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A NOTE ON THE LOCAL INJECTIVITY OF WEAKLY DIFFERENTIABLE MAPPINGS HAVING CONSTANT JACOBIAN SIGN

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Abstract

We give topological and analytical conditions in order that some weakly differentiable mappings with constant Jacobian sign to be locally injective, generalizing results from [10], [2], established for Sobolev mappings.

1. Introduction

The central result of this paper is the following theorem.

THEOREM 1. *Let $D \subset \mathsf{R}^n$ be open, $f : D \to \mathsf{R}^n$ continuous so that f satisfies condition (N) and suppose that f has at a.e. point $x \in D$ a quasidifferential L_x with* $\det L_x \geq 0$*. Then f is weakly sense-preserving. If, in addition, f is open, then f is sense-preserving.*

Throughout this paper we shall work with mappings $f : D \to \mathsf{R}^n$ defined on open sets from R^n. Condition (N) requires that f carries sets $A \subset D$ with $\mu_n(A) = 0$ to sets with $\mu_n(f(A)) = 0$, where μ_n denotes the Lebesgue measure from R^n. An $n \times n$ matrix L is called a quasidifferential of f at $x \in D$ if there exists $r_i \to 0$ so that for every $\varepsilon > 0$, there is $i_\varepsilon \in \mathsf{N}$ so that

$$\sup_{\|z-x\|=r_i} \|f(z) - f(x) - L(z-x)\| \leq \varepsilon \cdot r_i, \qquad \text{for} \quad i \geq i_\varepsilon.$$

A map f can have different quasidifferentials at the same point. The concepts weakly ”sense-preserving” and ”sense-preserving” refer to the topological degree, see Section 2 for the definitions. We call a mapping f open if it maps open sets to open sets. We wish to point out that the condition (N) assumption cannot be omitted even when $\det L_x > 0$ a.e. and $f \in \bigcap_{p<n} W^{1,p}(D, \mathsf{R}^n)$, see [5]. On the other hand, if $f \in W^{1,n}(D, \mathsf{R}^n)$ and either $\det L_x > 0$ a.e., $L_x = 0$ a.e. in the set where $\det L_x = 0$, or f is open, then condition (N) is

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automatically satisfied, see [2], [6], [8]. For results related to Theorem 1, see Theorem 1.5 and Theorem 2.4 from [5]. See also [3] for some other results concerning the maps with finite distortion.

Theorem 1 together with its slightly more general version given in Section 3 allow us to prove extensions of certain results of Putten [10] and Fonseca and Gangbo [2] on the local invertibility of Sobolev mappings. For simplicity we only formulate here the following result and refer the reader to Section 2 and Section 3 for further conclusions.

THEOREM 2. *Let $D \subset \mathsf{R}^n$ be open, $f : D \to \mathsf{R}^n$ continuous, open and satisfying condition (N), and suppose that f has at a.e. point $x \in D$ a quasidifferential L_x with* $\det L_x \geq 0$. *Then $B_f \subset \tilde{Z}_f \cup \tilde{S}_f$.*

Here, B_f consists of those points x in D for which f fails to be a local homeomorphism at x, $\tilde{S}_f$ is the set of the points x at which f does not have a quasidifferential, $\tilde{Z}_f$ is the set of points where f has only quasidifferentials L with $\det L = 0$.

2. Proofs of the main results

We begin with a Sard type lemma for quasidifferentiable maps.

LEMMA 1. *Let $D \subset \mathsf{R}^n$ be open and $f : D \to \mathsf{R}^n$ be continuous and open. Then $\mu_n(f(C_f)) = 0$.*

Here C_f is the set of points where f has a quasidifferential L with $\det L = 0$.

