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ON THE BOREL COHOMOLOGY OF FREE LOOP SPACES

IVER OTTOSEN*

Abstract

Let X be a space and let $K = H^*(X; \mathsf{F}_p)$ where p is an odd prime. We construct functors $\bar{\Omega}$ and ℓ which approximate cohomology of the free loop space ΛX as follows: There are homomorphisms $\bar{\Omega}(K) \to H^*(\Lambda X; \mathsf{F}_p)$ and $\ell(K) \to H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X; \mathsf{F}_p)$. These are isomorphisms when X is a product of Eilenberg-MacLane spaces of type $K(\mathsf{F}_p, n)$ for $n \geq 1$.

1. Introduction

Let X be a topological space and R a ring. The circle group T acts on the free loop space ΛX by rotation of loops. The associated Borel cohomology groups are called string cohomology of X [4]. We denote them as follows:

$$H^*_{st}(X; R) = H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X; R).$$

String cohomology as well as non equivariant cohomology of free loop spaces play a central role in geometry and topology. It is however often not possible to compute such cohomology groups.

When $R = \mathsf{F}_2 = \mathsf{Z}/2$, M. Bökstedt and I found functors of $H^*(X)$ which approximate $H^*_{st}(X)$ and $H^*(\Lambda X)$ [2]. The purpose of this paper is to generalize these functors to the case $R = \mathsf{F}_p = \mathsf{Z}/p$ where p is any of the odd primes. Certain algebra generators in string cohomology are more difficult to construct in the odd primary case. Hence method and strategy differs from [2] at various places.

The following application of the functors $\bar{\Omega}$ and ℓ will appear in the near future. There are two Bousfield cohomology spectral sequences. One converging to $H^*(\Lambda X)$ and the other converging to $H^*_{st}(X)$. The E_2 term of the first is isomorphic to the (non Abelian) derived functors of $\bar{\Omega}$ and the E_2 term of the second is isomorphic to the derived functors of ℓ.

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NOTATION. Fix an odd prime p. We use F_p-coefficients everywhere unless otherwise is specified. $\mathscr{A}$ denotes the mod p Steenrod algebra, $\mathscr{U}$ the category of unstable $\mathscr{A}$-modules and $\mathscr{K}$ the category of unstable $\mathscr{A}$-algebras. We let $\mathscr{A}lg$ denote the following category. An object in $\mathscr{A}lg$ is a non-negatively graded F_p-algebra A with the property that if $a \in A$ and $|a| = 0$ then $a = a^p$. The category of differential graded F_p-algebras is denoted DGA. For any $A \in \mathscr{A}lg$ we define $\sigma : A \to \mathsf{F}_p$ by $\sigma(x) = 1$ for $|x|$ odd and $\sigma(x) = 0$ for $|x|$ even. We also define $\hat{\sigma} : A \to \mathsf{F}_p$ by $\hat{\sigma}(x) = 1 - \sigma(x)$. The circle group is denoted T.

2. The approximation functor $\bar{\Omega}$

In this section we define a functor $\bar{\Omega} : \mathscr{F} \to \mathscr{A}lg$ which approximates the cohomology ring $H^*(\Lambda X)$ when applied to H^*X. Here $\mathscr{F}$ is a certain category which lies between $\mathscr{K}$ and $\mathscr{A}lg$. The functor $\bar{\Omega}$ lifts to an endofunctor on $\mathscr{K}$ which is nothing but an explicit description of Lannes' division functor $(- : H^*(\mathsf{T}))_{\mathscr{K}}$ introduced in [5].

DEFINITION 2.1. Let $\mathscr{F}$ denote the following category. An object in $\mathscr{F}$ is an object $A \in \mathscr{A}lg$ which is equipped with an F_p-linear map $\lambda : A \to A$ with the following properties:

- $|\lambda x| = p(|x| - 1) + 1$ for all $x \in A$.
- $\lambda x = x$ when $|x| = 1$ and $\lambda x = 0$ when $|x|$ is even.
- $\lambda(xy) = \lambda(x)y^p + x^p\lambda(y)$ for all $x, y \in A$.

