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A NOTE ON LÁRUSSON-SIGURDSSON'S PAPER

ARMEN EDIGARIAN*

1. Introduction

Let X be a complex manifold. We denote $\mathcal{A}_X$ the family of all mappings $f : \overline{\mathsf{D}} \to X$ which are holomorphic in a neighborhood of the closure $\overline{\mathsf{D}}$ of the unit disc D. A *disc functional* on X is a function $H : \mathcal{A}_X \to \mathsf{R} \cup \{-\infty\}$. The *envelope* of H is the function $E_H : X \to \mathsf{R} \cup \{-\infty\}$ defined by the formula

$$E_H(x) := \inf\{H(f) : f \in \mathcal{A}_X,\ f(0) = x\}, \qquad x \in X.$$

E. Poletsky [5], [6], [7], has shown that certain disc functionals on domains in C^n have plurisubharmonic envelopes. Later, for three classes of disc functionals plurisubharmonicity of envelopes on a class of complex manifolds were shown by F. Lárusson and R. Sigurdsson [4]. The paper is motivated by results from [4].

Let us consider the following two functionals.

Let $\phi : X \to \mathsf{R} \cup \{-\infty\}$ be an upper semi-continuous function. Define the functional $H_1 = H_1^\phi$ by the formula

$$H_1(f) = \frac{1}{2\pi} \int_0^{2\pi} \phi(f(e^{i\theta}))\, d\theta, \qquad f \in \mathcal{A}_X.$$

In [4] this functional is called the *Poisson functional*.

Let v be a plurisubharmonic function on X. We define the functional $H_2 = H_2^v$ as follows. If $f \in \mathcal{A}_X$ and $v \circ f$ is not identically $-\infty$, then

$$H_2(f) = \frac{1}{2\pi} \int_{\mathsf{D}} \big(\log|\cdot|\big)\, \Delta(v \circ f),$$

where Δu is the generalized Laplacian of a subharmonic function u. If $f \in \mathcal{A}_X$ and $v \circ f = -\infty$, then we put $H_2(f) = 0$. In [4] the functional H_2 is called the *Riesz functional*.

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Following [4], we define $\mathscr{P}$ as the class of complex manifolds X for which there exists a finite sequence of complex manifolds and holomorphic mappings

$$X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} \dots \xrightarrow{h_m} X_m = X, \qquad m \geq 0,$$

where X_0 is a domain in a Stein manifold and each h_i, $i = 1, \dots, m$, is either a covering or a finite branched covering. More on the class $\mathscr{P}$ could be found in [4].

For a complex manifold X we denote $\mathrm{PSH}(X)$ the set of all plurisubharmonic functions on X. We assume that the constant function $-\infty$ is plurisubharmonic.

Recall the following result from [4]

THEOREM 1.1. *Let X be a manifold in $\mathscr{P}$. If ϕ is an upper semi-continuous function on X, then $E_{H_1^\phi}$ is plurisubharmonic, and*

$$E_{H_1^\phi} = \sup\{u \in \mathrm{PSH}(X) : u \leq \phi\}.$$

If v is a continuous *plurisubharmonic function on X, then $E_{H_2^v}$ is plurisubharmonic, and*

$$E_{H_2^v} = \sup\{u \in \mathrm{PSH}(X) : u \leq 0, \mathscr{L}(u) \geq \mathscr{L}(v)\},$$

where $\mathscr{L}(u)$ is the Levi form $i\partial\bar{\partial}u$ of u.

In Theorem 1.1 the plurisubharmonicity of H_2 is obtained as a corollary from the plurisubharmonicity of H_1 (see [4]). Actually, this is the reason why in Theorem 1.1 the authors assumed the continuity of v. The main purpose of this note is to show the plurisubharmonicity of H_2 for *any* plurisubharmonic function v.

Let ϕ be a plurisuperharmonic function on a complex manifold X, $\phi \not\equiv +\infty$. We put $H^\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} \phi(f(e^{i\theta}))\,d\theta$ for $f \in \mathscr{A}_X$ such that $\phi \circ f \not\equiv +\infty$ and $H^\phi(f) = +\infty$ for $f \in \mathscr{A}_X$ such that $\phi \circ f \equiv +\infty$. Note that if $\phi \circ f \not\equiv +\infty$, then $\phi \circ f \in L^1(\mathsf{T})$, where T is the unit circle. According to our definition H^ϕ is *not* a disc functional, because it may take the value $+\infty$. Nevertheless, we may consider the envelope E_{H^ϕ} of H^ϕ. It is not difficult to see that $E_{H^\phi} < +\infty$. We have even more. Namely, we have the following results.

