MATH. SCAND. 93 (2003), 99–108

# ON THE TOPOLOGY OF SASAKIAN MANIFOLDS

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## Abstract

The notion of $q$-bisectional curvature of a Sasakian manifold $M$ is defined. It is proved that if $M$ has lower bounded $q$-bisectional curvature and contains a compact invariant submanifold tangent to the structure vector field then $M$ is compact. Myers and Frankel type theorems for Sasakian manifolds with lower bounded and positive $q$-bisectional curvature, respectively, are also given.

## 1. Introduction

Let $M$ be a complete connected Riemannian manifold. An important theorem, proved by Myers [11], asserts that if $M$ has sectional curvature $\geq k_0 > 0$ (or more generally, if all the eigenvalues of the Ricci tensor are $\geq (\dim M-1)k_0 > 0$) then $M$ is compact, its diameter is $\leq \pi/\sqrt{k_0}$ and has finite fundamental group.

Another remarkable theorem, due to Frankel [2], asserts that if $M$ has positive sectional curvature, then any two compact totally geodesic submanifolds $N$, $P$ of $M$ and such that $\dim N + \dim P \geq \dim M$, must intersect. He also proved that in the case of a complete connected Kähler manifold with positive sectional curvature the same conclusion holds if we replace the hypothesis *totally geodesic* by *analytic*. Such results were proved by Goldberg and Kobayashi [5] for Kähler manifolds with positive bisectional curvature. Frankel's theorems were extended by Gray [6] to nearly Kähler manifolds, by Marchiafava [10] to quaternionic Kähler manifolds and by Ornea [12] to locally conformal Kähler manifolds and to Sasakian manifolds in the case when the submanifolds $N$ and $P$ are invariant and tangent to the structure vector field of $M$.

Recently, Kenmotsu and Xia [7], [8] proved Frankel type theorems for Kähler manifolds in the more general case when $M$ has either partially positive sectional curvature or partially positive bisectional curvature.

Our purpose is to give Myers and Frankel type theorems for a Sasakian manifold $M$ under weaker conditions on the curvature of the manifold. The second section of this paper is devoted to the notion of $q$-bisectional curvature for such a manifold. We remark that if $q = 1$ then it is exactly the $F$-bisectional

Received June 18, 2001.

curvature of the manifold [13], [14] and so the class of Sasakian manifolds with positive (or lower bounded) $q$-bisectional curvature is richer than that of Sasakian manifolds with positive (or lower bounded) $F$-bisectional curvature. We prove that if $M$ has lower bounded $q$-bisectional curvature and contains a compact invariant submanifold tangent to the structure vector field of $M$ then $M$ must be compact (Theorem 2.3 and Corollary 2.4). In section 3 we prove a Myers type theorem for Sasakian manifolds with lower bounded $q$-bisectional curvature (Theorem 3.2). The last section is devoted to the proof of a Frankel type theorem for Sasakian manifolds with positive $q$-bisectional curvature and we obtain an extension of Ornea's Theorem 2, [12].

## 2. $q$-bisectional curvature

Let $M$ be a Sasakian manifold and denote by $F, \xi, \eta, g$ its fundamental tensor fields. For any vector fields $X, Y \in \mathscr{X}(M)$, orthogonal to $\xi$, the $F$-bisectional curvature $\mathscr{H}$ of $M$ is defined by

$$\mathscr{H}(X,Y) = \frac{\mathscr{R}(X,FX,Y,FY)}{\|X\|^2\|Y\|^2}, \tag{1}$$

where $\mathscr{R}$ is the Riemann-Christoffel curvature tensor of $M$. Then for any $X', Y' \in \mathscr{X}(M)$ such that

$$\operatorname{span}_{\mathsf{R}}\{X', FX'\} = \operatorname{span}_{\mathsf{R}}\{X, FX\}, \qquad \operatorname{span}_{\mathsf{R}}\{Y', FY'\} = \operatorname{span}_{\mathsf{R}}\{Y, FY\},$$

we obtain

$$\mathscr{H}\left(X', Y'\right) = \mathscr{H}\left(X, Y\right).$$

Moreover

PROPOSITION 2.1. *For any $X, Y \in \mathscr{X}(M)$, orthogonal to $\xi$, we have*

$$\mathscr{H}(X,Y) = \frac{1}{\|X\|^2\|Y\|^2}\big\{\mathscr{R}(X,Y,X,Y) + \mathscr{R}(X,FY,X,FY) + 2\left[g^2(X,Y) - \|X\|^2\|Y\|^2 + g^2(X,FY)\right]\big\}.$$