Furthermore A is equipped with an F_p-linear map $\beta : A \to A$ of degree 1 with the following properties:

- $\beta \circ \beta = 0$.
- $\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y)$ for all $x, y \in A$.

A morphism $f : A \to A'$ in $\mathscr{F}$ is a morphism in $\mathscr{A}lg$ such that $f(\lambda x) = \lambda' f(x)$ and $f(\beta x) = \beta' f(x)$.

REMARK 2.2. There are forgetful functors $\mathscr{K} \to \mathscr{F}$ and $\mathscr{F} \to \mathscr{A}lg$. For an object K in $\mathscr{K}$ the map $\lambda : K \to K$ is defined by $\lambda x = P^{(|x|-1)/2}x$ when $|x|$ is odd. The map β is the Bockstein operation.

We let $\Lambda(v)$ denote the object $H^*(\mathsf{T})$ in $\mathscr{K}$. There is an associative and commutative coproduct $\delta : \Lambda(v) \to \Lambda(v) \otimes \Lambda(v);\ v \mapsto 1 \otimes v + v \otimes 1$. It comes from the product on T and has counit $\gamma : \Lambda(v) \to \mathsf{F}_p$ coming from the unit $1 \to \mathsf{T}$.

Let $\perp : \mathscr{K} \to \mathscr{K}$ be the functor given by $A \mapsto \Lambda(v) \otimes A$. The coproduct and counit above define natural transformations $\delta : \perp \to \perp^2$ and $\gamma : \perp \to Id$

such that $(\perp, \delta, \gamma)$ is a comonad. A $\perp$-coalgebra is an object K in $\mathscr{K}$ equipped with a morphism $f : K \to \perp(K)$ such that the following diagrams commute:

$$\begin{array}{ccc} K & \xrightarrow{f} & \perp(K) \\ & \searrow^{id} & \downarrow \gamma \\ & & K \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{f} & \perp(K) \\ \downarrow f & & \downarrow \delta \\ \perp(K) & \xrightarrow{\perp(f)} & \perp^2(K). \end{array}$$

Examples of $\perp$-coalgebras are cohomology of T-spaces.

PROPOSITION 2.3. *If K is a $\perp$-coalgebra with structure map $f : K \to \perp(K)$ then K is a graded commutative DGA with degree -1 differential d given by*

$$f(x) = 1 \otimes x + v \otimes dx, \qquad x \in K.$$

Furthermore, $d(P^i x) = P^i dx$ for each $i \geq 0$ and $d(\beta x) = -\beta d(x)$. In particular $d(\lambda x) = (dx)^p$ and $d(\beta\lambda x) = 0$.

PROOF. By the left of the above diagrams f may be expanded as stated. By the right diagram $d \circ d = 0$. Since f is a morphism in $\mathscr{K}$ we see that d is F_p-linear, a derivation over the identity and that the stated relations hold.

PROPOSITION 2.4. *Assume that the functor $\perp : \mathscr{K} \to \mathscr{K}$ has a left adjoint $\top : \mathscr{K} \to \mathscr{K}$. Then there is a natural $\perp$-coalgebra structure $\eta : \top \to \perp\top$ on $\top$. For an object $B \in \mathscr{K}$ the map η_B is the image of the identity under the composite*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{K}}(\top(B), \top(B)) & & \mathrm{Hom}_{\mathscr{K}}(\top(B), \perp\top(B)) \\ \cong\downarrow & & \cong\uparrow \\ \mathrm{Hom}_{\mathscr{K}}(B, \perp\top(B)) & \xrightarrow{\delta_*} & \mathrm{Hom}_{\mathscr{K}}(B, \perp^2\top(B)) \end{array}$$

PROOF. This is formally the same as the proof of [11] Proposition 3.4.

