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LOCAL L^2 RESULTS FOR $\overline{\partial}$ ON A SINGULAR SURFACE

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Abstract

The Cauchy-Riemann equations are fundamental in complex analysis. This paper contributes to the understanding of these equations on singular spaces. Various methods have been used to overcome the problem of defining forms near singularities. One can blow up the singularity, restrict forms from smooth ambient spaces or work on the regular points. In this paper we use the latter approach to obtain square integrable solutions on singular surfaces. This can be briefly called the Kohn solution up to the singularity to contrast with results in terms of curvature, weights or different function spaces.

1. Introduction

The $\overline{\partial}$ equation is the main quantitative tool in the theory of several complex variables. It has been used extensively in analysis of domains in $\mathbf{C}^d$ and on complex manifolds. However, in the theory of several complex variables, one also needs to investigate complex spaces as they occur naturally as soon as one considers zero sets of holomorphic functions.

In this paper we address the question whether it is possible to solve the $\overline{\partial}$ equation with L^2 estimates in complex spaces. The main motivation is that L^2 estimates appear as the most promising tool to solve the Levi problem in Stein spaces.

The general philosophy is that one can study a complex Stein space by reducing to a closed k-dimensional subvariety X in $\mathbf{C}^d$ and viewing X as a branched cover over some suitable $\mathbf{C}^k$. Next one can remove a hypersurface S from X such that the projection $\pi : X \setminus S \to \mathbf{C}^k$ is unbranched. On $X \setminus S$ one can solve $\overline{\partial} u = \lambda$ using the classical L^2 theory of Hörmander [4]. Next one can use a detailed geometric analysis of the singular space to modify this u to obtain a solution with L^2-estimates, up to an explicit finite dimensional set of obstructions.

We analyze complex (not necessarily algebraic) surfaces in $\mathbf{C}^d$ with an isolated singularity at the origin. We solve the problem $\overline{\partial} u = \lambda$ where λ is a

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$\overline{\partial}$ closed (0, 1) form in a deleted neighborhood of the singular point 0 of X. More precisely we show the following:

THEOREM 1.1. *There exists a closed subspace H of finite codimension of the set of $\overline{\partial}$-closed* (0, 1) *forms λ in $L^{2,(0,1)}_{X^*\cap \mathsf{B}(0,\epsilon)}$ and a linear operator $T : H \to L^{2,(0,0)}_{X^*\cap \mathsf{B}(0,\delta)}$ for some $\delta < \epsilon$ and a constant C so that*

$$\overline{\partial}(T\lambda) = \lambda$$

$$\|T\lambda\|_{L^{2,(0,0)}_{X^*\cap \mathsf{B}(0,\delta)}} \leq C\|\lambda\|_{L^{2,(0,1)}_{X^*\cap \mathsf{B}(0,\epsilon)}}$$

The L^2 norms in the theorem are measured with respect to the induced metric by the imbedding of $X^* \hookrightarrow \mathsf{C}^d$.

The case of conic singularities was done by Fornæss in [1].

Our result complements results by Pardon [8], Pardon and Stern [9]. In the particular case that X is a complex projective surface with isolated singularities and the set of regular points of X is given the Hermitian (incomplete) metric induced by an embedding of X to a projective space Pardon and Stern [9] identified the $(0, q)$ L^2-$\overline{\partial}$-cohomology groups with Neumann boundary conditions of $X \setminus \operatorname{sing} X$ with certain sheaf cohomology groups of its blow-up $\tilde{X}$. Namely they proved

$$H_N^{(0,q)}(X \setminus \operatorname{sing} X) \cong H^q(\tilde{X}, \mathcal{O}(Z - |Z|))$$

where the left-hand side is the $(0, q)$ L^2-$\overline{\partial}$-cohomology with Neumann boundary conditions, $\tilde{X} \to X$ is a resolution of singularities of X, Z is the (unreduced) exceptional divisor and Z is supported along a divisor with normal crossings.

In the same paper, Pardon and Stern proved a local vanishing result for the (n, q) L^2-$\overline{\partial}$-cohomology groups of $U \setminus U \cap \operatorname{sing} X$, $U \subset X$, open, $n = \dim X$, $q \geq 1$ and X is any projective variety with isolated singularities. This allowed them to identify

$$H_N^{(n,q)}(X \setminus \operatorname{sing} X) \cong H^{n,q}(\tilde{X})$$

Applying an L^2-version of Serre's duality theorem to the left hand side of the above isomorphism and its algebraic version to the right hand side they obtained

$$H_D^{(0,q)}(X \setminus \operatorname{sing} X) \cong H^{0,q}(\tilde{X})$$

where the left-hand side is the $(0, q)$ L^2-$\overline{\partial}$-cohomology with Dirichlet boundary conditions.

In proving the above results essential use is made of the fact that X and hence $\tilde{X}$ is compact which ensures the finite dimensionality of certain cohomology groups and is also necessary in order to apply an L^2-version of Serre's duality theorem. A similar argument doesn't seem feasible when X is not compact. Also key ingredient in proving the result about the L^2-cohomology with Neumann boundary conditions is an explicit computation of the pull back of the Fubiny-Study metric under the blow-up map obtained by Hsiang and Pati [5] and Nagase [7]. It is not clear to us that such a computation can be generalized to higher dimensional varieties.

Similar results (as in [8], [9]) for complex algebraic surfaces in projective space with isolated singularities were obtained by Haskell [3], and Nagase [7].

Our point of view is to work without weights, without essential use of metrics and mainly on the original singular space. We plan to continue the investigation in future work.

There is a parallel program to investigate L^∞ estimates, see [2].

The paper is organized as follows: In Section 2 we describe the geometry of the variety near the singular point. In Section 3 we introduce the L^2 spaces and norms we will be using. Finally, in Section 4 we construct the solutions to the $\overline{\partial}$-equation.

2. Geometry of Varieties

We will investigate the surface X in $\mathbb{C}^d$ with an isolated singularity at the origin. We first need to make some standard remarks on the choice of generic coordinates.

LEMMA 2.1. *Let X be a 2-dimensional complex variety in a neighborhood of* (0) *in* $\mathbb{C}^d$, $0 \in X$. *Let* $\{\phi_j\}_{1\le j\le k}$ *be a finite family of linear functions. Then after arbitrarily small perturbations we may assume that* (0) *is isolated in* $X \cap \{\phi_i = \phi_j = 0\}$ *for any* $i \neq j$.

PROOF. We prove this by induction in k. If $k = 2$, perturb ϕ_1 first so that $Z = X \cap \{\phi_1 = 0\}$ has dimension 1. Then perturb ϕ_2 so that ϕ_2 is not equivalent to 0 on each irreducible component of Z.

Next, suppose the statement is true for k and let $\{\phi_j\}_{1\le j\le k+1}$ be given. We may assume that (0) is isolated in $X \cap \{\phi_i = \phi_j = 0\}$ for any $1 \le i < j \le k$. If we make any sufficiently small perturbations of $\phi_1, \dots \phi_k$ then this remains true. So, as above, we may assume that $X \cap \{\phi_i = 0\}$ has dimension 1 for any $i \le k$. But then, as in the first step we can perturb ϕ_{k+1} so that ϕ_{k+1} doesn't vanish identically on any irreducible branch in any $X \cap \{\phi_i = 0\}$, $i \le k$.

We can assume that the ϕ_i are the coordinate functions x_i. It follows that we can assume that X is a branched cover over $\mathbb{C}^2(x_i, x_j)$ for any pair $1 \le i < j \le d$. To simplify notation, let us consider the branching of X over $\mathbb{C}^2(x, y)$, $x = x_1$, $y = x_2$. The branching occurs over a one dimensional variety L in $\mathbb{C}^2(x, y)$. Then L is a union of finitely many irreducible curves L_i, $i = 1, \ldots, M$. After another small rotation, the Zariski tangent space of each curve L_i at 0 is different from the x- and y-axis. Then L_i can be parametrized by one complex variable t in a neighborhood of 0

$$\begin{aligned} x &= t^{n_i} \\ y &= \tilde{g}_i(t) = c_0 t^{n_i} + c_1 t^{n_i+1} + \cdots, \qquad c_0 \neq 0. \end{aligned}$$

Here $\tilde{g}_i$ is holomorphic in a neighborhood of 0, $\tilde{g}_i(0) = 0$.

REMARK 2.2. The above assumption implies that $\frac{dy}{dx}$ is bounded and bounded away from 0 along L_i near the origin.

We can assume that similar descriptions are valid with projections to all the $x_i x_j$-planes.

We call π the local projection, $\pi : X \to \mathbb{C}^2(x, y)$. Set $Y_i = \pi^{-1}(L_i)$. Then the Y_i are complex curves with finitely many irreducible components Y_i^k. Set $b(i, k) :=$ the branching number of π along Y_i^k. Then $b(i, k) \geq 1$ and for each i at least one $b(i, k) > 1$. Also let $c(i, k)$ denote the sheet number of the projection from Y_i^k to L_i. Then Y_i^k can be parametrized as follows:

$$\begin{aligned} x &= t^{n_i c(i,k)} \\ y &= \tilde{g}_i(t^{c(i,k)}) = c_0 t^{n_i c(i,k)} + c_1 t^{(n_i+1)c(i,k)} + \cdots \\ z_j &= h^j_{i,k}(t), \qquad j = 3, \ldots, d. \end{aligned}$$

where $h^j_{i,k}$ is holomorphic in a neighborhood of 0, $h^j_{i,k}(0) = 0$. Hence the parametrization of all L_i and Y_i^k are of the form:

$$\begin{aligned} x &= t^n \\ y &= g(t) = ct^n + \cdots, \ c \neq 0 \\ z_j &= h_j(t), \qquad j = 3, \ldots d. \end{aligned}$$

where the first two equations parametrize L_i and all equations parametrize Y_i^k.