THEOREM 1.2. *Let X be a complex manifold and let ϕ be a plurisuperharmonic function on X, $\phi \not\equiv +\infty$. Then $E_{H^\phi} < +\infty$ and E_{H^ϕ} is an upper semicontinuous function on X.*

THEOREM 1.3. *Let X be a manifold in $\mathscr{P}$ and let ϕ be a plurisuperharmonic function on X, $\phi \not\equiv +\infty$. Then E_{H^ϕ} is a plurisubharmonic function and*

$$E_{H^\phi} = \sup\{u \in \mathrm{PSH}(X) : u \le \phi\} \qquad \text{on } X. \tag{1}$$

By the Riesz representation, for a plurisubharmonic function v on a complex manifold X and a holomorphic mapping $f \in \mathscr{A}_X$ such that $v \circ f \not\equiv -\infty$ we have

$$H_2^v(f) = v(f(0)) - \frac{1}{2\pi}\int_0^{2\pi} v(f(e^{i\theta}))\,d\theta.$$

So,

$$H_2^v(f) = v(f(0)) + H_1^{-v}(f) \qquad \text{and} \qquad E_{H_2^v} = v + E_{H_1^{-v}} \tag{2}$$

As a simple corollary of Theorem 1.2 and equation (2) we have immediately the following.

COROLLARY 1.4. *Let X be a complex manifold and let v be a plurisubharmonic function on X. Then $E_{H_2^v}$ is an upper semicontinuous function in X.*

Using results from [4], Theorem 1.3, and equation (2) we have the following.

COROLLARY 1.5. *Let X be a manifold in $\mathscr{P}$ and let v be a plurisubharmonic function on X. Then $E_{H_2^v}$ is a plurisubharmonic function and*

$$E_{H_2^v} = \sup\{u \in \mathrm{PSH}(X) : u \le 0, \mathscr{L}(u) \ge \mathscr{L}(v)\} \qquad \text{on } X.$$

2. Proof of Theorem 1.2

The following two simple results (Lemma 2.1 and Lemma 2.2) play a crucial role in our considerations.

LEMMA 2.1. *Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a plurisuperharmonic function on Ω. Then for any $y_0 \in \Omega$ and any $\epsilon > 0$ there exists $r_0 > 0$ such that for any $y_1 \in \mathsf{B}(y_0, r)$, $r \in (0, r_0)$, we have*

$$\phi(y_0) \ge \frac{1}{b_n r^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\,d\lambda_n(y) - \epsilon,$$

where $\mathsf{B}_n(y_0, r) := \{y \in \mathbb{C}^n : \|y - y_0\| < r\}$, $\mathsf{B}_n := \mathsf{B}_n(0,1)$, $b_n := \lambda_n(\mathsf{B}_n)$, and λ_n is the Lebesgue measure in $\mathbb{C}^n$.

PROOF. Fix $y_0 \in \Omega$ and $\epsilon > 0$. We may assume that $\phi(y_0) \ne +\infty$. Put $\epsilon_1 := \frac{\epsilon}{2^{2n}-1}$. Since ϕ is a lower semicontinuous function, there exists $r_0 > 0$ such that

$$\phi(y) + \epsilon_1 \ge \phi(y_0), \qquad y \in \mathsf{B}_n(y_0, 2r) \subset\subset \Omega.$$

Fix $r \in (0, r_0)$ and $y_1 \in \mathsf{B}_n(y_0, r)$. We have

$$\begin{aligned}
\phi(y_0) &\geq \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_0,2r)} \phi(y)\, d\lambda_n(y) \\
&\geq \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y) \\
&\qquad + \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_0,2r)\setminus \mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y) \\
&\geq \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y) \\
&\qquad + \frac{1}{b_n(2r)^{2n}} (\phi(y_0) - \epsilon_1)(b_n(2r)^{2n} - b_n r^{2n}) \\
&\geq \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y) + (\phi(y_0) - \epsilon_1)\left(1 - \frac{1}{2^{2n}}\right) \\
&= \frac{1}{b_n(2r)^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y) + \phi(y_0) - \phi(y_0)\frac{1}{2^{2n}} - \epsilon_1\left(1 - \frac{1}{2^{2n}}\right).
\end{aligned}$$