DEFINITION 2.5. For $A \in \mathscr{F}$ we define $\bar{\Omega}(A)$ as the quotient of the free graded commutative and unital A-algebra on generators

$$dx \qquad \text{for} \quad x \in A$$

where $|dx| = |x| - 1$, by the ideal generated by the elements

(1) $\quad d(x + y) - dx - dy,$

(2) $\quad d(xy) - d(x)y - (-1)^{|x|} x d(y),$

(3) $\quad d(\lambda x) - (dx)^p,$

(4) $\quad d(\beta\lambda x).$

Note that $\bar{\Omega}(A)$ is non-negatively graded since $d(x^p) = 0$. We have defined a functor $\bar{\Omega} : \mathscr{F} \to \mathscr{A}lg$.

PROPOSITION 2.6. *The functor* $\bar{\Omega} : \mathscr{F} \to \mathscr{A}lg$ *lifts to a functor* $\bar{\Omega} : \mathscr{K} \to \mathscr{K}$. *Explicitely the* $\mathscr{A}$*-action on* $\bar{\Omega}(K)$ *is given by* $\theta(x) = \theta x$ *and* $\theta(dx) = (-1)^{|\theta|} d(\theta x)$ *for* $x \in K$ *and* $\theta \in \mathscr{A}$ *and the Cartan formula. The differential* d *on* $\bar{\Omega}(K)$ *is graded* $\mathscr{A}$*-linear.*

PROOF. Let dK denote the graded F_p-vector space given by $(dK)^n = K^{n+1}$. We write dx for the element in dK corresponding to x in K hence $d(x+y) = dx + dy$. We define an $\mathscr{A}$-module structure on dK by $P^i dx = dP^i x$ and $\beta dx = -d\beta x$. Let $S(dK)$ denote the free graded commutative algebra on the F_p-vector space dK. By the Cartan formula $S(dK)$ is an $\mathscr{A}$-algebra and the symmetric product $K \odot S(dK)$ is an $\mathscr{A}$-algebra. By definition $\bar{\Omega}(K) = K \odot S(dK)/I$ where I is the ideal generated by

$$1 \odot d(xy) - d(x) \odot y - (-1)^{|x|} x \odot d(y), \tag{5}$$

$$1 \odot (d(\lambda x) - (dx)^p), \tag{6}$$

$$1 \odot d(\beta \lambda x). \tag{7}$$

We verify that $\mathscr{A} \cdot I \subseteq I$ such that $\bar{\Omega}(K)$ is an $\mathscr{A}$-algebra. We have

$$P^n(1 \odot d(xy) - dx \odot y - (-1)^{|x|} x \odot dy)$$
$$= \sum_{i+j=n} (1 \odot d(P^i(x)P^j(y)) - dP^i x \odot P^j y - (-1)^{|x|} P^i x \odot dP^j y)$$

which is in I by (5) since the degree of P^i is even. Further

$$\beta(1 \odot d(xy) - dx \odot y - (-1)^{|x|} x \odot dy)$$
$$= -(1 \odot d(\beta(x)y) - d\beta x \odot y - (-1)^{|\beta x|} \beta x \odot dy)$$
$$- (-1)^{|x|}(1 \odot d(x\beta y) - dx \odot \beta y - (-1)^{|x|} x \odot d\beta y)$$

which is also in I by (5).

In any $\mathscr{A}$-algebra one has $P^i(a^p) = (P^{i/p}a)^p$ when $i = 0 \bmod p$ and zero otherwise, since this is a consequence of the Cartan formula alone. So by Lemma 2.7 we have the following relation in $S(dK)$ when $i = 0 \bmod p$:

$$P^i(d(\lambda x) - (dx)^p) = d(P^i \lambda x) - (P^{i/p} dx)^p = d(\lambda P^{i/p} x) - (dP^{i/p} x)^p.$$

For $i \neq 0 \bmod p$ we get zero. So P^i applied to an element of the form (6) lies in I. If we apply β to such an element we also land in I by (7). Finally Lemma 2.7 shows that $P^i(1 \odot d(\beta\lambda x)) \in I$ and trivially $\beta(1 \odot d(\beta\lambda x)) \in I$.