We need to choose tubular Stein neighborhoods, pinched at 0, for L_i in $\mathbb{C}^2_{x,y}$ and for Y_i^k in X. Let $\{P_i(x, y)\}_{i=1}^M$ be irreducible holomorphic functions, $L_i = \{P_i = 0\}$. We can assume that the P_i are Weierstrass polynomials, $P_i(x, y) = y^{m_i} + \sum_{j<m_i} a^i_j(x) y^j$, $a^i_j(0) = 0$. Hence we can write $P_i(x, y) = \Pi_{j=1}^{m_i}(y - y_j(x))$ with locally defined holomorphic functions $y_j(x)$.

If $i \neq j$, then $|P_i|^2(x, y) + |P_j|^2(x, y)$ is a nonnegative real analytic function that vanishes only at the origin. By the weak form of Lojasiewicz [6] there exist

$C > 0$, $N > 1$ such that

$$|P_i|^2(x, y) + |P_j|^2(x, y) \geq C\|(x, y)\|^N.$$

We will increase N whenever necessary.

For $(x, y) \in L_j$ we have

$$\begin{aligned}\operatorname{dist}((x, y), L_i) &\geq \frac{3}{C}|P_i|^2(x, y) = \frac{3}{C}(|P_i|^2(x, y) + |P_j|^2(x, y)) \\ &\geq 3\|(x, y)\|^N.\end{aligned}$$

We define

$$T_i := \{(x, y); d((x, y), L_i) < \|(x, y)\|^N\}.$$

If $(x, y) \in T_j$, let (x', y') be a point on L_j for which $\operatorname{dist}((x, y), L_j) = \|(x, y) - (x', y')\|$. Then

$$\begin{aligned}d((x, y), L_i) &\geq d((x', y'), L_i) - \|(x, y) - (x', y')\| \\ &\geq 3\|(x', y')\|^N - \|(x, y) - (x', y')\| \\ &\geq 3\|(x', y')\|^N - \|(x, y)\|^N\end{aligned}$$

Now,

$$\|(x', y')\| \geq \|(x, y)\| - \|(x, y) - (x', y')\| \geq (3/4)^{1/N}\|(x, y)\|$$

and so

$$d((x, y), L_i) \geq 5/4\|(x, y)\|^N.$$

Hence,

$$\alpha)\ \ L_i \setminus \{0\} \subset T_i$$

$$\beta)\ \ T_i \cap T_j = \emptyset, \qquad i \neq j$$

For our purposes though, it is more convenient to use tubular neighborhoods which are circular in the y direction. For this, recall that $L_i = \{P_i = 0\}$ where $P_i = \Pi_{j=1}^{m_i}(y - y_j(x))$. Then $\Pi_{k\neq j}(y_j(x) - y_k(x))^2 =: \phi_i(x)$ is a holomorphic function with an isolated zero at the origin. Hence $|\phi_i(x)| \geq |x|^N$ for some larger N. It follows that for $x \neq 0$, the discs $\{D_j(x) := \{y; |y - y_j(x)| < |x|^{\tilde{N}}\}\}$ are pairwise disjoint. We define the tubular neighborhoods

$$\hat{T}_i := \{(x, y); \exists t;\ x = t^{n_i}, |y - \tilde{g}_i(t)| < |x|^N\} \subset T_i \tag{1}$$

Then the $\hat{T}_i$ are Stein. Moreover, the $\pi^{-1}(\hat{T}_i)$ split into disjoint Stein neighborhoods Ω_i^k of the respective Y_i^k. To see this, we can use a continuity argument.

First of all, the Y_i^k only intersect at (0). Hence if $p \in L_i \setminus (0)$ and $B(p, \epsilon)$ is a small enough ball centered at p, then $\pi^{-1}((B(p, \epsilon))$ splits into disjoint neighborhoods of the respective points in $\pi^{-1}(p)$ in Y_i^k. Hence, if we start by replacing the $\hat{T}_i$ by thin enough neighborhoods of the L_i, then the Ω_i^k are disjoint. If we now continuously increase the $\hat{T}_i$, then if at some point two Ω_i^k overlap, the number of branches of π drops and this can only happen where π branches.

NOTATION. For variable A, B (complex or real valued objects) we write $A \sim B$ if there exist positive constants C_1, C_2, independent of objects that will be specified later on, such that $C_1|A| \leq |B| \leq C_2|A|$. Similarly, for E, F real or complex valued objects we write $E \lesssim F$ if there exists a positive constant C such that $|E| \leq C|F|$ where C is independent of the objects and will be specified later on.

Pick any $p = p(t) \in Y_i^k \setminus 0$, $b(i, k) > 1$. Then the tangent plane to X at p is spanned by two vectors, one is tangent to Y_i^k, the other is a vector $(0, 0, z_3, \ldots, z_d)$ perpendicular to the (x, y) plane because Y_i^k is in the branching locus and this second one projects to a unique point $Z = [z_3 : \ldots : z_d] \in \mathbb{P}^{d-3}$.

LEMMA 2.3. *We can parametrize $Z = Z(t)$, t being the parameter for Y_i^k with $Z(t) = (z_3(t), \ldots, z_d(t))$ and with some $z_j = z^{i,k}(t) \equiv 1$. Moreover, $Z(t)$ is holomorphic and extends holomorphically across $t = 0$.*

PROOF. The case d=3 is trivial, so we assume that $d > 3$. Let $\{H_1, \ldots, H_\nu\}$ be functions generating the ideal sheaf of X at every point in a neighborhood of 0. Then $[z_3 : \ldots : z_d]$ at $p \in Y_i^k \setminus 0$ is uniquely determined by the equations

$$\sum_{j=3}^{n} \frac{\partial H_\alpha}{\partial x_j} z_j = 0 \qquad \forall \alpha.$$

Since the solutions of the above system form a 1-dimensional complex subspace of $\mathbb{C}^{d-2}$, the rank of $\left(\frac{\partial H_\alpha}{\partial x_j}\right)_{\alpha, j; j \geq 3}$ is $d - 3$. Notice that each entry $f_j^\alpha(t) := \frac{\partial H_\alpha}{\partial x_j}$ is holomorphic in t on Y_i^k, including across $t = 0$.

Not all $\frac{\partial H_\alpha}{\partial x_j} \equiv 0$ for $j \geq 3$, $\alpha \leq \nu$, on Y_i^k. Therefore, there will exist $k_1 \in \{3, \ldots, d\}$, $\alpha_1 \in \{1, \ldots, \nu\}$ such that $\frac{\partial H_{\alpha_1}}{\partial x_{k_1}}$ vanishes to smallest order at $t = 0$. After rearranging the coordinates making sure that $x_3{}' =: x_{k_1}$ and letting α_1 correspond to the first equation, we obtain as the first equation of the system;

$$f_3^{3,1}(t) z_3{}' + \cdots + f_d^{3,1}(t) z_d{}' = 0$$

where $f_j^{3,1}(t)$ are holomorphic functions of t, and $|f_j^{3,1}(t)| \lesssim |f_3^{3,1}(t)|$ for $j = 3, \dots, d$ and $f_3^{3,1}(t) \neq 0$ when $t \neq 0$.

We can solve the above equation with respect to $z_3{}'$ and substitute the corresponding expression in the remaining equations of the system. We obtain;

$$f_4^{4,1}(t)z_4{}' + \cdots + f_d^{4,1}(t)z_d{}' = 0$$
$$f_4^{5,1}(t)z_4{}' + \cdots + f_d^{5,1}(t)z_d{}' = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_4^{\nu+2,1}(t)z_4{}' + \cdots + f_d^{\nu+2,1}(t)z_d{}' = 0$$

where $f_j^{i,1}(t)$ are holomorphic functions of t (due to the fact that $|f_j^{3,1}(t)| \lesssim |f_3^{3,1}(t)|$, and f_j^{α} are holomorhic functions in t).

Not all $f_k^{j,1}(t) \equiv 0$, $4 \leq j \leq \nu + 2$, $4 \leq k \leq d$ on Y_i^k. Therefore there will exist $x_2 \in \{4, \dots, d\}$, $\alpha_2 \in \{4, \dots, \nu + 2\}$ such that $f_{k_2}^{\alpha_2,1}(t)$ vanishes to smallest order at $t = 0$. Rearranging the coordinates making sure that $x_3{}'' = x_3'$ and $x_4{}'' = x_{k_2}'$ and reordering the equations so that α_2 corresponds to the second equation we get:

$$f_3^{3,2}(t)z_3{}'' + f_4^{3,2}(t)z_4{}'' + \cdots + f_d^{3,2}(t)z_d{}'' = 0$$
$$f_4^{4,2}(t)z_4{}'' + \cdots + f_d^{4,2}(t)z_d{}'' = 0$$

where $|f_j^{3,2}(t)| \lesssim |f_3^{3,2}(t)|$ for $j = 3, \dots, d$ and $|f_j^{4,2}(t)| \lesssim |f_4^{4,2}(t)|$ for $j = 4, \dots, d$ and $f_j^{3,2}(t)$, $f_j^{4,2}(t)$ are holomorphic functions of t.

Continuing this way we can finally obtain the following system;

$$\sum_{j=3}^{d} f_j^{3,d-3}(t)z_j = 0$$
$$\sum_{j=4}^{d} f_j^{4,d-3}(t)z_j = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{j=d-1}^{d} f_j^{d-1,d-3}(t)z_j = 0$$

for holomorphic functions $f_i^j(t)$, $f_j^j(t)$ not identically equal to 0, $|f_i^j(t)| \lesssim |f_j^j(t)|, i > j$. At this point, by rank considerations all the remaining equations are identically zero. We obtain from the last equation that $z_{d-1} = c_{d-1}(t)z_d$,

$c_{d-1}(t) := -\frac{f_d^{d-1,d-3}(t)}{f_{d-1}^{d-1,d-3}(t)}$. Moving upwards in the above system we can compute $z_j = c_j(t)z_d$, $c_j(t) = \mathscr{O}(1)$ on Y_i^k for $j = 3, \dots, d-1$.

We write $z_j =: z$ as in the Lemma.

