be chosen such that $\beta = \alpha h^{-1}$. Then $d(\beta g, \beta) = d(\alpha h^{-1} g, \alpha h^{-1}) = d(\alpha g^h, \alpha) \leq M$. So g acts on Γ as a bounded automorphism.

For the automorphism group of a connected locally finite graph we have the following result due to Trofimov.

THEOREM 1 ([11], cf. [12, Theorem 3]). *Let Γ be a locally finite connected graph and suppose $G = \mathrm{Aut}(\Gamma)$ acts transitively on $V\Gamma$. The subgroup of bounded automorphisms is closed in G when G has the permutation topology.*

Finally we come to the main result of this note, the partial answer to the question of Ghahramani, Runde and Willis.

THEOREM 2. *Let G be a totally disconnected locally compact group. Assume furthermore that G is compactly generated. Then the subgroup $B(G)$ of* FC^{-}-*elements in G is closed in G.*

PROOF. We apply Corollary 1 to find a locally finite connected graph Γ that G acts on and a continuous map $\pi : G \to \mathrm{Aut}(\Gamma)$. From the result of Trofimov, Theorem 1, we know that the subgroup $B(\Gamma)$ of bounded automorphisms in $\mathrm{Aut}(\Gamma)$ is closed in $\mathrm{Aut}(\Gamma)$. The pre-image of $B(\Gamma)$ under π is then closed and from Lemma 4 it follows that $B(G) = \pi^{-1}(B(\Gamma))$.

REMARKS. 1. In the thread of arguments leading to Theorem 2 one notes that the assumption that G is a totally disconnected locally compact group is only used to guarantee the existence of a compact open subgroup. Theorem 2 thus also holds with the weaker assumption that G contains a compact open subgroup.

2. Trofimov deduces Theorem 1 as a consequence of [10, Proposition 2.3] which he in turn deduces as a consequence of the main result of [9]. In [4] there is a short direct proof of [10, Proposition 2.3] (see [4, Lemma 5]). It might be possible to find a direct proof of Theorem 2, not passing through the realm of graphs, by finding a purely group theoretic analogue of [10, Proposition 2.3].

3. The group of FC^{-}-elements has been discussed by various authors. The present note is perhaps most closely related to [13]. Theorem 2 extends [13, Proposition 2] where it is shown that if G is a compactly generated totally disconnected locally compact group such that $B(G)$ is dense in G then $G = B(G)$.

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