PROOF. By the Lemma, pg. 93, [1], on a contact manifold we have

$$\mathscr{R}(FY,Y,X,FX) = \mathscr{R}(Y,X,X,Y) + \mathscr{R}(FY,X,X,FY) - 2\mathscr{P}(X,Y,X,FY), \tag{2}$$

where

$$\mathscr{P}(X,Y,Z,U) = d\eta(X,Z)\,g(Y,U) - d\eta(X,U)\,g(Y,Z) - d\eta(Y,Z)\,g(X,U) + d\eta(Y,U)\,g(X,Z).$$

But $M$ is Sasakian, hence $d\eta = \Omega$, with $\Omega(X, Y) = g(X, FY)$, and then

$$\mathscr{P}(X, Y, X, FY) = g^2(X, Y) + g^2(X, FY) - \|X\|^2\|Y\|^2. \tag{3}$$

Now, from (1) and taking into account (2), (3), we obtain the announced formula.

Let $T_x M$ be the tangent space to the Sasakian manifold $M$ at the point $x$ and we denote by $\mathscr{S} = \{X_1, \dots, X_q\} \subset T_x M$ an orthonormal system of vectors orthogonal to $\xi$. Then the vectors of the system $F\mathscr{S} = \{FX_1, \dots, FX_q\}$ are orthogonal to $\xi$. $\mathscr{S}$ is called an *$F$-orthonormal $q$-system* of tangent vectors at $x$ if $\mathscr{S} \cup F\mathscr{S}$ is orthonormal. We remark that $q \leq \left[\frac{1}{2}\dim M\right]$ and for such a system $\mathscr{S}$ and for any tangent vector $X \in T_x M$, orthogonal to $\xi$, we can consider the scalar

$$\mathscr{H}_q(X, \mathscr{S}) = \sum_{i=1}^{q} \mathscr{H}(X, X_i).$$

Now, taking into account Proposition 2.1, we obtain

PROPOSITION 2.2. *Let $X$ be a unit tangent vector at $x \in M$ and $\mathscr{S}$ be an $F$-orthonormal $q$-system at $x$. If $\mathscr{S}' \subset T_x M$ is an orthonormal system such that* $\operatorname{span}_{\mathsf{R}} \mathscr{S}' = \operatorname{span}_{\mathsf{R}} \mathscr{S}$ *then:*

a) *$\mathscr{S}'$ is an $F$-orthonormal $q$-system*

b) $\mathscr{H}_q\left(X, \mathscr{S}'\right) = \mathscr{H}_q(X, \mathscr{S})$.

From Proposition 2.2 it follows that $\mathscr{H}_q(X, \mathscr{S})$ is depending only on the subspace of $T_x M$ spanned by $\mathscr{S}$, but not on the $F$-orthonormal $q$-system $\mathscr{S}$. We call $\mathscr{H}_q(X, \mathscr{S})$ the *$q$-bisectional curvature* of $M$ at the point $x$ and we remark that for $q = 1$ it is exactly the $F$-bisectional curvature of $M$.

In the following of this section we shall construct $F$-orthonormal systems and these will be used in order to give information about the topology of the manifold.

Let $N$ be a $2r$-dimensional ($r \geq 1$) submanifold of the complete connected Sasakian manifold $M$ and we assume $N$ to be invariant (i.e. $FT_x N \subseteq T_x N$ for any $x \in N$) and it is tangent to $\xi$. If $\{e_1, \dots, e_r, Fe_1, \dots, Fe_r, \xi\}$ is an adapted basis of $T_x N$ then $\mathscr{B} = \{e_1, \dots, e_r\}$ is, obviously, an $F$-orthonormal $r$-system. Moreover, if $\gamma : [0, \infty) \to M$ is the geodesic starting from $x$ and orthogonal to $N$ at $x$ then the system $\tilde{\mathscr{B}}$, obtained from $\mathscr{B}$ by parallel translation along $\gamma$ is an $F$-orthogonal $r$-system, too. Indeed, if $E_i$ is obtained by parallel translation of $e_i$ along $\gamma$ then we have

$$\nabla_{\gamma'} E_i = 0, \qquad E_i(\gamma(t)) = E_i(t) = e_i,$$

hence $E_i$ is normal to $\gamma$. Similar equalities hold for the vector fields $\tilde{E}_i$, obtained from $Fe_i$ by parallel translation along $\gamma$, and therefore $\tilde{E}_i$ are normal to $\gamma$, too. But by using the well-known equality, true on a Sasakian manifold, ([1], Theorem, pg. 73)

$$(\nabla_X F)\, Y = g(X,Y)\xi - \eta(Y)X,$$

we have

$$\nabla_{\gamma'}(FE_i) = 0,$$

and because $FE_i(\gamma(t)) = Fe_i$, it follows $\tilde{E}_i = FE_i$, which proves that $\tilde{\mathscr{B}}$ is an $F$-orthonormal $r$-system.