Hence we can write X as a multivalued graph (unbranched Riemann domain) over the x, z plane in some neighborhood of Y_i^k. In fact we can describe X near Y_i^k by

$$\begin{aligned} &y - g\big(x^{\frac{1}{n}}\big) = \big(z_j - h_j\big(x^{\frac{1}{n}}\big)\big)^a \phi \\ &z_\ell - h_\ell\big(x^{\frac{1}{n}}\big) = \big(z_j - h_j\big(x^{\frac{1}{n}}\big)\big)\psi_\ell \qquad \ell \neq j, \end{aligned} \tag{2}$$

for $a \geq 2$, ϕ, ψ_ℓ depending on x, z_j. The above lemma implies that each

$$\psi_\ell = \mathscr{O}(1) \qquad \text{on} \quad Y_i^k \tag{3}$$

Since the projection to the (x, y) plane has locally $b(i, k) =: b$ preimages we can assume that $a = b$ and $\phi = \phi(x^{\frac{1}{n}}, z - h_j(x^{\frac{1}{n}})) \neq 0$, when $z = h_j(x^{\frac{1}{n}})$, $x \neq 0$.

For such (x, z) close to 0 in $\mathbb{C}^2$ let $y_\alpha(x, z)$ denote the y_α coordinates of points in X. Then $H(x, y, z) := \Pi(y - y_\alpha)$ is holomorphic in a neighborhood of 0 and vanishes on X.

Points in Y_i^k satisfy $H(x(t), y(t), z(t)) = 0$.

LEMMA 2.4. *Suppose $b(i, k) =: b > 1$. There exist integers $m \geq 1$, $r \geq 0$ so that on Y_i^k, $\frac{\partial H}{\partial x} \sim \frac{\partial H}{\partial y} \sim t^m$, $\frac{\partial H}{\partial z} = \cdots = \frac{\partial^{b-1} H}{\partial z^{b-1}} = 0$ and $\frac{\partial^b H}{\partial z^b} \sim t^r$ (for t close to 0).*

PROOF. The fact that $\frac{\partial H}{\partial z} = \cdots = \frac{\partial^{b-1} H}{\partial z^{b-1}} = 0$ follows since Y_i^k is in the branch locus. Since $\frac{\partial^b H}{\partial z^b}(x(t), y(t), z(t))$ is holomorphic in t, it follows that $\frac{\partial^b H}{\partial z^b} \sim t^r$ for some integer $r \geq 0$. By the chain rule, $\frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt} = 0$. But $\frac{dx}{dt} \sim \frac{dy}{dt}$ by Remark 2.1 and $\frac{\partial H}{\partial z} = 0$ since $b > 1$. Since $\nabla H \neq 0$ on Y_i^k, $\frac{\partial H}{\partial x} \sim \frac{\partial H}{\partial y} \neq 0$. Since $\frac{\partial H}{\partial x}(x(t), y(t), z(t))$ is holomorphic in t it follows that $\frac{\partial H}{\partial x} \sim t^m$ for some integer $m \geq 0$.

Next we want to show that X is nearly flat as a graph over the (x, z) plane in the y-direction near those Y_i^k for which $b > 1$. More precisely, we want uniform bounds on the slopes.

LEMMA 2.5. *Suppose that $b>1$. For a large enough N, if $(x, y, z_3, \dots, z_d) \in X^* \cap B(0, \epsilon)$ for some small $\epsilon > 0$, $X^* := X \backslash 0$ and $d((x, y, z_3, \dots, z_d), Y_i^k) < \|(x, y, z_3, \dots, z_d)\|^N$, then $\frac{\partial H}{\partial x} \sim \frac{\partial H}{\partial y} \sim x^{\frac{m}{n}}$, $|x|^{m/n} \sim |y|^{m/n} \gtrsim \left|\frac{\partial H}{\partial z}\right|$. The constants in the estimates are arbitrarily close to those of Lemma 2.4.*

PROOF. Recall that if $(x, y, z_3, \dots, z_d) =: p \in Y_i^k$, then $\frac{\partial H}{\partial z} = 0$ and $\frac{\partial H}{\partial x} \sim \frac{\partial H}{\partial y} \sim t^m \sim x^{\frac{m}{n}} \sim y^{\frac{m}{n}}$. The second derivatives of H are uniformly bounded. Let $p' = (x', y', z_3', \dots, z_d')$ be a point on Y_i^k closest to $(x, y, z_3, \dots, z_d)$. Then

$$\left|\frac{\partial H}{\partial z}(x, y, z_3, \dots, z_d)\right| \lesssim \|(x, y, z_3, \dots, z_d) - (x', y', z_3', \dots, z_d')\| \\ \lesssim \|(x, y, z_3, \dots, z_d)\|^N \sim \|(x', y', z_3', \dots, z_d')\|^N.$$

Also
$$\left|\frac{\partial H}{\partial x}(p) - \frac{\partial H}{\partial x}(p')\right| \lesssim \|p'\|^N.$$

If N is large,
$$\|p'\|^N \ll \left|\frac{\partial H}{\partial x}(p')\right|$$

Hence,
$$\frac{\partial H}{\partial x}(p) \sim \frac{\partial H}{\partial x}(p') \qquad \text{and} \qquad \frac{\partial H}{\partial z}(p) \lesssim \frac{\partial H}{\partial x}(p)$$

similarly:
$$\frac{\partial H}{\partial x}(p) \sim \frac{\partial H}{\partial y}(p)$$

We remind the reader that $z =: z_j$. We pick $\ell \neq j$ and want to study the slopes of X near Y_i^k as a graph over the (x, z) plane in the z_ℓ-direction. This slope can be unbounded when we approach the origin. Let $z_{\ell,\alpha}$ denote the z_ℓ-coordinates of points in X with given (x, z). We define $H^\ell(x, z, z_\ell) =: \Pi(z_\ell - z_{\ell,\alpha})$. The treatment is similar to $H(x, y, z)$. In analogy to Lemma 2.4 we have

LEMMA 2.6. *There exist integers m_ℓ, r_ℓ, s_ℓ such that on Y_i^k, $\frac{\partial H^\ell}{\partial x} \sim t^{m_\ell}$, (or $\frac{\partial H^\ell}{\partial x} \equiv 0$), $\frac{\partial H^\ell}{\partial z_\ell} \sim t^{r_\ell}$, $r_\ell \geq 0$, $\frac{\partial H^\ell}{\partial z} \sim t^{s_\ell}$, $s_\ell \geq r_\ell$ or $\frac{\partial H^\ell}{\partial z} \equiv 0$.*

PROOF. We use the fact that all partial derivatives of H^ℓ are holomorphic functions of t in Y_i^k. The estimate $s_\ell \geq r_\ell$ follows since $H^\ell(x, z, h_\ell + (z - h_j)\psi_\ell) \equiv 0$, so
$$\frac{\partial H^\ell}{\partial z} = -\frac{\partial H^\ell}{\partial z_\ell}\,\psi_\ell$$

Recall that
$$\psi_\ell = \mathcal{O}(1) \qquad \text{on} \quad Y_i^k$$

as seen in (3).

The unboundedness of the slope of X over the (x, z) plane in the z_ℓ-direction arises if $m_\ell < r_\ell$. In analogy to Lemma 2.5 we obtain from Lemma 2.6:

LEMMA 2.7. *For a large enough N, if $(x, y, z_3, \dots, z_d) \in X^* \cap B(0, \epsilon)$ and $d((x, y, z_3, \dots, z_d), Y_i^k) < \|(x, y, z_3, \dots, z_d)\|^N$, we can write $z_{\ell,\alpha} = Z_{\ell,\alpha}(x, z)$, $\ell \neq j$, with $\frac{\partial Z_{\ell,\alpha}}{\partial z} = \mathscr{O}(1)$. If $m_\ell \geq r_\ell$ then $\frac{\partial Z_{\ell,\alpha}}{\partial x} = \mathscr{O}(1)$, if $m_\ell < r_\ell$, then $\frac{\partial Z_{\ell,\alpha}}{\partial x} \sim t^{m_\ell - r_\ell} = x^{\frac{m_\ell - r_\ell}{n}}$.*

We now come back to Ω_i^k, the Stein neighborhoods of Y_i^k as introduced after (1). We set $p := (x, y, z_3, \dots, z_d)$ and for some small $\delta > 0$

$$V_i^k := \{p \in X^* \cap B(0, \delta); d(p, Y_i^k) < \|p\|^{\tilde{N}}\}$$

for a large enough $\tilde{N}$, with $\tilde{N}$ as in Lemma 2.6 and 2.7. We want to show that for large enough N, the Ω_i^k are contained in V_i^k.

We can locally write $y = Y(x, z)$ near Y_i^k as long as we are close enough that Lemma 2.5 applies. Since $H(x, Y(x, z), z) \equiv 0$ we get $Y_x = -H_x/H_y \sim 1$ and $|Y_z| = |-H_z/H_y| \lesssim 1$. Similarly for $\ell \neq j$, as in Lemma 2.7 we can write $z_{\ell,\alpha} = Z_{\ell,\alpha}(x, z)$ and

$$\frac{\partial Z_{\ell,\alpha}}{\partial z} = \mathscr{O}(1), \qquad \left|\frac{\partial Z_{\ell,\alpha}}{\partial x}\right| \lesssim |x|^{\frac{-s}{n}}$$

for some $s \geq 0$.

Let $(x, y, z_{j,1}), \dots, (x, y, z_{j,c(i,k)})$ denote the (x, y, z_j) coordinates of points in Y_i^k with given (x, y). Then $\Pi_{\alpha\neq\beta}(z_{j,\alpha} - z_{j,\beta})^2$ is holomorphic on $L_i \setminus (0)$ and by the removable singularity theorem it vanishes to finite order. Increasing N if necessary, it follows that for every point $(x, y) \in L_i$ the $(z_{j,\alpha}, z_{j,\beta})$ are at least a distance of $|x|^N$ apart. Similarly for given x, the y_γ for which $(x, y_\gamma) \in L_i$ are at least $|x|^N$ apart. It follows that we can find neighborhoods of the form $\{(x, z); |z - h_j(x^{1/n})| < |x|^N\}$ over which X is a graph

$$y = Y(x, z) = g(x^{1/n}) + (z - h_j)^b \phi(x^{1/n}, z - h_j(x^{1/n})) \tag{4}$$
$$z_{\ell,\alpha} = Z_{\ell,\alpha}(x, z) = h_\ell(x^{1/n}) + (z - h_j)\psi_\ell(x^{1/n}, z - h_j(x^{1/n}))$$

These graphs are contained in V_i^k.