We verify that $\bar{\Omega}(K) \in \mathcal{U}$. We must show that $P^i dx = 0$ if $2i > |x| - 1$. This holds if $2i > |x|$ since $K \in \mathcal{U}$. If $2i = |x|$ we have $P^i dx = dP^i x = d(x^p) = 0$. We must also show that $\beta P^i dx = 0$ when $2i + 1 > |x| - 1$. This holds if $2i + 1 > |x|$ since $K \in \mathcal{U}$ and if $2i + 1 = |x|$ we have $\beta P^i dx = -d\beta P^i x = -d\beta\lambda x = 0$. Since the action on products are by the Cartan formula we have shown that $\bar{\Omega}(K) \in \mathcal{U}$.

Finally we check that $\bar{\Omega}(K) \in \mathcal{K}$. The Cartan formula holds by definition. For $|x|$ odd we have $P^{|dx|/2}(dx) = d\lambda x = (dx)^p$ and the result follows.

LEMMA 2.7. *For any unstable $\mathcal{A}$-algebra K and $x \in K$ the following equations hold.*

$$P^i \lambda x = \begin{cases} \lambda(P^{\frac{i}{p}} x), & i = 0 \mod p \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

$$P^i \beta\lambda x = \begin{cases} \beta\lambda(P^{\frac{i}{p}} x), & i = 0 \mod p \\ (\beta P^{\frac{i-1}{p}} x)^p, & i = 1 \mod p \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

PROOF. We just prove (8) since the proof of (9) is similar. When $|x|$ is even both sides in the equation are zero. Assume that $|x|$ is odd. By the instability condition $P^i \lambda x = 0$ when $2i > p(|x| - 1) + 1$. When i is divisible by p this inequality implies $2i \geq p(|x| - 1) + p$ or $\frac{2i}{p} \geq |x|$ and since $|x|$ is odd $\frac{2i}{p} > |x|$. So $P^{i/p} x = 0$ and the equation holds in this case. If $2i = p(|x| - 1)$ then $P^i \lambda x = \lambda^2 x = \lambda(P^{i/p} x)$.

Finally assume that $2i < p(|x| - 1)$. Then we can apply the Adem relation:

$$P^i P^{\frac{|x|-1}{2}} x = \sum_{t=0}^{[\frac{i}{p}]} (-1)^{i+t} \binom{(p-1)\left(\frac{|x|-1}{2} - t\right) - 1}{i - pt} P^{i + \frac{|x|-1}{2} - t} P^t x.$$

The instability condition shows that $P^{i+\frac{|x|-1}{2}-t} P^t x = 0$ unless $i \leq pt$. But the binomial coefficient is zero when $i < pt$. So we get zero when $i \neq 0 \mod p$ and the term corresponding to $t = i/p$ when $i = 0 \mod p$.

PROPOSITION 2.8. *The functor $\bar{\Omega} : \mathcal{K} \to \mathcal{K}$ is left adjoint to $\perp : \mathcal{K} \to \mathcal{K}$; $B \mapsto H^*(\mathsf{T}) \otimes B$. Thus there is an equivalence of functors $\bar{\Omega} \cong (- : H^*(\mathsf{T}))_{\mathcal{K}}$. The differential $d : \bar{\Omega}(A) \to \bar{\Omega}(A)$, associated to the natural $\perp$-coalgebra structure, is given by $d(x) = dx$ for $x \in A$.*

PROOF. We can define natural maps as follows where $x \in A$:

$$F : \mathrm{Hom}_{\mathscr{K}}(\bar{\Omega}(A), B) \rightleftarrows \mathrm{Hom}_K(A, \perp(B)) : G$$
$$F(f)(x) = 1 \otimes f(x) + v \otimes f(dx),$$
$$G(g)(x) = \gamma \circ g(x), \qquad G(g)(dx) = (\alpha \otimes 1) \circ g(x)$$

where $\alpha : \Lambda(v) \to \mathsf{F}_p$ is the additive map of degree -1 given by $v \mapsto 1$ and $1 \mapsto 0$. It is easy to verify that $F \circ G = id$ and $G \circ F = id$. The description of d follows by using these explicit adjunction formulas in the composite defining η in Proposition 2.4.