THEOREM 2.3. *Let $M$ be a complete connected Sasakian manifold of dimension $2n+1 \geq 5$. If for some $r \geq 1$ there exists a $2r+1$-dimensional compact invariant submanifold $N$, tangent to $\xi$ and such that*

$$\liminf_{t\to\infty} \int_0^t \mathscr{H}_r\big(\gamma'(s), \tilde{\mathscr{B}}\big)\, ds > 0, \tag{4}$$

*for any $x \in N$ and for any $F$-orthonormal $r$-system $\mathscr{B}$ of $T_xM$ then the manifold $M$ is compact.*

For the proof of this theorem we drew one's inspiration from [8].

PROOF. If $M$ is not compact then, by Theorem 1 of [3], there exists $x \in N$ and a geodesic $\gamma : [0,\infty) \to M$, orthogonal to $N$ at $x$, and such that

$$\text{distance}(\gamma(t), N) = \text{length}\, \gamma_{|[0,t]},$$

hence $\gamma$ has no conjugate points. By putting $\mathscr{H}(t) = \frac{1}{2r}\mathscr{H}_r(\gamma'(t), \tilde{\mathscr{B}})$ and taking into account (4), from [15] (see also [8]) it follows that the scalar Jacobi equation

$$f'' + \mathscr{H}(t) f = 0$$

has a solution $\Phi : [0,\infty) \to \mathsf{R}$, satisfying the conditions $\Phi(0) = 1$, $\Phi'(0) = 0$ and $\Phi(t_0) = 0$ for some $t_0 > 0$.

In the following we shall use the well-known expressions of the index form $I$ of the geodesic $\gamma$ along $\gamma_{|[a,b]}$ (see [9], t. II, Theorems 5.4 and 5.5, pg. 81)

$$\begin{aligned} I_a^b(X,Y) &= \int_a^b \big[g(X',Y') - \mathscr{R}(X,\gamma',Y,\gamma')\big]\, dt \\ &= g(X',Y)(b) - g(X',Y)(a) \\ &\qquad - \int_a^b \big[g(X'',Y) + \mathscr{R}(X,\gamma',Y,\gamma')\big]\, dt \end{aligned} \tag{5}$$

for all vector fields $X$ and $Y$ along $\gamma$. Indeed, we consider it for the vector fields $X_i$, $Y_i$, defined along $\gamma_{|[0,t_0]}$, by

$$X_i(t) = \Phi(t)E_i(t), \qquad Y_i(t) = \Phi(t)\tilde{E}_i(t). \tag{6}$$

They are tangent to $N$ at $\gamma(0)$ and $X_i(t_0) = Y_i(t_0) = 0$. Moreover we have

$$X_i' = \Phi' E_i, \qquad Y_i' = \Phi' \tilde{E}_i, \tag{7}$$

hence $X_i'(0) = Y_i'(0) = 0$ and $X_i'' = \Phi'' E_i$, $Y_i'' = \Phi'' \tilde{E}_i$. Then we have

$$g(X_i'', X_i) + \mathscr{R}(X_i, \gamma', X_i, \gamma') = \Phi''\Phi + \Phi^2 \mathscr{R}(E_i, \gamma', E_i, \gamma'), \tag{8}$$

$$g(Y_i'', Y_i) + \mathscr{R}(Y_i, \gamma', Y_i, \gamma') = \Phi''\Phi + \Phi^2 \mathscr{R}(\tilde{E}_i, \gamma', \tilde{E}_i, \gamma'). \tag{9}$$