We can write

$$\phi(x^{1/n}, z - h_j(x^{1/n})) = \phi(x^{1/n}, 0) + \theta(x^{1/n}, z - h_j(x^{1/n}))$$

$$|(z - h_j(x^{1/n}))^b \phi| \lesssim |z - h_j(x^{1/n})| \text{ if } |z - h_j(x^{1/n})| < \|x\|^N$$

$$|\phi(x^{1/n}, z - h_j(x^{1/n}))| \lesssim \frac{1}{|z - h_j(x^{1/n})|^{b-1}}$$

On X we have $\frac{\partial Y}{\partial z}\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0$. By differentiating repeatedly this equation with respect to z and taking into account that on Y_i^k, $\frac{\partial H}{\partial z}=\cdots=\frac{\partial^{b-1}H}{\partial z^{b-1}}=0$, $\frac{\partial^b H}{\partial z^b}\sim x^{\frac{r}{n}}$ and $\frac{\partial H}{\partial y}\sim x^{\frac{m}{n}}$ (Lemma 2.4) we obtain $\frac{\partial^b Y}{\partial z^b}|_{Y_i^k\backslash(0)}\sim x^{\frac{r-m}{n}}$. Notice also that $\phi(x^{1/n},0)=\frac{1}{b!}\frac{\partial^b Y}{\partial z^b}$ on Y_i^k. Combining these observations we conclude that for $x\neq 0$ we have:

$$(*)\qquad \phi(x^{1/n},0)=x^{\rho/n}u(x^{1/n})$$

for $\rho\in\mathbf{Z}$ and a unit u.

Since a holomorphic function on a disc takes its maximum on the boundary, we get for $x\neq 0$

$$|\phi(x^{1/n},z-h_j(x^{1/n}))|\lesssim\frac{1}{\|x\|^{N(b-1)}}$$

$$|\theta(x^{1/n},z-h_j(x^{1/n}))|\lesssim\frac{1}{\|x\|^{\tilde{N}}}$$

$$|\theta_z(x^{1/n},z-h_j(x^{1/n}))|\lesssim\frac{1}{\|x\|^{\hat{N}}}\ \text{ if }\ |z-h_j(x^{1/n})|<\|x\|^{N+1}$$

$$|\theta(x^{1/n},z-h_j(x^{1/n}))|\lesssim\frac{|z-h_j(x^{1/n})|}{\|x\|^{\hat{N}}}\ \text{ if }\ |z-h_j(x^{1/n})|<\|x\|^{N+1}$$

Suppose $|z-h_j(x^{1/n})|=|x|^{N'}$, $x\neq 0$, $N'\geq N+1$. Then

$$\begin{aligned}|\phi(x^{1/n},z-h_j)|&=|\phi(x^{1/n},0)+\theta(x^{1/n},z-h_j)|\\&\geq|\phi(x^{1/n},0)|-|\theta(x^{1/n},z-h_j)|\end{aligned}$$

We have seen that

$$|\theta(x^{1/n},z-h_j)|\lesssim\frac{|z-h_j(x^{1/n})|}{|x|^{\hat{N}}}=\frac{|x|^{N'}}{|x|^{\hat{N}}}$$

$$|\phi(x^{1/n},0)|\sim|x|^{\rho/n}$$

Hence

$$|\phi(x^{1/n},z-h_j)|\gtrsim|x|^{\rho/n}-\frac{|x|^{N'}}{|x|^{\hat{N}}}$$

We can increase N' such that for $|x|$ small

$$|x|^{\rho/n}-\frac{|x|^{N'}}{|x|^{\hat{N}}}\geq\frac{1}{2}|x|^{\rho/n},$$

hence

$$|\phi(x^{1/n}, z - h_j)| \gtrsim \frac{1}{2}|x|^{\rho/n}$$
$$|y - g(x^{1/n})| = (|x|^{N'})^b |\phi(x^{1/n}, z - h_j(x^{1/n}))| \gtrsim \frac{1}{2}|x|^{bN' + \frac{\rho}{n}} \gtrsim |x|^N$$

if N is big enough such that $N \geq bN' + \frac{\rho}{n}$. The above estimates imply:

LEMMA 2.8. $\Omega_i^k \subset V_i^k$.

3. L^2 Spaces

Here we describe forms and L^2 norms on X^*. We use the notation $\|\cdot\|_{L^2_{x,y,V}}$ to denote the L^2 norm in an open set $V \subset X$ using the standard metric in the (x, y) coordinates. Similarly, we use the notation $\|\cdot\|_{L^2_{X,V}}$ for the L^2 norm in V given by the induced metric on X. Usually the set V is supressed from the notation, and we write $\|\cdot\|_{L^2_{x,y}}$, $\|\cdot\|_{L^2_X}$ respectively. We will also similarly use the norms $\|\cdot\|_{L^2_{x,z,V}}$ and $\|\cdot\|_{L^2_{x,z}}$.

LEMMA 3.1. *Let λ be a (p, q) form on $V \subset X^*$. Then*

$$\|\lambda\|_{L^2_{x,y,V}} \leq \|\lambda\|_{L^2_{X,V}} \quad \text{if } (p, q) = (0, 0), (1, 0) \text{ or } (0, 1)$$
$$\|\lambda\|_{L^2_{x,y,V}} = \|\lambda\|_{L^2_{X,V}} \quad \text{if } (p, q) = (2, 0) \text{ or } (0, 2)$$
$$\|\lambda\|_{L^2_{x,y,V}} \geq \|\lambda\|_{L^2_{X,V}} \quad \text{if } (p, q) = (2, 1), (1, 2) \text{ or } (2, 2)$$

For $(p, q) = (1, 1)$ there is no inequality.

PROOF. The case of functions, i.e. $(0, 0)$ forms is obvious. We prove the Lemma for $(0, 1)$ forms, the others go similarly.

We consider a form defined on a small open set V in X^* where π is biholomorphic, $U := \pi(V)$. The projection $\pi : \mathbb{C}^n \to \mathbb{C}^2$ can be viewed as composition of projections from $\mathbb{C}^{k+1} \to \mathbb{C}^k$. The restriction of π to the tangent plane of X then becomes a composition of projections of the form $P \times \mathbb{C} \to P$ for a 2-dimensional plane P. Hence we may assume that we are in $C^3(x, y, z)$. The set of points on X with vertical tangent over the (x, y)-plane has zero measure, so we ignore these points. Then, either this tangent plane will be parallel to the (x, y)-plane (in which case the norms are equal) or the tangent plane will intersect the (x, y)-plane along a line. Without loss of generality, we may assume that this line is the x-axis.

Let $z = ay$ be the equation for the tangent plane, for $a \neq 0$, $a \in \mathbb{C}$. Set

$$u = x, \qquad v = y\sqrt{1 + |a|^2}$$

Then $\{u, v\}$ form an orthonormal system of coordinates for the tangent plane to X.

Let us assume $\lambda' =: (\pi^{-1})^* \lambda = \alpha d\overline{x} + \beta d\overline{y}$ in $U \subset \mathbb{C}^2_{x,y}$. Then $\lambda = \pi^* \lambda'$ is the corresponding $(0, 1)$-form on $V \subset X^*$.

Then $\lambda = (\alpha \circ \pi) d\overline{u} + (\beta \circ \pi) \frac{d\overline{v}}{\sqrt{1+|a|^2}}$.

$$\begin{aligned}
\|\lambda\|^2_{L^2_{X,V}} = \int_V |\lambda|^2 \, dV_X &= \int_V \left(|\alpha \circ \pi|^2 + \frac{|\beta \circ \pi|^2}{(1 + |a|^2)} \right) du \wedge d\overline{u} \wedge dv \wedge d\overline{v} \\
&= \int_U \left(|\alpha|^2 + \frac{|\beta|^2}{(1 + |a|^2)} \right) (1 + |a|^2) \, dx \, d\overline{x} \, dy \, d\overline{y} \\
&= \int_U (|a|^2 + 1)|\alpha|^2 + |\beta|^2 \, dx \, d\overline{x} \, dy \, d\overline{y} \\
&\geq \int_U (|\alpha|^2 + |\beta|^2) \, dx \, d\overline{x} \, dy \, d\overline{y} =: \|\lambda\|^2_{L^2_{x,y,V}}
\end{aligned}$$

4. Solutions of $\overline{\partial}$

Let λ be a $\overline{\partial}$-closed $(0, 1)$ form on $V_\epsilon := X^* \cap \mathrm{B}(0, \epsilon)$, $\|\lambda\|_{L^2_{X,V_\epsilon}} < \infty$. We want to solve the $\overline{\partial}$-problem, $\overline{\partial} f = \lambda$, $\|f\|_{L^2_{X,V_\delta}} \leq C_{\delta,\epsilon} \|\lambda\|_{L^2_{X,V_\epsilon}}$ for suitable $0 < \delta < \epsilon$. We suppress explicit mention of the neighborhoods which will be shrunk several times in the proof.

4.1. Piecewise solutions

Removing the branching curves Y_i from X, $Y = \cup_i Y_i = \cup_{i,k} Y_i^k$, we get an unbranched, Stein Riemann domain $X' = X \setminus Y$ over $\mathbb{C}^2_{x,y}$. We have the Hörmander solution u_H on X', solving $\overline{\partial} u_H = \lambda$ in the L^2 norm of the x, y coordinate plane. So all norms are calculated with respect to the euclidean metric in the x, y plane. Our first goal is to analyze the singularities of u_H at the curves Y_i^k. Assume first that π is unbranched at $Y_i^k \setminus (0)$.