So,

$$\phi(y_0) + \epsilon \geq \frac{1}{b_n r^{2n}} \int_{\mathsf{B}_n(y_1,r)} \phi(y)\, d\lambda_n(y).$$

LEMMA 2.2. *Let* $\phi : \mathsf{T} \times \mathsf{B}_n \to [-\infty, +\infty)$ *be an integrable function. Then*

(3)
$$\frac{1}{2\pi b_n} \int_0^{2\pi} \int_{\mathsf{B}_n} \phi(e^{i\theta}, y)\, d\theta\, d\lambda_n(y) = \frac{1}{2\pi b_n} \int_0^{2\pi} \int_{\mathsf{B}_n} \phi(e^{i\theta}, e^{i\theta} y)\, d\theta\, d\lambda_n(y).$$

Therefore, there exists $y_0 \in \mathsf{B}_n$ *such that*

(4)
$$\frac{1}{2\pi b_n} \int_0^{2\pi} \int_{\mathsf{B}_n} \phi(e^{i\theta}, y)\, d\theta\, d\lambda_n(y) \geq \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}, e^{i\theta} y_0)\, d\theta.$$

PROOF. Easily follows from measure theory.

Recall also the following result (see Lemma 2.3 in [4]).

THEOREM 2.3. *Let* X *be a complex manifold and let* $f_0 \in \mathscr{A}_X$. *Then there exist* $r > 1$, *an open neighborhood* V *of* $x_0 = f_0(0)$, *and* $f \in \mathscr{O}(\mathsf{D}_r \times V, X)$ *such that*

(i) $f(z, x_0) = f_0(z)$ *for all* $z \in \mathsf{D}_r$,

(ii) $f(0, x) = x$ *for all* $x \in V$,

where $\mathsf{D}_r := \{z \in \mathsf{C} : |z| < r\}$.

LEMMA 2.4. *Let* $x_0 \in X$, $\beta \in \mathsf{R}$, *and assume that* $E_H(x_0) < \beta$. *Then there exist a neighborhood* V *of* x_0 *in* X, $r > 1$, *and* $f \in \mathscr{O}(\mathsf{D}_r \times \mathsf{B}_n(r) \times V, X)$, *such that* $f(0, 0, x) = f(0, y, x) = x$, $y \in \mathsf{B}_n(r)$, *and*

$$\frac{1}{b_n} \int_{\mathsf{B}_n} H(f(\cdot, y, x))\, d\lambda_n(y) < \beta \qquad \textit{for all} \quad x \in V. \tag{5}$$

PROOF OF LEMMA 2.4. By definition there exists $f_0 \in \mathscr{A}_X$ such that $f_0(0) = x_0$ and $H(f_0) < \beta$. According to Theorem 2.3 there exist $\widetilde{r} > 1$, an open neighborhood $\widetilde{V}$ of x_0, and $\widetilde{f} \in \mathscr{O}(\mathsf{D}_r \times \widetilde{V}, X)$ such that $\widetilde{f}(z, x_0) = f_0(z)$ for all $z \in \mathsf{D}_r$ and $\widetilde{f}(0, x) = x$ for all $x \in \widetilde{V}$.

Let (U, ζ) be a local coordinate centered at x_0. We may assume that $U \subset \widetilde{V}$ and $\zeta : U \to \mathsf{B}_n$ and $\zeta(x_0) = 0$. Consider the function

$$F(w) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\widetilde{f}(e^{i\theta}, \zeta^{-1}(w)))\, d\theta, \qquad w \in \mathsf{B}_n.$$

Note that F is a plurisuperharmonic function in B_n. Fix an $\epsilon > 0$ such that $H(f_0) < \beta - \epsilon$. Then there exists $r > 0$ such that

$$\frac{1}{b_n} \int_{\mathsf{B}_n} F(y_1 + ry)\, d\lambda_n(y) \le F(0) + \epsilon,$$

for any $y_1 \in \mathsf{B}_n(r)$. Put $f(z, y, x) := \widetilde{f}(z, \zeta^{-1}(\zeta(x) + rzy))$ (use here (3)) and $V := \zeta^{-1}(\mathsf{B}_n(r))$.