By using Gauss formula we have

$$g(X', Y) = g(h(X, Y), \gamma')$$

for all $X$, $Y$ tangent to $N$ and normal to $\gamma$. But $N$ is invariant and tangent to $\xi$ and then we have

$$h(Fe_i, Fe_i) = -h(e_i, e_i). \tag{10}$$

Now, from (5), (8), (9) and (10) we give

$$\begin{aligned}\sum_{i=1}^{r}\left[I_0^{t_0}(X_i, X_i) + I_0^{t_0}(Y_i, Y_i)\right] &= -2r\int_0^{t_0} \Phi[\Phi'' + \mathscr{H}(t)\Phi]\,dt \\ &\quad - 2r\int_0^{t_0} \|\gamma'\|^2\,dt \\ &= -2r\int_0^{t_0} \|\gamma'\|^2\,dt < 0,\end{aligned}$$

hence $I_0^{t_0}(X_i, X_i) < 0$ or $I_0^{t_0}(Y_i, Y_i) < 0$ for some $i \in \{1, 2, \ldots, r\}$ and therefore $\gamma_{|[0,t_0]}$ has a conjugate point. But this contradicts the hypothesis that $\gamma$ has no conjugate points.

We say that $M$ has *lower bounded $q$-bisectional curvature* at the point $x \in M$ if there exists $k_0 \in \mathsf{R}$ such that $\mathscr{H}_q(X, \mathscr{S}) \geq k_0$ for any unit tangent vector $X \in T_x M$ and for any $F$-orthonormal $q$-system $\mathscr{S}$. If $k_0 = 0$ then we say that $M$ has *nonnegative $q$-bisectional curvature* and taking into account (1) we remark that if $M$ has nonnegative $F$-bisectional curvature then its $q$-bisectional curvature is also nonnegative for any $q \leq \left[\frac{1}{2}\dim M\right]$. Hence the family of Sasakian manifolds with nonnegative $q$-bisectional curvature is richer than the one containing all Sasakian manifolds with nonnegative $F$-bisectional curvature.

By using the above notions, from Theorem 2.3 we deduce

COROLLARY 2.4. *Let $M$ be a complete connected Sasakian manifold with positive lower bounded $r$-bisectional curvature, $r \leq \left[\frac{1}{2}\dim M\right]$. If $M$ contains a $2r+1$-dimensional compact invariant submanifold, tangent to $\xi$, then $M$ is compact.*

## 3. A Myers type theorem

Let $M$ be a $2n+1$-dimensional Sasakian manifold and we denote by $M^*$ its universal covering space. It is well-known (see for instance [9], t. I, pg. 162) that on $M^*$ there is a Riemannian metric $g^*$ such that the projection $\pi : M^* \to M$ is an isometric immersion and we define the 1-form $\eta^*$ on $M^*$ by $\eta^*_p = \pi^*_p \eta_{\pi(p)}$, where $\pi^*_p$ is the codifferential of $\pi$ at the point $p \in M^*$. If $\xi^{**}$ is its dual vector field with respect to $g^*$, i.e. the only vector field satisfying

$$\eta^*(X^*) = g^*(X^*, \xi^{**}), \tag{11}$$

for any $X^* \in \mathscr{X}(M^*)$, then $\xi^{**}$ is nowhere zero, hence we can consider its associated unit vector field $\xi^*$ and we have $\pi_{*,p}\,\xi^*_p = \xi_{\pi(p)}$, where $\pi_{*,p}$ is the differential of $\pi$.

Let $\nabla^*$ be the Levi-Civita connection on $M^*$, associated with the metric $g^*$. Then we can define the morphism $F^* : \mathscr{X}(M^*) \to \mathscr{X}(M^*)$ by

$$F^*X^* = -\nabla^*_{X^*}\xi^*$$

for any $X^* \in \mathscr{X}(M^*)$ and a straightforward computation shows that the tensor fields $F^*, \xi^*, \eta^*, g^*$ define a Sasakian structure on $M^*$ (see for instance [1], Theorem, pg. 73).

If $\mathscr{S} = \{e_1, \ldots, e_q\}$ is an $F$-orthonormal $q$-system of local vector fields in $M$ then we consider the 1-forms $\omega_1, \ldots, \omega_{2q}$, defined by

$$\omega_i(X) = g(e_i, X), \qquad \omega_{q+i}(X) = g(Fe_i, X)$$

for any $X \in \mathscr{X}(M)$ and $i \in \{1, \ldots, q\}$. We obtain $2q$ local 1-forms $\omega^*_1, \ldots, \omega^*_{2q}$, defined by

$$\omega^*_{j,p} = \pi^*_p \omega_{j,\pi(p)}$$

for $j \in \{1, \ldots, 2q\}$. Their dual local vector fields $e^*_1, \ldots, e^*_{2q}$, given by formulae similar to (11), satisfy

$$g(\pi_* e^*_j, \pi_* Y^*) = (\pi^*\omega_j)(Y^*) = \omega_j(\pi_* Y^*) = g(e_j, \pi_* Y^*),$$

and taking into account $\pi_*$ is injective, we deduce $e_j = \pi_* e_j^*$. Hence $\mathscr{S}^* = \{e_1^*, \ldots, e_q^*\}$ is an $F$-orthonormal $q$-system of local vector fields in $M^*$ and by a straightforward computation we deduce