LEMMA 4.1. *The solution u_H extends in $L^2_{x,y}$ across all those $Y_i^k \setminus (0)$, on which π is unbranched, as a solution to $\overline{\partial} u_H = \lambda$.*

PROOF. Assume that π is unbranched (away from 0) on Y_i^k. Then there is for any point $p \in Y_i^k \setminus (0)$ a small neighborhood W_p and a solution v_p on W_p, $\overline{\partial} v_p = \lambda$ and $v_p \in L^2_{x,y,W_p}$. Then $u_H - v_p$ is holomorphic and in $L^2_{x,y,W_p \setminus Y_i^k}$. Hence $u_H - v_p$ extends holomorphically as a function f_p on W_p. Therefore u_H extends as $v_p + f_p$ across Y_i^k. In other words, the Hörmander solution u_H extends across all Y_i^k where π is unbranched.

4.2. Estimates in $\mathbb{C}^2_{x,y}$

We now consider a pair (i, k) so that π branches on Y_i^k. For this it is convenient to compare u_H with the Hörmander solution U_i^k on Ω_i^k.

More precisely, in Ω_i^k we can use $x, z =: z_j$ as coordinates and we will let U_i^k be the Hörmander solution with respect to these coordinates as described below: Let $p : \mathbb{C}^d \to \mathbb{C}^3(x, y, z)$ and $\pi' : \mathbb{C}^3(x, y, z) \to \mathbb{C}^2(x, y)$ be the projections, so $\pi = \pi' \circ p$. Let $\hat{X}$, $\hat{\Omega}_i^k$ and $\hat{Y}_i^k$ denote the projections in $\mathbb{C}^3(x, y, z)$ of X, Ω_i^k and Y_i^k respectively. Note that p, π and π' are finite covers on $\Omega_i^k \setminus Y_i^k$, $\hat{\Omega}_i^k \setminus \hat{Y}_i^k$. Hence we can write $\lambda = p^*\hat{\lambda}$ for a form $\hat{\lambda}$ on $\hat{\Omega}_i^k \setminus \hat{Y}_i^k$, but the form $\hat{\lambda}$ lives naturally on a finitely sheeted Riemann domain over this set, i.e. the form $\hat{\lambda}$ might be multiple valued. Notice that according to Lemma 2.5, $\hat{\Omega}_i^k$ is a finitely sheeted unbranched cover graph with bounded slope over $\mathbb{C}^2(x, z)$. Hence the L^2-norms in $\hat{X}$ and $\hat{\Omega}_i^k(x, z)$ (or equivalently $\Omega_i^k(x, z)$) are comparable.

We let $\hat{U}_i^k$ denote the Hörmander solution of $\overline{\partial}\hat{U}_i^k = \hat{\lambda}$ on $\hat{\Omega}_i^k$ in the (x, z) coordinates, and set $U_i^k = \hat{U}_i^k \circ p$ on Ω_i^k.

The function $v_i^k := u_H - U_i^k$ is holomorphic on $\Omega_i^k \setminus Y_i^k$ and is in L^2 with respect to the (x, y) coordinates since

$$
\begin{aligned}
\|v_i^k\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} &\leq \|u_H\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} + \|U_i^k\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} \\
&= \|u_H\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} + \|\hat{U}_i^k\|_{L^2_{x,y,\hat{\Omega}_i^k\setminus \hat{Y}_i^k}} \\
&\lesssim \|u_H\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} + \|\hat{U}_i^k\|_{L^2_{\hat{X},\hat{\Omega}_i^k\setminus \hat{Y}_i^k}} \qquad \text{(by Lemma 3.1)} \\
&\sim \|u_H\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} + \|\hat{U}_i^k\|_{L^2_{x,z,\hat{\Omega}_i^k\setminus \hat{Y}_i^k}} \\
&= \|\lambda\|_{L^2_{x,y,\Omega_i^k\setminus Y_i^k}} + \|\hat{\lambda}\|_{L^2_{x,z,\hat{\Omega}_i^k\setminus \hat{Y}_i^k}} \\
&\leq 2\|\lambda\|_{L^2_{X,\Omega_i^k\setminus Y_i^k}} < \infty
\end{aligned}
$$

The function v_i^k has at most a local singularity like $1/(z - h_j(x^{1/n}))^{b-1}$ since it is integrable in the L^2 sense in the x, y direction.

Note that the curve $x = t^n$, $z = h_j(t)$ can also be described by $\psi(x, z) = 0$ for a function vanishing to first order at all regular points of the curve.

We can write, shrinking Ω_i^k, and using $x^{1/n}$ and ψ as local coordinates

$$
v_i^k = \sum_{m \geq 1-b, s \geq -s_0} a_{m,s}^{i,k} x^{s/n} \psi^m(x, z)
$$

on Ω_i^k with $s_0 \in \mathbb{N}$ suitably chosen.

The set $(\psi = 0) \cap X$ in $\mathbf{C}^d$, consists of $Y_i^k \cup \Sigma$ for some finite union Σ of complex curves intersecting Y_i^k at (0) only. Let μ be a holomorphic function in a neighborhood of 0 in $\mathbf{C}^d$, $\mu_{|\Sigma} \equiv 0$ and $\mu_{|Y_i^k}$ vanishing only at 0. Raising μ to a high power if necessary, we may assume that $\frac{\mu}{\psi^{b-1}}$ extends holomorphically across Σ except at 0.

Note that if m is large enough, then $x_{|Y_i^k}^{m+\frac{1}{n}} = t^{mn+1}$ is the restriction of a holomorphic function on the germ, $\mathbf{C}_0^d$, of $\mathbf{C}^d$ at 0. Also we can write $\mu_{|Y_i^k} = x^{s/n} u(x^{\frac{1}{n}})$ for a unit u. Again we may assume $u \equiv 1$. Finally we get:

LEMMA 4.2. *There exists an integer $M \gg 1$ so that if $\sigma = 0, 1, \ldots, n-1$, then the functions $x_{|Y_i^k}^{M+\sigma/n}$ extend to μ_σ on $\mathbf{C}_0^d$ so that they vanish to high order at Σ, in the sense that $\frac{\mu_\sigma}{\psi^{b-1}}$ is holomorphic across Σ except at* 0.

We want to decompose v_i^k into a sum of three terms of a certain type. The first two can be absorbed into u and U_i^k respectively, while the third gives finitely many possible obstructions to solving $\overline{\partial}$ in the first place. We start by giving a preliminary decomposition of v_i^k and then proceed with estimates.

Inductively, we write, using that

$$x^{M+\frac{\sigma}{n}} = \mu_\sigma + \mathcal{O}(\psi) = \mu_\sigma + \psi \sum_{m \geq 0, s \geq -s_1} c_{m,s}^{i,k,\sigma} x^{s/n} \psi^m$$

on Ω_i^k:

$$
\begin{aligned}
(5) \qquad v_i^k &= \sum_{m \geq 1-b, s \geq -s_0} a_{m,s}^{i,k} x^{s/n} \psi^m (x, z) \\
&= \psi^{1-b} \sum_{m \geq 0, s \geq -s_0} \tilde{a}_{m,s}^{i,k} x^{s/n} \psi^m (x, z) \\
&= \psi^{1-b} \sum_{\sigma=0}^{n-1} x^{\sigma/n} \sum_{p \in \mathbf{Z}, m \geq 0} \hat{a}_{\sigma,p,m}^{i,k} x^p \psi^m \\
&= \psi^{1-b} \sum_{\sigma=0}^{n-1} x^{M+\sigma/n} \sum_{p \in \mathbf{Z}, m \geq 0} \hat{b}_{\sigma,p,m}^{i,k} x^p \psi^m \\
&= \psi^{1-b} \sum_{\sigma=0}^{n-1} \mu_\sigma \sum_{p \in \mathbf{Z}, m \geq 0} \hat{b}_{\sigma,p,m}^{i,k} x^p \psi^m \\
&\qquad + \psi^{1-b} \sum_{\sigma=0}^{n-1} (x^{M+\sigma/n} - \mu_\sigma) \sum_{p \in \mathbf{Z}, m \geq 0} \hat{b}_{\sigma,p,m}^{i,k} x^p \psi^m
\end{aligned}
$$

$$
\begin{aligned}
&= \psi^{1-b}\Bigg[\sum_{\sigma=0}^{n-1}\mu_\sigma \sum_{p\in\mathsf{Z}} b^{i,k}_{\sigma,p,1-b}x^p + \mathscr{O}(\psi)\Bigg] \\
&= \psi^{1-b}\Bigg[\sum_{\sigma=0}^{n-1}\mu_\sigma \sum_{p\in\mathsf{Z}} b^{i,k}_{\sigma,p,1-b}x^p\Bigg] \\
&\qquad + \psi^{2-b}\Bigg[\sum_{\sigma=0}^{n-1}x^{M+\sigma/n} \sum_{p\in\mathsf{Z}} b^{i,k}_{\sigma,p,2-b}x^p + \mathscr{O}(\psi)\Bigg] \\
&= \sum_{m=1-b}^{-1}\psi^m \sum_{\sigma=0}^{n-1}\mu_\sigma \sum_{p\in\mathsf{Z}} b^{i,k}_{\sigma,p,m}x^p + \sum_{m\geq 0, s\in\mathsf{Z}} B^{i,k}_{m,s}x^{s/n}\psi^m \\
&=: \sum_{m=1-b}^{-1}\sum_{\sigma=0}^{n-1}\psi^m\mu_\sigma v^{i,k}_{\sigma,m}(x) + v^{i,k}_0(x^{1/n},\psi)
\end{aligned}
$$

Next we go through the steps above giving $L^2_{x,y}$ estimates of the various terms. (Our decomposition is not orthogonal.)

LEMMA 4.3. *We have the estimates*

$$
\begin{aligned}
\|\psi^m\mu_\sigma v^{i,k}_{\sigma,m}\|_{L^2_{x,y,\tilde{\Omega}^k_i}} &\lesssim \|v^k_i\|_{L^2_{x,y,\Omega^k_i}} \\
\|v^{i,k}_0\|_{L^2_{x,y,\tilde{\Omega}^k_i}} &\lesssim \|v^k_i\|_{L^2_{x,y,\Omega^k_i}}
\end{aligned}
$$

where $\tilde{\Omega}^k_i$ are thinner neighborhoods than the Ω^k_i of the same type.