PROOF. By usual covering arguments (see for instance [1]), it clearly sufficies to prove that, given $x \in C_f$, there is a sequence of radii r_i tending to zero so that

$$\lim_{i\to\infty} \frac{\mu_n(f(\bar{B}(x, r_i)))}{\mu_n(B(x, r_i))} = 0.$$

For this, let L be a quasidifferential with $\det L = 0$ and (r_i) a corresponding sequence of radii. Because L is not injective, there is an $(n-1)$-dimensional plane T so that $f(x) + L(z - x) \in T$ for all $z \in \mathsf{R}^n$. Now, given $\varepsilon > 0$,

$$\|f(z) - f(x) - L(z - x)\| \leq \varepsilon r_i$$

and

$$\|f(z) - f(x)\| \leq \|f(z) - f(x) - L(z - x)\| + \|L(z - x)\| \leq (1 + \|L\|)r_i,$$

when i is sufficiently large and $\|z - x\| = r_i$. Thus $f(S(x, r_i))$ is contained in a product type set U obtained from an $(n-1)$-ball of radius no more than $(1 + \|L\|)r_i$ in T and an interval of lenght $2\varepsilon r_i$.

As f is open and continuous, also $f(B(x, r_i)) \subset U$. The claim follows because ε was arbitrary.

The topological degree is widely used in the proofs, and we use the notations from [7] and [2]. If $D \subset \mathsf{R}^n$ is open, bounded, $f : \bar{D} \to \mathsf{R}^n$ is continuous and $p \notin f(\partial D)$, we denote by $d(f, D, p)$ the topological degree of f, on D, at the point p. We say that $f : D \to \mathsf{R}^n$ is weakly sense-preserving (sense-preserving) if $d(f, Q, y) \geq 0 \quad (d(f, Q, y) > 0)$ for every domain Q with $\bar{Q} \subset D$.

Proof of Theorem 1. Let $U \subset\subset D$ be open and $x_0 \in U$ such that $f(x_0) \notin f(\partial U)$. Pick an open ball W with $f(x_0) \in W$ and so that $\bar{W} \cap f(\partial U) = \emptyset$. We shall associate to each point $x \in U \setminus \tilde{S}_f$ a sequence of balls $B(x, r_i)$ shrinking to x and corresponding open sets V_i with $f(S(x, r_i)) \subset V_i$ as follows.

If $x \in \tilde{Z}_f$, we use the same argument as in the proof of Lemma 1 to select a sequence (r_i) tending to zero and open sets V_i with $f(\bar{B}(x, r_i)) \subset V_i$ and so that

$$\frac{\mu_n(V_i)}{\mu_n(B(x, r_i))} \leq \frac{\mu_n(W)}{2 \cdot \mu_n(U)}.$$

Then $d(f, B(x, r_i), y) = 0$ for every $y \notin V_i$.

Given $x \in U \setminus (\tilde{Z}_f \cup \tilde{S}_f)$, let L_x be a quasidifferential of f at x with $\det L_x \neq 0$. Let (r_i) be a sequence as in the definition of quasidifferentiability, corresponding to L_x.

We can find as in [12], pages 325–334, a sequence of numbers $\lambda_i > 0$ tending to zero, so that

$$f(S(x, r_i)) \subset V_i := f(x) + (L_x(B(0, (1 + \lambda_i)r_i)) \setminus L_x(B(0, (1 - \lambda_i)r_i))),$$

with

$$d(f, B(x, r_i), y) = 0 \qquad \text{if} \quad y \notin f(x) + L_x(B(0, (1 + \lambda_i)r_i)),$$

and

$$d(f, B(x, r_i), y) = 1 \qquad \text{if} \quad y \in f(x) + L_x(B(0, (1 - \lambda_i)r_i)),$$

and so again

$$\frac{\mu_n(V_i)}{\mu_n(B(x, r_i))} \leq \frac{\mu_n(W)}{2\mu_n(U)}.$$

Our balls in this case will be the balls $B(x, r_i)$.