PROOF OF THEOREM 1.2. Let $x_0 \in X$ be fixed. Let us show that $E_{H^\phi}(x_0) < +\infty$. Assume that (U, ζ) is a local coordinate centered at x_0, i.e. $\zeta(x_0) = 0$. We may assume that $\zeta : U \to \zeta(U) = \mathsf{B}_n(2)$. Take an $x_1 \in U$ such that $\phi(x_1) < +\infty$. Consider the superharmonic function $u := \phi \circ f$, where $f(z) := \zeta^{-1}\left(z \frac{\zeta(x)}{\|\zeta(x)\|}\right)$. Note that $f(0) = x_0$ and $u\big(\|\zeta(x)\|\big) = \phi(x_1) < +\infty$. Hence, $H(f) < +\infty$.

Now, let $\beta > E_H(x_0)$ be fixed. According to Lemma 2.4 there exist a neighborhood V of x_0 in X, $r > 1$, and $f \in \mathscr{O}(\mathsf{D}_r \times \mathsf{B}_n(r) \times V, X)$, such that $f(0, 0, x) = x$ and

$$\frac{1}{b_n} \int_{\mathsf{B}_n} H(f(\cdot, w, x))\, d\lambda_n(y) < \beta \qquad \text{for all} \quad x \in V.$$

Fix $x \in V$. By Lemma 2.2 there exists $y_0 \in \mathsf{B}_n$ such that

$$\frac{1}{b_n} \int_{\mathsf{B}_n} H(f(\,\cdot\,, y, x))\, d\lambda_n(y) \geq H(g),$$

where $g(z) = f(z, zy_0, x)$. It suffices to note that $g(0) = x$.

3. Proof of Theorem 1.3

From [4] it follows that it suffices to prove Theorem 1.3 for domains in C^n. So, in this section we assume that X is a domain in C^n and ϕ is a plurisuperharmonic function on X, $\phi \not\equiv +\infty$. Moreover, the equality (1) follows from the plurisubharmonicity of E_{H^ϕ} (see also [5], [6]).

For the proof of Theorem 1.3 it suffices to show that

$$E_H(h(0)) \leq \frac{1}{2\pi} \int_0^{2\pi} E_H(h(e^{i\theta}))\, d\theta \tag{6}$$

for every $h \in \mathscr{A}_X$ such that $\phi \circ h \not\equiv +\infty$ (since we know that E_H is upper semi-continuous).

The idea of the proof of (6) goes back to E. Poletsky ([5], [6]) and proceeds as follows. It suffices to show that for every $\epsilon > 0$ and $v \in C(X, \mathsf{R})$ with $v \geq E_H$ there exists $g \in \mathscr{A}_X$ such that $g(0) = h(0)$ and

$$H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta}))\, d\theta + \epsilon.$$

For the construction of g, first we show that there exists $r > 1$ and $F \in C^\infty(\mathsf{D}_r \times \mathsf{T}, X)$ such that $F(\cdot, w) \in \mathscr{A}_X$, $F(0, w) = h(w)$ for all $w \in \mathsf{T}$, and

$$\frac{1}{2\pi} \int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{i\theta}))\, d\theta + \epsilon.$$

Next we show that there exist $s \in (1, r)$ and $G \in \mathscr{O}(\mathsf{D}_s \times \mathsf{D}_s, X)$ such that $G(0, w) = h(w)$ for all $w \in \mathsf{D}_s$ and

$$\frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta}))\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta + \epsilon.$$

Finally, we show that there exists $\theta_0 \in [0, 2\pi)$ such that if g is defined by the formula $g(z) = G(e^{i\theta_0} z, z)$ then

$$H(g) \leq \frac{1}{2\pi} \int_0^{2\pi} H(G(\cdot, e^{i\theta}))\, d\theta.$$

As we see, main steps of the proof completely coincide with the proof of plurisubharmonicity of $E_{H_1^\phi}$ for an upper semi-continuous function ϕ (see the discussion before Lemma 2.3 in [4]). But the proofs of these steps turn out to be very technical and complicated.

Let us start with the following simple result, which follows from the measure theory.