PROOF. We only show that

$$
\|\psi^{1-b}\mu_\sigma v^{i,k}_{\sigma,1-b}\|_{L^2_{x,y,\tilde{\Omega}^k_i}} \lesssim \|v^k_i\|_{L^2_{x,y,\Omega^k_i}}.
$$

The other inequalities follow by finite induction with similar details.

We can write $(x,y) = \Phi(x,\psi)$ for some finite mapping Φ. Then $\mathscr{J}(\Phi) = \frac{\partial y}{\partial\psi} = \frac{1}{(\partial\psi/\partial y)} = \frac{1}{\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}}$.

$$
\begin{aligned}
&\psi = (z-h_j)\tilde{\kappa}(x,z) \\
&\tilde{\kappa}(x,z) \sim x^{t/n}, \qquad t\in\mathsf{Z} \\
&\frac{\partial\psi}{\partial z} = \tilde{\kappa} + (z-h_j)\frac{\partial\tilde{\kappa}}{\partial z} \sim x^{t/n}
\end{aligned}
$$

We have seen in Section 2 that we can find neighborhoods of the form $\{(x,z) : |z - h_j(x^{\frac{1}{n}})| < |x|^N\}$ over which X is a graph. In particular

$$y - g(x^{\frac{1}{n}}) = (z - h_j(x^{\frac{1}{n}}))^b \phi(x^{\frac{1}{n}}, z - h_j(x^{\frac{1}{n}}))$$

Differentiating with respect to y we obtain

$$1 = (z - h_j)^{b-1}[b\phi + (z - h_j)\phi_z]\,\frac{\partial z}{\partial y}$$

Hence,

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{1}{(z-h_j)^{b-1}\,[b\phi + (z-h_j)\phi_z]} \\ &\sim \frac{\tilde{\kappa}^{b-1}}{\psi^{b-1}[bx^{\frac{\varrho}{n}}u + b\theta + (z-h_j)\theta_z]} \\ &\sim \frac{x^{\frac{t(b-1)}{n}}}{\psi^{b-1}x^{\frac{\varrho}{n}}}\end{aligned}$$

Recall that $\theta(x^{1/n}, 0) = 0$, $|\theta_z| \lesssim \frac{1}{|x|^{\hat{N}}}$, if we shrink our neighborhood a bit more.

Thus,

$$\mathscr{J}(\Phi) = \frac{1}{\frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}} \sim \psi^{b-1}x^{\frac{r}{n}}, \qquad r \in \mathsf{Z} \tag{6}$$

REMARK 4.4. We can assume that $\mathscr{J}(\Phi) \sim \psi^{b-1}x^{\frac{r}{n}}$, $\frac{\partial\psi}{\partial z} \sim x^{\frac{t}{n}}$ for a thinner neighborhood $\Omega_i^{k\prime}$.

Hence,

$$\begin{aligned}\left\|\upsilon_i^k\right\|^2_{L^2_{x,y,\Omega_i^k}} &= \left\|\upsilon_i^k \circ \Phi\,\mathscr{J}(\Phi)\right\|^2_{L^2_{x,\psi,\Omega_i^k}} \\ &\gtrsim \left\|\psi^{1-b}\sum_{\sigma=0}^{n-1} x^{M+\frac{\sigma}{n}} \sum_{p\in\mathsf{Z},m\geq 0} \hat{b}^{i,k}_{\sigma,p,m}x^p\psi^m\psi^{b-1}x^{r/n}\right\|^2_{L^2_{x,\psi,\Omega_i^{k\prime}}} \\ &\geq \sum_{\sigma=0}^{n-1}\sum_{m\geq 0}\left\|\psi^{1-b}x^{M+\sigma/n}\sum_{p\in\mathsf{Z}}\hat{b}^{i,k}_{\sigma,p,m}x^p\psi^m\psi^{b-1}x^{r/n}\right\|^2_{L^2_{x,\psi,\tilde{\Omega}_i^k}} \\ &\gtrsim \left\|\psi^{1-b}x^{M+\sigma/n}\sum_{p\in\mathsf{Z}}\hat{b}^{i,k}_{\sigma,p,0}x^p\,\mathscr{J}(\Phi)\right\|^2_{L^2_{x,\psi,\tilde{\Omega}_i^k}}\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \psi^{1-b} x^{M+\frac{\sigma}{n}} \sum_{p\in\mathbf{Z}} b^{i,k}_{\sigma,p,1-b} x^p \Big\|^2_{L^2_{x,y,\tilde{\Omega}^k_i}} \\
&\sim \Big\| \psi^{1-b} \mu_\sigma \sum_{p\in\mathbf{Z}} b^{i,k}_{\sigma,p,1-b} x^p \Big\|^2_{L^2_{x,y,\tilde{\Omega}^k_i}} \\
&= \big\| \psi^{1-b} \mu_\sigma \upsilon^{i,k}_{\sigma,1-b} \big\|^2_{L^2_{x,y,\tilde{\Omega}^k_i}}
\end{aligned}
$$

To pass from the 1st line to the second one we need to shrink the original neighborhood Ω^k_i into $\Omega^{k'}_i$, of the same type as Ω^k_i, to guarantee that $\mathcal{J}(\Phi) \sim \psi^{b-1} x^{\frac{r}{n}}$ holds there. To pass from the 2nd line in the above estimate to the 3rd one we need to replace the neighborhood $\Omega^{k'}_i$ by a smaller one which is circular with respect to the variables (x, ψ). Therefore the different monomials in (x, ψ) will be orthogonal. Recall also that $b^{i,k}_{\sigma,p,1-b} = \hat{b}^{i,k}_{\sigma,p,0}$.

4.3. *Estimates in X near the branch locus*

LEMMA 4.5.

$$
\|\upsilon^{i,k}_0\|_{L^2_{X,\hat{\Omega}^k_i}} \lesssim \|\upsilon^k_i\|_{L^2_{x,y,\Omega^k_i}}
$$

where $\hat{\Omega}^k_i$ are thinner neighborhoods than the $\tilde{\Omega}^k_i$ of the same type.

PROOF. Recall that

$$
\upsilon^{i,k}_0(x^{\frac{1}{n}}, \psi) = \sum_{m\geq 0, s\geq -s_2} B^{i,k}_{m,s} x^{\frac{s}{n}} \psi^m
$$

The argument goes in two steps. First we use the fact that $\upsilon^{i,k}_0 \in L^2_{x,y,\tilde{\Omega}^k_i}$ (Lemma 4.3) to obtain estimates on the coefficients $B^{i,k}_{m,s}$. Secondly, we use these estimates to show that $\upsilon^{i,k}_0 \in L^2_{X,\hat{\Omega}^k_i}$. There are two cases for the second step. The first case is when X is a graph with bounded slope over the (x, z) coordinates. This correspond to $m_\ell \geq r_\ell$, $\ell \neq j$ in Lemma 2.7. The second case is when $m_\ell < r_\ell$. In that case we will use (z, w) as coordinates where $w =: z_s$, $s \neq j$ maximizing $m_s - r_s$.

Step 1. We use again the coordinate change $\Phi(x, \psi)=(x, y)$, with $\mathcal{J}(\Phi) \sim \psi^{b-1} x^{r/n}$ as in (5). So,

$$
\|\upsilon^{i,k}_0\|_{L^2_{x,y,\tilde{\Omega}^k_i}} = \|\upsilon^{i,k}_0 \mathcal{J}(\Phi)\|_{L^2_{x,\psi,\tilde{\Omega}^k_i}} \sim \Big\| \sum_{m\geq 0, s\geq -s_2} B^{i,k}_{m,s} x^{\frac{r+s}{n}} \psi^{b-1+m} \Big\|_{L^2_{x,\psi,\tilde{\Omega}^k_i}}
$$

Hence,

$$\begin{aligned}
&\Big\| \sum_{m\geq 0, s\in \mathsf{Z}} B^{i,k}_{m,s} x^{\frac{r+s}{n}} \psi^{b-1+m} \Big\|^2_{L^2_{x,\psi,\tilde{\Omega}^k_i}} \\
&\qquad = \sum_{m\geq 0, s\in \mathsf{Z}} \big|B^{i,k}_{m,s}\big|^2 \int_{\tilde{\Omega}^k_i} |x|^{\frac{2(r+s)}{n}} |\psi|^{2b-2+2m}\, dx\, d\overline{x}\, d\psi\, d\overline{\psi} \\
&\qquad = \sum_{m\geq 0, s\in \mathsf{Z}} \big|B^{i,k}_{m,s}\big|^2 \frac{\int_{|x|<\eta} |x|^{\frac{2(r+s)}{n}+(2b+2m)N}}{2b+2m} \\
&\qquad = \sum_{m\geq 0, s\in \mathsf{Z}} \frac{|B^{i,k}_{m,s}|^2}{2b+2m} * \frac{\eta^{\frac{2(r+s)}{n}+(2b+2m)N+2}}{\frac{2(r+s)}{n}+(2b+2m)N+2}
\end{aligned}$$

Since the L^2 norms are finite, we must have $\frac{2(r+s)}{n} + (2b+2m)N > -2$ if $B^{i,k}_{m,s} \neq 0$. Since $2b + 2m \geq 2b \geq 2$ we can increase N so that the powers $\frac{2(r+s)}{n} + (2b+2m)N > 0$ whenever $B^{i,k}_{m,s} \neq 0$.

All together we have obtained

(7)
$$\big\|v^k_i\big\|_{L^2_{x,y,\Omega^k_i}} \gtrsim \big\|v^{i,k}_0\big\|_{L^2_{x,y,\tilde{\Omega}^k_i}} \sim \sum_{m\geq 0, s\in \mathsf{Z}} \frac{|B^{i,k}_{m,s}|^2}{2b+2m} * \frac{\eta^{\frac{2(r+s)}{n}+(2b+2m)N+2}}{\frac{2(r+s)}{n}+(2b+2m)N+2}$$

Step 2.