By the Vitali covering theorem (see [9], page 26), we can select pairwise disjoint balls B_j from the above two collections of balls so that

$$\mu_n\left((U \setminus \tilde{S}_f) \setminus \bigcup_1^{\infty} B_j\right) = 0.$$

Let $K = U \setminus \bigcup_1^{\infty} B_j$. Then $\mu_n(K) = 0$, and hence $\mu_n(f(K)) = 0$, as f satisfies condition (N). Consider the sets V_j associated to the balls B_j. From the definition of the sets V_j we notice that

$$d(f, B(x, r_j), y) \geq 0$$

whenever $y \notin V_j$. Moreover,

$$\sum_1^{\infty} \mu_n(V_j) \leq \frac{\mu_n(W)}{2\mu_n(U)} \cdot \sum_1^{\infty} \mu_n(B_j) \leq \frac{\mu_n(W)}{2}.$$

Thus there is at least one point y with

$$y \in W \setminus \left(f(K) \cup \bigcup_1^{\infty} V_j\right).$$

Notice that $f^{-1}(y) \cap U \subset \bigcup_1^{\infty} B_j$. Let $I = \{j \in \mathsf{N} \mid f^{-1}(y) \cap B_j \neq \emptyset\}$. Then I is finite and we obtain

$$d(f, U, f(x_0)) = d(f, U, y) = d\left(f, \bigcup_{j \in I} B_j, y\right) = \sum_{j \in I} d(f, B_j, y) \geq 0.$$

It follows that f is weakly sense-preserving.

Suppose then that f is open. By Lemma 1, it follows that $\mu_n(f(\tilde{Z}_f)) = 0$. Let again x_0, U, and W be as in the first part of the proof. Let B_0 be an open ball centered at x_0 so that $f(B_0) \subset W$. Because f is open and $\mu_n(f(\tilde{Z}_f \cup \tilde{S}_f)) = 0$, there is a point $x \in B_0$ with $f(x) \in W \setminus (f(\tilde{Z}_f \cup \tilde{S}_f))$. We can find a radius $r > 0$ so that $\bar{B}(x, r) \subset B_0$, $f(x) \notin f(S(x, r))$ and $d(f, B(x, r), f(x)) = 1$. By the first part of the proof, we conclude that

$$\begin{aligned} d(f, U, f(x_0)) &= d(f, U, f(x)) \\ &= d(f, U \setminus \bar{B}(x, r), f(x)) + d(f, B(x, r), f(x)) \geq 1, \end{aligned}$$

as desired.

Our next result deals with local invertibility.

THEOREM 3. *Let $D \subset \mathsf{R}^n$ be open, $f : D \to \mathsf{R}^n$ continuous, nonsingular and satisfying condition (N) and $\mu_n(f(\tilde{Z}_f)) = 0$. Suppose that f has at a.e. point $x \in D$ a quasidifferential L_x at x with* $\det L_x \geq 0$. *Then, for every $x \notin \tilde{Z}_f \cup \tilde{S}_f$, there is $r_x > 0$ and a neighborhood Q_x of x so that*

$$f|Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f) : Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f) \to B(f(x), r_x) \setminus f((\tilde{Z}_f \cup \tilde{S}_f) \cap Q_x)$$

is a homeomorphism and for every $y \in B(f(x), r_x)$, $f^{-1}(y) \cap Q_x$ is connected.

Here, a map $f : D \to \mathsf{R}^n$ is nonsingular if $\operatorname{int} f(U) \neq \emptyset$ for every open nonempty set $U \subset D$. We denote by $N(y, f, A)$ the number of elements of the set $A \cap f^{-1}(y)$.

PROOF OF THEOREM 3. Since f is nonsingular and $\mu_n(f(\tilde{Z}_f \cup \tilde{S}_f)) = 0$, we see that $\operatorname{int} f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f)) = \emptyset$, hence $\operatorname{int}(\tilde{Z}_f \cup \tilde{S}_f) = \emptyset$. Let $x \notin \tilde{Z}_f \cup \tilde{S}_f$. Then f has a quasidifferential L_x at x with $\det L_x > 0$, and we find $\delta_x > 0$ so that $f(x) \notin f(S(x, \delta_x))$ and $d(f, B(x, \delta_x), f(x)) = 1$. Let $r_x > 0$ be so that $\bar{B}(f(x), r_x) \cap f(S(x, \delta_x)) = \emptyset$ and let Q_x be the component of $f^{-1}(B(f(x), r_x))$ containing x. Then $Q_x \subset B(x, \delta_x)$, Q_x is a neighbourhood of x and

$$d(f, B(x, \delta_x), y) = d(f, B(x, \delta_x), f(x)) = 1 \quad \text{for every } \; y \in B(f(x), r_x),$$