LEMMA 3.1. *Let $h \in \mathscr{A}_X$ be such that $\phi \circ h \not\equiv +\infty$ and, therefore, $\phi \circ h \in L^1(\mathsf{T})$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_I \phi \circ h(w)\, d\sigma(w) < \epsilon$$

for any measurable set $I \subset \mathsf{T}$ with $\sigma(I) < \delta$, where σ is the arc length measure on T.

LEMMA 3.2 (cf. Lemma 5.5 in [5], Lemma 2.5 in [4]). *Let $h \in \mathscr{A}_X$ be such that $\phi \circ h \not\equiv +\infty$, $\epsilon > 0$, and $v \in C(X, \mathsf{R})$ with $v \geq E_H$. Then there exist $r > 1$ and $F \in C^\infty(\mathsf{D}_r \times X)$ such that $F(\cdot, w) \in \mathscr{O}(\mathsf{D}_r, X)$, $F(0, w) = h(w)$ for all $w \in \mathsf{T}$, and*

$$\frac{1}{2\pi}\int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} v(h(e^{i\theta}))\, d\theta + \epsilon. \tag{7}$$

PROOF OF LEMMA 3.2. Let $w_0 \in \mathsf{T}$. Put $x_0 = h(w_0)$. From Lemma 2.4 it follows that there exist $r_0 > 1$, $f_0 \in \mathscr{O}(\mathsf{D}_{r_0} \times \mathsf{B}_n(r_0) \times V_0, X)$ such that $f_0(0, 0, x) = x$, $x \in V_0$, and

$$\frac{1}{b_n}\int_{\mathsf{B}_n} H(f_0(\,\cdot\,, y, x))\, d\lambda_n(y) < v(x_0) \qquad \text{for all} \quad x \in V_0.$$

By replacing V_0 by a smaller neighborhood of x_0 we get

$$\frac{1}{b_n}\int_{\mathsf{B}_n} H(f_0(\,\cdot\,, y, x))\, d\lambda_n(y) \leq v(x) + \frac{\epsilon}{4}, \qquad x \in V_0.$$

We can take an open arc $I_0 \subset \mathsf{T}$ containing w_0 such that $h(w) \in V_0$ for all $w \in I_0$. Define $F_0 : \mathsf{D}_{r_0} \times \mathsf{B}_n(r_0) \times I_0 \to X$ by $F_0(z, y, w) = f(z, y, h(w))$. By replacing r_0 by a smaller number in $(1, \infty)$ and I_0 by a smaller open arc containing w_0, we may assume that $F_0(\mathsf{D}_{r_0} \times \mathsf{B}_n(r_0) \times I_0)$ is relatively compact in X.

Using compactness argument, we see that there exist a covering $\{I_\nu\}_{\nu=1}^N$ of T by open arcs, $r_\nu > 1$, $F_\nu \in C^\infty(\mathsf{D}_{r_\nu} \times \mathsf{B}_n(r_\nu) \times I_\nu, X)$ such that

a) $F_\nu(\cdot, \cdot, w) \in \mathscr{O}(\mathsf{D}_{r_\nu} \times \mathsf{B}_n(r_\nu), X)$,

b) $F_\nu(0, 0, w) = h(w)$,

c) $F_\nu(\mathsf{D}_{r_\nu} \times \mathsf{B}_n(r_\nu) \times I_\nu)$ is relatively compact in X,

d)
$$\frac{1}{b_\nu} \int_0^{2\pi} H(F_\nu(\,\cdot\,, y, w))\, d\lambda_n(y) < v(h(w)) + \frac{\epsilon}{4},$$

for $w \in I_\nu$, $\nu = 1, \dots, N$.

Put $r := \min_\nu r_\nu$. Let $M \subset X$ be a compact set such that $\cup_{\nu=1}^N F_\nu(\mathsf{D}_{r_\nu} \times \mathsf{B}_n(r_\nu) \times I_\nu) \subset M$ and let $C > \sup_M |v|$.