Case (i). We use the coordinate change

(8)
$$\begin{aligned}
&\Psi(x,\psi) = (x,z) \\
&\psi = (z-h_j)\tilde{\kappa}(x,z) \\
&\tilde{\kappa}(x,z) \sim x^{t/n}, \qquad t \in \mathsf{Z} \\
&\frac{\partial \psi}{\partial z} = \tilde{\kappa} + (z-h_j)\frac{\partial \tilde{\kappa}}{\partial z} \sim x^{t/n} \\
&\mathscr{J}(\Psi) = \frac{1}{\frac{\partial \psi}{\partial z}} \sim x^{-t/n}
\end{aligned}$$

(9)
$$\begin{aligned}
\big\|v^{i,k}_0\big\|^2_{L^2_{X,\hat{\Omega}^k_i}} &\sim \big\|v^{i,k}_0\big\|^2_{L^2_{x,z,\hat{\Omega}^k_i}} \\
&= \big\|v^{i,k}_0(x,\psi)\mathscr{J}(\Psi)\big\|^2_{L^2_{x,\psi,\hat{\Omega}^k_i}}
\end{aligned}$$

$$\begin{aligned}
&\sim \left\| \upsilon_0^{i,k}(x,\psi)x^{-t/n} \right\|^2_{L^2_{x,\psi,\hat{\Omega}_i^k}} \\
&= \sum_{m\geq 0, s\in \mathbf{Z}} \left|B^{i,k}_{m,s}\right|^2 \int_{\hat{\Omega}_i^k} |x|^{\frac{2(s-t)}{n}} |\psi|^{2m}\, dx\, d\overline{x}\, d\psi\, d\overline{\psi} \\
&= \sum_{m\geq 0, s\in \mathbf{Z}} \left|B^{i,k}_{m,s}\right|^2 \frac{\int_{|x|<\eta} |x|^{\frac{2(s-t)}{n}+(2m+2)\tilde{N}} dx d\overline{x}}{2m+2} \\
&= \sum_{m\geq 0, s\in \mathbf{Z}} \left|B^{i,k}_{m,s}\right|^2 \frac{\eta^{\frac{2(s-t)}{n}+(2m+2)\tilde{N}+2}}{(2m+2)\left(\frac{2(s-t)}{n}+(2m+2)\tilde{N}+2\right)}
\end{aligned}$$

where we choose $\tilde{N} = 2bN$ and N sufficiently big to guarantee that $N - \frac{r+t}{n} > 0$.

We notice that the right hand side of (7) is of the same order of magnitude as the right hand side of (9) hence,

$$\left\| \upsilon_0^{i,k} \right\|_{L^2_{X,\hat{\Omega}_i^k}} \lesssim \left\| \upsilon_0^{i,k} \right\|_{L^2_{x,y,\tilde{\Omega}_i^k}} \lesssim \left\| v_i^k \right\|_{L^2_{x,y,\Omega_i^k}}.$$

Case (ii). We can write X as a graph (possibly multisheeted but unbranched) over (z_j, z_s), with $s \neq j$ chosen as in the beginning of the proof of Lemma 4.4 in a neighborhood of Y_i^k. We get $x = X(z_j, z_s)$, $y = Y(z_j, z_s)$, $z_m = Z_m(z_j, z_s)$, $z_m \neq z_j, z_s$.

On Y_i^k:

$$\begin{aligned}
z_s &= Z_s(x, z_j) \\
x &= \tilde{Z}_j(z_s, z_j) \\
z_s &= Z_s(\tilde{Z}_j(z_s, z_j), z_j)
\end{aligned}$$

Differentiating the last equation with respect to z_s we obtain:

$$\begin{aligned}
&1 = \frac{\partial Z_s}{\partial x}\frac{\partial \tilde{Z}_j}{\partial z_s} \sim x^{\frac{m_s - r_s}{n}} \frac{\partial \tilde{Z}_j}{\partial z_s} \qquad \text{(Lemma 2.7)} \\
&\frac{\partial \tilde{Z}_j}{\partial z_s} \sim x^{\frac{r_s - m_s}{n}} = o(|x|) \\
&0 = \frac{\partial Z_s}{\partial x}\frac{\partial \tilde{Z}_j}{\partial z_j} + \frac{\partial Z_s}{\partial z_j} \\
&\frac{\partial Z_s}{\partial z_j} = \mathscr{O}(1) \qquad \text{(Lemma 2.7)} \\
&\frac{\partial \tilde{Z}_j}{\partial z_j} = \mathscr{O}\left(x^{\frac{r_s - m_s}{n}}\right) = o(|x|)
\end{aligned}$$

Next, for $\ell \neq s, j$,

$$z_\ell = Z_\ell(x, z_j) = Z_\ell(\tilde{Z}_j(z_s, z_j), z_j)$$

$$\frac{\partial z_\ell}{\partial z_s} = \frac{\partial Z_\ell}{\partial x}\frac{\partial \tilde{Z}_j}{\partial z_s} = \mathcal{O}\big(x^{\frac{m_s - r_s}{n}}\big)\mathcal{O}\big(x^{\frac{r_s - m_s}{n}}\big) = \mathcal{O}(1)$$

$$\frac{\partial z_\ell}{\partial z_j} = \frac{\partial Z_\ell}{\partial x}\frac{\partial \tilde{Z}_j}{\partial z_j} + \frac{\partial Z_\ell}{\partial z_j} = \mathcal{O}\big(x^{\frac{m_s - r_s}{n}}\big)\mathcal{O}\big(x^{\frac{r_s - m_s}{n}}\big) + \mathcal{O}(1) = \mathcal{O}(1)$$

The estimates show that the tangent space of X over the (z_j, z_s) plane has uniformly bounded slope. As before this remains true in a tubular neighborhood.

In this case we use the coordinate change

$$(x, \psi) \xrightarrow{\Psi} (x, z) \xrightarrow{\Lambda} (z_j, z_s)$$

Since $\mathcal{J}(\Lambda) = -\frac{\partial z_s}{\partial x} \sim x^{\frac{m_s - r_s}{n}}$ we have

$$\mathcal{J}(\Lambda \circ \Psi) \sim x^{\frac{m_s - r_s - t}{n}}$$

Using similar arguments as in Case (i) we obtain

$$\|v_0^{i,k}\|_{L^2_{X,\hat{\Omega}_i^k}} \lesssim \|v_i^k\|_{L^2_{x,y,\Omega_i^k}}$$

We have therefore reduced (5) to the case when

$$v_i^k = \sum_{m=1-b}^{-1} \psi^m \sum_{\sigma=0}^{n-1} \mu_\sigma \sum_{p \in \mathsf{Z}} b^{i,k}_{\sigma,p,m} x^p$$

4.4. *Estimates in $\mathsf{C}^2_{x,y}$ on X*

LEMMA 4.6. *There exists an integer $K > 0$ so that if $p < -K$, then $b^{i,k}_{\sigma,p,m} = 0$.*

PROOF. From Lemma 4.3 we know that for each $m = 1 - b, \dots, -1$, $\sigma = 0, \dots, n-1$ the functions

$$\psi^m \mu_\sigma \sum_{p \in \mathsf{Z}} b^{i,k}_{\sigma,p,m} x^p \in L^2_{x,y,\tilde{\Omega}_i^k}$$

We have for ψ, μ_σ estimates of the form $\psi \sim (y-g)^{\frac{1}{b}} x^{s/n}$ and $\mu_\sigma \sim x^{M+\sigma/n}$. Hence,

$$(y-g)^{m/b} x^{M+\frac{ms+\sigma}{n}} \sum_{p \in \mathbb{Z}} b^{i,k}_{\sigma,p,m} x^p \in L^2_{x,y,\tilde{\Omega}^k_i}.$$

Shrinking to a thinner neighborhood $\check{\Omega}^k_i = \{|y-g| < |x|^N, |x| < R\}$ we get:

$$\begin{aligned}\|\upsilon^k_i\|^2_{L^2_{x,y,\tilde{\Omega}^k_i}} &\gtrsim \sum_{p \in \mathbb{Z}} \big|b^{i,k}_{\sigma,p,m}\big|^2 \int_{\check{\Omega}^k_i} |x|^{2p+2M+\frac{2(ms+\sigma)}{n}} |y-g|^{2m/b}\, dx\, d\overline{x}\, dy\, d\overline{y} \\ &= \sum_{p \in \mathbb{Z}} \big|b^{i,k}_{\sigma,p,m}\big|^2 \frac{\int_{|x|<R} |x|^{2p+2M+\frac{2(ms+\sigma)}{n}+(2m/b+2)N}}{2+2m/b}\end{aligned}$$

Hence $2p + 2M + \frac{2(ms+\sigma)}{n} + (2m/b+2)N > -2$ if $b^{i,k}_{\sigma,p,m} \neq 0$.

REMARK 4.7. For later use, we observe that we have obtained the following estimate:

$$\begin{aligned}\|\upsilon^k_i\|^2_{L^2_{x,y,\tilde{\Omega}^k_i}} \\ \gtrsim \sum_{p \in \mathbb{Z}} \big|b^{i,k}_{\sigma,p,m}\big|^2 \frac{R^{2p+2M+\frac{2(ms+\sigma)}{n}+(2m/b+2)N+2}}{(2+2m/b)\big(2p+2M+\frac{2(ms+\sigma)}{n}+(2m/b+2)N+2\big)}\end{aligned} \tag{10}$$

Next, we would like to show that

$$\psi^m \mu_\sigma \sum_{p > \tilde{K}} b^{i,k}_{\sigma,p,m} x^p \in L^2_{x,y,X} \tag{$**$}$$

for a large enough $\tilde{K}$ and for $m = 1-b, \ldots, -1, \sigma = 0, \ldots, n-1$.

LEMMA 4.8. *There exists an integer T so that if $p \in X \setminus \tilde{\Omega}^k_i$ close to the origin, then $|\psi^m \mu_\sigma(p)| < 1/|x|^T$ for all $m = 1-b, \ldots, -1$, $\sigma = 0, \ldots, n-1$.*

PROOF. The μ_σ are holomorphic functions on $\mathbb{C}^d_0$ which vanish along $\Sigma = \{(\psi = 0) \cap X\} \setminus Y^k_i$ to higher order than ψ^{b-1}. Hence, $|\psi^m \mu_\sigma| < 1$ on some tubular neighborhood $\Omega = \{p \in X; d(p, \Sigma) < \|p\|^N\}$. It remains to consider points on X outside $\Omega \cup \tilde{\Omega}^k_i$. But there $|\psi| \geq |x|^\tau$ for some τ. Since μ_σ is bounded, the Lemma follows.