PROPOSITION 3.1. *Let $M$ be a Sasakian manifold and $M^*$ its universal covering. If $\mathscr{H}_q$, $\mathscr{H}_q^*$ are the $q$-bisectional curvatures of $M$ and $M^*$ respectively, then*

$$\mathscr{H}_q^*(X^*, \mathscr{S}^*) = \mathscr{H}_q(\pi_* X^*, \mathscr{S}),$$

*for any unit vector field $X^* \in \mathscr{X}(M^*)$ and for any $F$-orthonormal $q$-system $\mathscr{S}$ of $M$.*

Now we shall prove a Myers type theorem for Sasakian manifolds, namely

THEOREM 3.2. *Let $M$ be a complete connected Sasakian manifold with lower bounded $q$-bisectional curvature $\mathscr{H}_q \geq k_0 > 0$. Then:*

a) *$M$ is compact;*

b) *the diameter of $M$ is at most equal to $\pi\sqrt{\frac{2q}{2q+k_0}}$;*

c) *$M$ has finite fundamental group.*

PROOF. For two arbitrary points $x$ and $y$ of $M$ denote by $\gamma : [a, b] \to M$ the minimizing geodesic joining $x$ to $y$. We assume that $\gamma$ is parametrized by its arc length $s$ and $b$ is such that $\gamma(b)$ is the first conjugate point of $\gamma$. If $\mathscr{S} = \{e_1, \ldots, e_q\} \subset T_x M$ is an $F$-orthonormal $q$-system normal to $\gamma$ at $\gamma(a)$ then, as above, by parallel translation of $\mathscr{S}$ along $\gamma$, we obtain another $F$-orthonormal $q$-system $\tilde{\mathscr{S}} = \{E_1, \ldots, E_q\}$ of vector fields defined along $\gamma$ and normal to $\gamma$.

Now, if $\Phi : [a, b] \to \mathsf{R}$ is a nonzero differentiable function such that $\Phi(a) = \Phi(c) = 0$ for some $c \in (a, b)$, then definitions similar to (6) give the vector fields $X_i$, $Y_i$ along $\gamma$ and by using (7) we obtain

$$\begin{aligned}(12)\quad &\sum_{i=1}^{q}\big[I_a^c(X_i, X_i) + I_a^c(Y_i, Y_i)\big]\\ &= \int_a^c \left\{2q\,\Phi'^2 - \Phi^2 \sum_{i=1}^{q}\big[\mathscr{R}(E_i, \gamma', E_i, \gamma') + \mathscr{R}(FE_i, \gamma', FE_i, \gamma')\big]\right\} ds.\end{aligned}$$

But $\gamma_{|[a,c]}$ has no conjugate points and taking into account Proposition 2.1, from (12) we deduce

$$0 < \int_a^c \big\{2q\Phi'^2 - \Phi^2[\mathscr{H}_q(\gamma', \tilde{\mathscr{S}}) + 2q]\big\}\, ds \leq \int_a^c \big[2q\Phi'^2 - (k_0 + 2q)\Phi^2\big]\, ds$$

For $\Phi(s) = \sin \pi \frac{s-a}{c-a}$, from the above inequalities we obtain

$$(c-a)^2 < \frac{2q}{2q+k_0}\pi^2.$$

But $c$ is arbitrary in $(a, b)$, hence

$$b - a \leq \pi \sqrt{\frac{2q}{2q+k_0}}$$

and because $\gamma$ is parametrized by its arc length, it follows

$$\text{distance}(x, y) = \text{distance}(\gamma(a), \gamma(b)) \leq \pi \sqrt{\frac{2q}{2q+k_0}}. \tag{13}$$

Thus b) is proved. From (13) we also deduce that $M$ is bounded and because it is complete, a) is proved too (see [9], t. I, Theorem 4.1, pg. 172). Now, because $M$ is compact, it is well-known that $M^*$ is complete and connected ([9], t. I, pg. 76) and taking into account Proposition 3.1, it follows that $M^*$ satisfies the hypothesis of Theorem 3.2. Hence, by a) it follows that $M^*$ is compact and then the fundamental group of $M$ is finite.