By Lemma 3.1 there exists a $\delta > 0$ such that for any measurable set $I \subset \mathsf{T}$ with $\sigma(I) < \delta$ we have
$$\int_I \phi \circ h\, d\sigma < \frac{\epsilon}{4}.$$

There exist a subset $A \subset \{1, \dots, N\}$ and disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A$, such that $\sigma(\mathsf{T} \setminus \cup J_\nu) < \min\{\delta, \frac{\epsilon}{2C}\}$. By possibly removing some arc I_ν from the covering of T, we may assume that $A = \{1, \dots, N\}$. We take disjoint open arcs K_ν such that $J_\nu \subset K_\nu \subset I_\nu$. Now, we take a function $\rho \in C^\infty(\mathsf{T})$ such that

- $0 \le \rho \le 1$,
- $\rho(w) = 1$ for $w \in \cup J_\nu$,
- $\rho(w) = 0$ for $w \in \mathsf{T} \setminus \cup K_\nu$,

Note that
$$\int_{J_\nu} \frac{1}{b_n} \int_{\mathsf{B}_n} H(F_\nu(\cdot, y, w))\, d\sigma(w)\, d\lambda_n(y) \le \int_{J_\nu} v(h(w))\, d\sigma(w) + \frac{\epsilon}{4}\sigma(J_\nu).$$

Hence, there exists $y_\nu \in \mathsf{B}_n$ such that
$$\int_{J_\nu} H(F_\nu(\cdot, y_\nu, w))\, d\sigma(w)\, d\lambda_n(y) \le \int_{J_\nu} v(h(w))\, d\sigma(w) + \frac{\epsilon}{4}\sigma(J_\nu).$$

We define $F : \mathsf{D}_r \times \mathsf{T} \to X$ by
$$F(z, w) = \begin{cases} F_\nu(\rho(w)z, y_\nu, w), & z \in \mathsf{D}_r,\ w \in K_\nu, \\ h(w), & z \in \mathsf{D}_r,\ w \in \mathsf{T} \setminus \cup K_\nu. \end{cases}$$

The choice of ρ ensures that $F \in C^\infty(\mathsf{D}_r \times \mathsf{T}, X)$, $F(\cdot, w) \in \mathcal{O}(\mathsf{D}_r, X)$, and $F(0, w) = h(w)$, $w \in \mathsf{T}$. Since ϕ is a plurisuperharmonic function,

$$H(F(\cdot, w)) \le \phi(F(0, w) = \phi(h(w)), \quad w \in \mathsf{T}. \tag{8}$$

If we combine the inequalities we already have, then we get

$$\frac{1}{2\pi}\int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta \le \sum_\nu \frac{1}{2\pi}\int_{J_\nu} H(F_\nu(\cdot, y_\nu, w))\, d\sigma(w) + \frac{\epsilon}{4}$$
$$\le \sum_\nu \frac{1}{2\pi}\int_{J_\nu} v\circ h\, d\sigma + \frac{\epsilon}{2} \le \frac{1}{2\pi}\int_{\mathsf{T}} v\circ h\, d\sigma + \epsilon,$$

and we have proved (7).

Recall the following result (see Lemma 2.6 in [4], cf. Lemma 5.6 in [5] and Lemma 6 in [1]).

LEMMA 3.3. *Let $r > 1$, $h \in \mathcal{O}(\mathsf{D}_r, X)$, and $F \in C^\infty(\mathsf{D}_r \times \mathsf{T}, X)$, such that $F(\cdot, w) \in \mathcal{O}(\mathsf{D}_r, X)$, and $F(0, w) = h(w)$ for all $w \in \mathsf{T}$. Then there exist $s \in (1, r)$, a natural number j_0, and a sequence $F_j \in \mathcal{O}(\mathsf{D}_s \times A_j, X)$, $j \ge j_0$, where A_j is an open annulus containing T, such that:*

(i) *$F_j \to F$ uniformly on $\mathsf{D}_s \times \mathsf{T}$ as $j \to \infty$,*
(ii) *there is an integer $\ell_j \ge j$ such that the map $(z, w) \mapsto F_j(zw^{\ell_j}, w)$ can be extended to a map $G_j \in \mathcal{O}(\mathsf{D}^2_{s_j}, X)$, where $s_j \in (1, s)$, and*
(iii) *$G_j(0, w) = h(w)$ for all $w \in \mathsf{D}_{s_j}$.*