PROPOSITION 2.9. *For any space X there is a morphism in $\mathscr{K}$ (and in DGA)*

$$e : \bar{\Omega}(H^*X) \to H^*(\Lambda X); \qquad e(x) = ev_0^*(x); \qquad e(dx) = dev_0^*(x)$$

where $ev_0 : \Lambda X \to X; \omega \mapsto \omega(1)$. This morphism is natural in X and it is an isomorphism if $X = K(\mathsf{F}_p, n)$ with $n \geq 0$. If H_*X is of finite type and Y is any space then there is a commutative diagram

$$\begin{array}{ccc} \bar{\Omega}(H^*X) \otimes \bar{\Omega}(H^*Y) & \xrightarrow{\cong} & \bar{\Omega}(H^*X \otimes H^*Y) \\ \downarrow{\scriptstyle e\otimes e} & & \downarrow{\scriptstyle e} \\ H^*(\Lambda X) \otimes H^*(\Lambda Y) & \xrightarrow{\cong} & H^*(\Lambda(X \times Y)) \end{array}$$

where the lower horizontal map is the Künneth isomorphism.

PROOF. The proof of Proposition 3.9 in [11] goes through with the obvious changes. Thus the isomorphism statement is a consequence of [5] 1.11.

3. The approximation functor ℓ

In this section we describe the functor $\ell : \mathscr{F} \to \mathscr{A}lg$ which gives an approximation to $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$ when applied to H^*X. We also define a natural transformation $Q : \ell \to \bar{\Omega}$ which corresponds to the map $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X) \to H^*(\Lambda X)$ induced by the quotient. We do however not go into the topological interpretations here.

DEFINITION 3.1. For $A \in \mathscr{F}$ we define $\ell(A)$ as the free graded commutative F_p-algebra on generators $\phi(x)$, $q(x)$, $\delta(x)$ for $x \in A$ and u of degrees

$$|\phi(x)| = p|x| - \sigma(x)(p-1), \qquad |q(x)| = p|x| - 1 - \sigma(x)(p-3),$$
$$|\delta(x)| = |x| - 1, \qquad |u| = 2$$

modulo the ideal generated by

$$\phi(x+y)-\phi(x)-\phi(y)+\sigma(x)\sum_{i=0}^{p-2}(-1)^i\delta(x)^i\delta(y)^{p-2-i}\delta(xy), \tag{10}$$

$$\delta(x+y)-\delta(x)-\delta(y), \tag{11}$$

$$q(x+y)-q(x)-q(y)+\hat{\sigma}(x)\sum_{i=1}^{p-1}(-1)^i\frac{1}{i}\delta(x^iy^{p-i}), \tag{12}$$

$$(-1)^{\sigma(a)\hat{\sigma}(c)}\delta(a)\delta(bc)+(-1)^{\sigma(b)\hat{\sigma}(a)}\delta(b)\delta(ca) + (-1)^{\sigma(c)\hat{\sigma}(b)}\delta(c)\delta(ab), \tag{13}$$

$$\phi(ab)-(-u^{p-1})^{\sigma(a)\sigma(b)}\phi(a)\phi(b), \tag{14}$$

$$q(ab)-(-u^{p-1})^{\sigma(a)\sigma(b)}(u^{\sigma(b)}q(a)\phi(b)+(-u)^{\sigma(a)}\phi(a)q(b)), \tag{15}$$

$$q(x)^p-u^{p-1}q(\lambda x)-\phi(\beta\lambda x), \tag{16}$$

$$\delta(a)\phi(b)-\delta(ab^p)-\delta(a\lambda b)+\delta(ab)\delta(b)^{p-1}, \tag{17}$$

$$\delta(a)q(b)-\delta(ab^{p-1})\delta(b)-\delta(a\beta\lambda b), \tag{18}$$

$$\delta(x)u, \tag{19}$$

$$q(\beta\lambda x), \tag{20}$$

$$\delta(x^p) \tag{21}$$

where $a, b, c, x, y \in K$ and $|x| = |y|$.