## 4. A Frankel type theorem

THEOREM 4.1. *Let $M$ be a complete connected Sasakian manifold with positive $q$-bisectional curvature. If $N$ and $P$ are two compact invariant submanifolds of $M$, tangent to $\xi$ and such that* $\dim N + \dim P \geq \dim M + 2q - 1$, *then* $N \cap P \neq \emptyset$.

PROOF. If $N \cap P = \emptyset$ then there is a geodesic $\gamma : [0, l] \to M$, parametrized by the arc length, joining two points $x_0 \in N$, $y_0 \in P$ and realizing the minimum of the distance between $N$ and $P$. We denote by $V_{y_0}$ the subspace of $T_{y_0}M$, obtained by parallel translation of $T_{x_0}N$ along $\gamma$ at the point $y_0$. From the hypothesis concerning the dimensions and because $N$ and $P$ are tangent to $\xi$, it follows that $\dim\left(V_{y_0} \cap T_{y_0}P\right) \geq 2q$. But $N$ and $P$ are invariant, hence $V_{y_0} \cap T_{y_0}P$ is invariant under $F$ and because $\xi_{y_0} \in V_{y_0} \cap T_{y_0}P$, it follows that $\dim\left(V_{y_0} \cap T_{y_0}P\right) \geq 2q + 1$. Moreover, we can find an $F$-orthonormal $q$-system $\mathcal{S}_{y_0} = \{e_1, \ldots, e_q\} \subset V_{y_0} \cap T_{y_0}P$. By parallel translation of $\mathcal{S}_{y_0}$ along $\gamma$, we obtain a system of $q$ unit vector fields $\tilde{\mathcal{S}} = \{E_1, \ldots, E_q\}$, defined along $\gamma$ and such that $a$ vectors of $\tilde{\mathcal{S}} \cup F\tilde{\mathcal{S}}$ are from $\tilde{\mathcal{S}}$ and $b$ $(a + b = 2q)$ of them are from $F\tilde{\mathcal{S}}$.

If $a \geq q$ then we can assume that these vector fields are $E_1, \ldots, E_q$ and because $P$ is invariant, it follows that $FE_1, \ldots, FE_q$ are tangent to $P$ at $y_0$, too. The same argument used if $b \geq q$ shows that always we can find an $F$-orthonormal $q$-system $\tilde{\mathscr{S}}$ of vector fields along $\gamma$, tangent to $N$ at $x_0$ and tangent to $P$ at $y_0$. Then, by a computation similar to the one used in [8] (the proof of Theorem 3.2.) and taking into account (10), we obtain

$$\begin{aligned}(14)\quad &\sum_{i=1}^{q}\left[I_0^l(E_i, E_i) + I_0^l(FE_i, FE_i)\right]\\ &\qquad = -\int_0^l \sum_{i=1}^{q}\left[\mathscr{R}(E_i, \gamma', E_i, \gamma') - \mathscr{R}(FE_i, \gamma', FE_i, \gamma')\right] ds.\end{aligned}$$

Now, by using Proposition 2.1 and taking into account

$$\sum_{i=1}^{q}\left[g^2(\gamma', E_i) + g^2(\gamma', FE_i)\right] \leq \|\gamma'\|^2 = 1,$$

from (14) it follows

$$\begin{aligned}&\sum_{i=1}^{q}\left[I_0^l(E_i, E_i) + I_0^l(FE_i, FE_i)\right]\\ &\qquad = -ql + \int_0^l \sum_{i=1}^{q}\left[g^2(\gamma', E_i) + g^2(\gamma', FE_i)\right] ds - \int_0^l \mathscr{H}_q(\gamma', \tilde{\mathscr{S}})\, ds\\ &\qquad \leq l(1-q) - \int_0^l \mathscr{H}_q(\gamma', \tilde{\mathscr{S}})\, ds < 0\end{aligned}$$

We deduce that $I_0^l(E_i, E_i) < 0$ or $I_0^l(FE_i, FE_i) < 0$ for some $i \in \{1, \ldots, q\}$, contradicting the hypothesis that $\gamma$ has minimal length. Hence $N$ and $P$ must have nonempty intersection.

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