Because of Lemma 4.6 we can put $\tilde{K} = T$ in $(**)$.

LEMMA 4.9. *For each* $m = 1 - b, \ldots, -1$, $\sigma = 0, \ldots, n-1$ *we have*

$$\left\| \psi^m \mu_\sigma \sum_{p>T} b^{i,k}_{\sigma,p,m} x^p \right\|_{L^2_{x,y,X}} \lesssim \| \upsilon^k_i \|_{L^2_{x,y,\Omega^k_i}}$$

PROOF. Lemma 4.3 shows that the estimate holds when the $L^2_{x,y,X}$-norm of the left hand-side is replaced by the $L^2_{x,y,\tilde{\Omega}^k_i}$-norm so it suffices to consider $X \setminus \tilde{\Omega}^k_i$. Hence it suffices to show that $\left\| \sum_{p>T} b^{i,k}_{\sigma,p,m} x^{p-T} \right\|_{L^2_{x,y,P_\eta}} \lesssim \| \upsilon^k_i \|_{L^2_{x,y,\tilde{\Omega}^k_i}}$ where P_η is the bidisc $P_\eta =: \{|x|, |y| < \eta\}$. But, by Remark 4.7

$$\left\| \sum_{p>T} b^{i,k}_{\sigma,p,m} x^{p-T} \right\|^2_{L^2_{x,y,P_\eta}} = \pi^2 \eta^2 \sum_{p>T} \left| b^{i,k}_{\sigma,p,m} \right|^2 \frac{\eta^{2p-2T+2}}{p-T+1} \lesssim \| \upsilon^k_i \|_{L^2_{x,y,\tilde{\Omega}^k_i}}$$

4.5. Estimates on X

We have decomposed υ^k_i in the following way;

$$\begin{aligned} \upsilon^k_i &= \sum_{m=1-b}^{-1} \sum_{\sigma=0}^{n-1} \sum_{-K \le p \le T} b^{i,k}_{\sigma,p,m} \psi^m \mu_\sigma x^p \\ &\qquad + \sum_{m=1-b}^{-1} \sum_{\sigma=0}^{n-1} \sum_{p>T} b^{i,k}_{\sigma,p,m} \psi^m \mu_\sigma x^p + \upsilon^{i,k}_0 \\ &= A^i_k + B^i_k + \upsilon^{i,k}_0 . \end{aligned}$$

Next we modify the Hörmander solution u_H by subtracting from it $\sum_{(i,k)} B^i_k$ over all branching loci Y^k_i (with $b(i,k) > 1$). All together we show:

PROPOSITION 4.10. *There exist finitely many* $\overline{\partial}$*-closed* $(0, 1)$ *forms* $\{\nu_j\}_{j=1}^A \subset L^2_{X^*}$ *so that if* λ *is any* $\overline{\partial}$*-closed form in* $L^2_{X^*}$ *(near* 0*), then there exist unique constants* $\{c_j\}$ *and a function* $u \in L^2_{x,y,X}$ *for which* $\overline{\partial} u = \lambda - \sum c_j \nu_j$*. Moreover,* $u \in L^2_{X^*,\mathrm{loc}}$.

PROOF. Let u_H be the Hörmander solution in $L^2_{x,y,X}$. Then $u_H \in L^2_{X^*,\mathrm{loc}}$ except possibly on the branch-loci Y^k_i with $b(i,k) > 1$. For each such Y^k_i we also considered local Hörmander solutions $U^k_i = U^k_i$ on neighborhoods Ω^k_i. Note that these were found in suitable (x, z_j) coordinates over which X is locally a graph perhaps without uniform bound on the slope as we approach the singularity. Nevertheless, $U^k_i \in L^2_{X,\mathrm{loc}}$.

We found a decomposition of $\upsilon_i^k = u_H - U_i^k$;

$$\begin{aligned}\upsilon_i^k &= \sum_{m=1-b}^{-1}\sum_{\sigma=0}^{n-1}\sum_{-K\le p\le T} b^{i,k}_{\sigma,p,m}\psi^m\mu_\sigma x^p \\ &\qquad + \sum_{m=1-b}^{-1}\sum_{\sigma=0}^{n-1}\sum_{p>T} b^{i,k}_{\sigma,p,m}\psi^m\mu_\sigma x^p + \upsilon_0^{i,k} \\ &= A_k^i + B_k^i + \upsilon_0^{i,k}.\end{aligned}$$

Let $u =: u_H - \sum_{i,k} B_k^i$. By Lemma 4.8 u is also a solution to $\overline{\partial}u = \lambda$ in $L^2_{x,y}$ outside the branching loci. Hence u is in $L^2_{X^*,\mathrm{loc}}$ and solves $\overline{\partial}u = \lambda$ except possibly on the curves $\cup Y_i^k$ with $b(i,k) > 1$.

Next we investigate u near each branching curve Y_i^k. On $\Omega_i^k \setminus Y_i^k$ (after shrinking) we have

$$\begin{aligned}u = u_H - \sum_{j,\ell} B_\ell^j = \upsilon_i^k + U_i^k - \sum_{j,\ell} B_\ell^j &= A_k^i + B_k^i + \upsilon_0^{i,k} + U_i^k - \sum_{(j,\ell)} B_\ell^j \\ &= A_k^i + \Big[U_i^k - \sum_{(j,\ell)\ne(i,k)} B_\ell^j + \upsilon_0^{i,k}\Big] \\ &= A_k^i + \tilde{U}_i^k.\end{aligned}$$

The function $U_i^k \in L^2_{X,\mathrm{loc}}$ on Ω_i^k and solves $\overline{\partial}U_i^k = \lambda$ on Ω_i^k (across Y_i^k). The B_ℓ^j are holomorphic on Ω_i^k since they are only singular on Ω_j^ℓ. Since $\upsilon_0^{i,k}$ is holomorphic on Ω_i^k it follows that $\tilde{U}_i^k \in L^2_{X^*,\mathrm{loc}}$ on Ω_i^k and solves $\overline{\partial}\tilde{U}_i^k = \lambda$ there. The A_k^i are possible obstructions to solving $\overline{\partial}$-globally.

We consider the operator

$$\Phi(\lambda) = \{A_k^i\}_{i,k} \cong \mathbb{C}^A$$

since there are finitely many pairs (i,k). Pick $\{\nu_j\}$ so that $\{\Phi(\nu_j)\}$ is a basis for the range of Φ. Then for a given λ there are constants $\{c_j\}$ so that $\Phi(\hat{\lambda}) = 0$ if $\hat{\lambda} = \lambda - \sum c_j\,\nu_j$. For each such $\hat{\lambda}$ the function $u = \tilde{U}_i^k$ on Ω_i^k so we are done.

Everything that has been done for $x = x_1$, $y = x_2$ can be done for any pair x_i, x_j with $i < j$. We let $H_{x_ix_j}$ denote the closed subset of finite codimension of the $\overline{\partial}$-closed (0, 1) forms in L^2_X for which all the corresponding $\{c_j\}$ vanish. Then the forms from $H_{x_ix_j}$ are solvable with solutions in $L^2_{x_i,x_j}$ which are in $L^2_{X^*,\mathrm{loc}}$. Suppose next that $\lambda \in \cap_{i<j} H_{x_ix_j} =: H$. Hence there exist solutions

$\{u_{x_ix_j}\}$ with $\overline{\partial} u_{x_ix_j} = \lambda$. The differences $u_{x_ix_j} - u_{x_rx_s} =: f_{ijrs}$, are holomorphic on X^* and belong to $L^2_{X^*,\mathrm{loc}}$.

Let $(\tilde{X}, X, \tau)$ be the normalization of X i.e. $\tilde{X}$ is a normal holomorphic variety and $\tau\colon \tilde{X} \to X$ is a finite branched holomorphic covering map of branching order one which is a biholomorphism on X^*. Let U be a neighborhood of 0 in X. We consider the pull back $\tau^* f_{ijrs}$ in $\tau^{-1}(U \setminus 0)$. Then $\tau^* f_{ijrs}$ is holomorhic in $\tau^{-1}(U \setminus 0)$ and belongs to $L^2_{\tilde{X}^*,\mathrm{loc}}$. Since $\tilde{X}$ is a normal variety $\tau^* f_{ijrs}$ extends holomorphically to a full neighborhood of 0. Hence it is bounded in a full neighborhood of 0. So, the f_{ijrs} are bounded in a neighborhood of 0. But then,

$$\begin{aligned}\|u_{x_1x_2}\|_{L^2_X} &\lesssim \sum \|u_{x_1x_2}\|_{L^2_{x_i,x_j}} = \sum \|u_{x_1x_2} - u_{x_ix_j} + u_{x_ix_j}\|_{L^2_{x_i,x_j}} \\ &\lesssim \sum \|u_{x_ix_j}\|_{L^2_{x_i,x_j}} + \sum \|f_{12ij}\|_{L^2_{x_i,x_j}} \\ &\lesssim \|\lambda\|_{L^2_X} + \sum \|f_{12ij}\|_{L^2_{x_i,x_j}} \lesssim \|\lambda\|_{L^2_X}\end{aligned}$$

Hence $u_{x_1x_2}$ is a global solution on X^*.

We conclude:

THEOREM 4.11. *There exists a closed subspace H of finite codimension of the set of $\overline{\partial}$-closed* $(0,1)$ *forms λ in $L^{2,(0,1)}_{X^*\cap \mathsf{B}(0,\epsilon)}$ and a linear operator $T\colon H \to L^{2,(0,0)}_{X^*\cap \mathsf{B}(0,\delta)}$ for some $\delta < \epsilon$ and a constant C so that*

$$\overline{\partial}(T\lambda) = \lambda$$
$$\|T\lambda\|_{L^{2,(0,0)}_{X^*\cap \mathsf{B}(0,\delta)}} \le C\|\lambda\|_{L^{2,(0,1)}_{X^*\cap \mathsf{B}(0,\epsilon)}}$$

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