and hence $B(f(x), r_x) \subset f(B(x, \delta_x))$. Let $y \in B(f(x), r_x) \setminus f(\tilde{Z}_f \cup \tilde{S}_f)$ and suppose that there exist $x_1, x_2 \in B(x, \delta_x)$, $x_1 \neq x_2$ so that $f(x_1) = f(x_2) = y$. Then $x_1, x_2 \notin \tilde{Z}_f \cup \tilde{S}_f$ and we can find $r_1, r_2 > 0$ so that

$$\bar{B}(x_1, r_1) \cup \bar{B}(x_2, r_2) \subset B(x, \delta_x), \qquad \bar{B}(x_1, r_1) \cap \bar{B}(x_2, r_2) = \emptyset,$$

$$y \notin f(S(x_1, r_1) \cup S(x_2, r_2)) \quad \text{and} \quad d(f, B(x_1, r_1), y) = d(f, B(x_2, r_2), y) = 1.$$

From Theorem 1 we see that f is weakly sense-preserving; hence

$$d(f, B(x, \delta_x) \setminus (\bar{B}(x_1, r_1) \cup \bar{B}(x_2, r_2)), y) \geq 0,$$

and we obtain that

$$1 = d(f, B(x, \delta_x), y) = d(f, B(x, \delta_x) \setminus (\bar{B}(x_1, r_1) \cup \bar{B}(x_2, r_2)), y) + d(f, B(x_1, r_1), y) + d(f, B(x_2, r_2), y) \geq 2$$

which represents a contradiction. We proved that f is injective on

$$(B(x, \delta_x) \cap f^{-1}(B(f(x), r_x))) \setminus f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f)),$$

and hence f is injective on $Q_x \setminus f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f))$.

Let now $z \in Q_x$ be so that f is open at z and suppose that there is $w \neq z$, $w \in B(x, \delta_x)$ so that $f(z) = f(w)$. We can find U_1, U_2 open, disjoint so that $z \in U_1$, $w \in U_2$, $\bar{U}_1 \cup \bar{U}_2 \subset B(x, \delta_x)$, and using the openness of f at z and the continuity of f at w, we can assume that

$$f(U_2) \subset f(U_1) \subset B(f(x), r_x).$$

Since $\operatorname{int} f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f)) = \emptyset$, we can find a point $a \in U_2 \setminus f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f))$, hence we can find a point $b \in U_1 \setminus f^{-1}(f(\tilde{Z}_f \cup \tilde{S}_f))$ so that $f(a) = f(b)$, which represents a contradiction, since we proved that f is injective on

$$B(x, \delta_x) \cap f^{-1}(B(f(x), r_x) \setminus f(\tilde{Z}_f \cup \tilde{S}_f)).$$

We proved that

$$(1) \qquad N(f(z), f, B(x, \delta_x)) = 1 \qquad \text{if} \quad z \in Q_x \quad \text{and} \quad f \text{ is open at } z.$$

Since f is open at every point $z \in Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f)$, it results that

$$f^{-1}(f(z)) \cap B(x, \delta_x) = \{z\} \qquad \text{for every} \quad z \in Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f).$$

We see that f is injective on $Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f)$,

$$\begin{aligned} Q_x \cap (\tilde{Z}_f \cup \tilde{S}_f) &= Q_x \cap f^{-1} f(Q_x \cap (\tilde{Z}_f \cup \tilde{S}_f)), \; Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f) \\ &= Q_x \cap f^{-1}(f(Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f))), \end{aligned}$$

and

$$f|Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f) : Q_x \setminus (\tilde{Z}_f \cup \tilde{S}_f) \to B(f(x), r_x) \setminus f(Q_x \cap (\tilde{Z}_f \cup \tilde{S}_f))$$

is a homeomorphism.

Now, let $y \in B(f(x), r_x)$ and let A_1, A_2 be two different components of $f^{-1}(y) \cap B(x, \delta_x)$. Then A_1 and A_2 are compact and $A_1 \cup A_2 \subset B(x, \delta_x)$.