LEMMA 3.4. *Let h and F satisfy the conditions of Lemma 3.2. Then for every $\epsilon > 0$ there exist $s \in (1, r)$ and $G \in \mathcal{O}(\mathsf{D}_s \times \mathsf{D}_s, X)$ such that $G(0, w) = h(w)$ for all $w \in \mathsf{D}_s$, and*

$$\frac{1}{2\pi}\int_0^{2\pi} H(G(\cdot, e^{i\theta}))\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} H(F(\cdot, e^{i\theta}))\, d\theta + \epsilon.$$

PROOF OF LEMMA 3.4. For any fixed $z, w \in \mathsf{T}$ there exists $r(z, w) > 0$ such that

$$\frac{1}{b_n}\int_{\mathsf{B}_n} \phi(y_1 + ry)\, d\lambda_n(y) \le \phi(F(z, w)) + \frac{\epsilon}{2}$$

for $y_1 \in \mathsf{B}(F(z, w), r)$, $r \in (0, r(z, w))$. Hence, for any fixed $z, w \in \mathsf{T}$ we have

$$\limsup_{m\to\infty}\limsup_{k\to\infty} \frac{1}{b_n}\int_{\mathsf{B}_n} \phi\left(F_k(z, w) + \frac{1}{m}y\right) d\lambda_n(y) \le \phi(F(z, w)) + \frac{\epsilon}{2}.$$

By Fatou's theorem, we have

$$\limsup_{m\to\infty}\limsup_{k\to\infty} \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{b_n}\int_{\mathsf{B}_n} \phi\left(F_k(e^{i\theta}, e^{i\tau}) + \frac{1}{m}y\right) d\lambda_n(y)\right] d\theta\, d\tau$$

$$\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\limsup_{m\to\infty}\limsup_{k\to\infty}\frac{1}{b_n}\int_{\mathsf{B}_n}\phi\left(F_k(e^{i\theta},e^{i\tau})+\frac{1}{m}y\right)d\lambda_n(y)\right]d\theta\,d\tau$$

$$\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(F(e^{i\theta},e^{i\tau}))\,d\theta\,d\tau+\frac{\epsilon}{2}.$$

Hence, there exist m_0 and k_0 such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{1}{b_n}\int_{\mathsf{B}_n}\phi\left(F_{k_0}(e^{i\theta},e^{i\tau})+\frac{1}{m_0}y\right)d\lambda_n(y)\right]d\theta\,d\tau$$

$$\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(F(e^{i\theta},e^{i\tau}))\,d\theta\,d\tau+\epsilon.$$

So, there exists $y_0 \in \mathsf{B}_n$ such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi\left(F_{k_0}(e^{i\theta},e^{i\tau})+\frac{1}{m_0}e^{i\theta}y_0\right)d\theta\,d\tau$$

$$\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(F(e^{i\theta},e^{i\tau}))\,d\theta\,d\tau+\epsilon.$$

Put $G(z,w)=G_{k_0}(z,w)+\frac{1}{m_0}zw^{\ell_{k_0}}y_0$, where G_{k_0} is given by Lemma 3.3 (iii).

LEMMA 3.5. *Let $s>1$ and $G\in\mathscr{O}(\mathsf{D}_s\times\mathsf{D}_s,X)$. Then there exists $g\in\mathscr{O}(\mathsf{D}_s,X)$ such that $g(0)=G(0,0)$ and*

$$H(g)\leq\frac{1}{2\pi}\int_0^{2\pi}H(G(\cdot,e^{i\theta}))\,d\theta.$$

PROOF OF LEMMA 3.5. Note that

$$\frac{1}{2\pi}\int_0^{2\pi}H(G(\cdot,e^{i\theta}))\,d\theta=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{i\tau},e^{i\theta}))\,d\tau\,d\theta$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{i\tau},e^{i\theta+i\tau}))\,d\tau\,d\theta.$$

So, there exists $\theta_0\in[0,2\pi)$ such that

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(G(e^{i\tau},e^{i\theta+i\tau}))\,d\tau\,d\theta\geq\frac{1}{2\pi}\int_0^{2\pi}\phi(G(e^{i\tau},e^{i\theta_0}e^{i\tau}))\,d\tau.$$

Put $g(z)=G(z,e^{i\theta_0}z)$.

REMARK 3.6. In a forthcoming paper [2], the author will continue the study of plurisubharmonicity of the Poisson functional.

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INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
REYMONTA 4/526
30-059 KRAKÓW
POLAND
E-mail: edigaria@im.uj.edu.pl