From Theorem 1, f is sense-preserving, hence it is quasiopen (Lemma 5.5, page 147 in [11]). Let Q_1, Q_2 be open so that $A_1 \subset Q_1$, $A_2 \subset Q_2$ and $\bar{Q}_1 \cup \bar{Q}_2 \subset B(z, \delta_x)$. Since $f(A_1) = f(A_2) = \{y\}$ and f is quasiopen, we can find a point $w \in f(Q_1) \cap f(Q_2) \cap B(f(x), r_x)$, and since f is sense-preserving, we have that

$$d(f, Q_1, w) \geq 1, \quad d(f, Q_2, w) \geq 1, \quad d(f, B(x, \delta_x) \setminus (\bar{Q}_1 \cup \bar{Q}_2), w) \geq 0.$$

We obtain that

$$\begin{aligned} 1 &= d(f, B(x, \delta_x), f(x)) = d(f, B(x, \delta_x), w) \\ &= d(f, B(x, \delta_x) \setminus (\bar{Q}_1 \cup \bar{Q}_2), w) + d(f, Q_1, w) + d(f, Q_2, w) \geq 2, \end{aligned}$$

which represents a contradiction.

We therefore proved that $f^{-1}(y) \cap B(f(x), r_x)$ is connected for every $y \in B(f(x), r_x)$.

PROOF OF THEOREM 2. We apply (1) and Lemma 1 to see that $B_f \subset \tilde{Z}_f \cup \tilde{S}_f$.

3. Further results

An $n \times n$ matrix L is called the approximate differential of f at x if the maps $f_h : B(0,1) \to \mathsf{R}^n$, defined by $f_h(y) = \frac{f(x+hy)-f(x)}{h}$ for $y \in B(0,1)$ and $h > 0$, converge to L in measure on $B(0,1)$. Then L is unique, and such a matrix L is called the weak differential of f at x if in addition L is a quasidifferential of f at x. We also say that f is weakly differentiable at x and we denote $J_f(x) = \det L$. We put $Z_f = \{x \in D \mid f$ is weakly differentiable at x and $J_f(x) = 0\}$ and $S_f = \{x \in D \mid f$ is not weakly differentiable at $x\}$. The version of Theorem 1 for weakly differentiable mappings is:

THEOREM 4. *Let $D \subset \mathsf{R}^n$ be open, $f : D \to \mathsf{R}^n$ continuous, nonsingular and satisfying condition (N), weakly differentiable a.e. so that $J_f(x) \geq 0$ a.e. in D. Then f is sense-preserving.*

PROOF. We see from Theorem 1 that f is weakly sense-preserving. We remark that in the proof of the sense-preserveness of the map f from Theorem 1, we used the openness of f only to state that f is nonsingular and that $\mu_n(f(\tilde{Z}_f)) = 0$. Since we see from [2], Theorem 5.6, page 110, that $\mu_n(f(Z_f)) = 0$, without the openness assumption, we argue as is Theorem 1 to find that f is sense-preserving.

Using Sard's lemma for weakly differentiable mappings from [2], Theorem 5.6, page 110, instead of our version of Sard's lemma from Lemma 1, and replacing the set $\tilde{Z}_f$ by Z_f and the set $\tilde{S}_f$ by S_f, we can formulate Theorem 2 and Theorem 3 in this setting. We give only the version of Theorem 3.

THEOREM 5. *Let $D \subset \mathsf{R}^n$ be open, $f : D \to \mathsf{R}^n$ continuous, open, satisfying condition (N), weakly differentiable a.e., so that $J_f(x) \geq 0$, a.e. in D. Then $B_f \subset Z_f \cup S_f$.*

We formulate now some simple conditions for nonsingularity, which can be used to obtain new versions of the theorems where it appears this property. First, a map $f : D \to \mathsf{R}^n$ is called a light map if $\dim f^{-1}(y) \leq 0$ for every $y \in \mathsf{R}^n$. We know from [4], page 92, that every continuous, light map is nonsingular. Also, if $f : D \to \mathsf{R}^n$ is continuous, so that $\operatorname{int}(\tilde{Z}_f \cup \tilde{S}_f) = \emptyset$ or $\operatorname{int}(Z_f \cup S_f) = \emptyset$, then f is nonsingular.

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