REMARK 3.2. We have some immediate consequences of these relations: By (10), (11) and (20) we have $\phi(0) = q(0) = \delta(0) = 0$. By (14) and (15) we have $q(a^n) = n\phi(a)^{n-1}q(a)$ such that $q(a^p) = 0$. By (21) we have $\delta(1) = 0$ so by (21) and (17) we find $\delta(\lambda b) = \delta(b)^p$. By (18) and $\delta(1) = 0$ we have $\delta(\beta\lambda b) = 0$. By (14), (15) and (17) the algebra $\ell(A)$ is unital with unit $\phi(1)$.

Since $\delta(x^p) = q(x^p) = 0$ we see that $\ell(A)$ is non-negatively graded. We have defined a functor $\ell : \mathscr{F} \to \mathscr{A}lg$.

LEMMA 3.3. *Let $K \in \mathscr{F}$ and $x, y \in K$ with $|x| = |y| = n$. The following relations hold in $\bar{\Omega}(K)$:*

$$\sum_{i=1}^{p-1}(-1)^{i+1}\frac{1}{i}d(x^iy^{p-i}) = (x+y)^{p-1}d(x+y)-x^{p-1}dx-y^{p-1}dy, \qquad n \text{ even} \tag{22}$$

$$(23)\quad \sum_{j=0}^{p-2}(-1)^{j+1}(dx)^j(dy)^{p-2-j}d(xy) = (d(x+y))^{p-1}(x+y)-(dx)^{p-1}x-(dy)^{p-1}y, \qquad n \text{ odd}.$$

PROOF. We verify (22) and omit the proof of (23) which is similar. Since d is a derivation we have

$$\sum_{i=1}^{p-1}(-1)^{i+1}\frac{1}{i}d(x^iy^{p-i}) = \sum_{i=1}^{p-1}(-1)^{i+1}(x^{i-1}y^{p-1}dx - x^iy^{p-i-1}dy).$$

By splitting the sum in two at the minus sign and substituting $j = i-1$ in the first of the resulting sums we see that the above equals the following:

$$\sum_{j=0}^{p-2}(-1)^jx^jy^{p-j-1}dx + \sum_{i=1}^{p-1}(-1)^ix^iy^{p-i-1}dy = \sum_{t=0}^{p-1}(-1)^tx^ty^{p-t-1}(dx+dy) - x^{p-1}dx - y^{p-1}dy.$$

For $0 \le t \le p-1$ we have that $t!$ is invertible in F_p and also

$$\binom{p-1}{t}t! = (p-1)(p-2)\dots(p-t) = (-1)^tt! \mod p.$$

Thus we have $\binom{p-1}{t} = (-1)^t$. Substituting this in the above and using the binomial formula the result follows.

PROPOSITION 3.4. *For $A \in \mathscr{F}$ there is a natural morphism in $\mathscr{A}lg$ as follows:*

$$Q : \ell(A) \to \bar{\Omega}(A);\quad \phi(x) \mapsto x^p + \lambda x - x(dx)^{p-1}, \quad \delta(x) \mapsto dx,\quad q(x) \mapsto x^{p-1}dx + \beta\lambda x,\quad u \mapsto 0.$$

Furthermore, $\mathrm{Im}(Q) \subseteq \ker(d : \bar{\Omega}(A) \to \bar{\Omega}(A))$.

PROOF. We check that the formulas for Q map the relations (10)-(21) to zero. Formula (23) and the additivity of $x \mapsto x^p$ shows that (10) is mapped to zero. It is trivial that (11) is mapped to zero. By (22) and the additivity of $x \mapsto \beta\lambda x$ it follows that (12) is mapped to zero.

Taking the derivative of products and permuting factors we find the following equations:

$$d(a)d(bc) = d(a)d(b)c + (-1)^{\sigma(b)} d(a)bd(c),$$

$$d(b)d(ca) = (-1)^{\sigma(a)(\hat{\sigma}(b)+\hat{\sigma}(c))} ad(b)d(c) + (-1)^{\sigma(c)+\hat{\sigma}(a)(\hat{\sigma}(b)+\sigma(c))} d(a)d(b)c,$$

$$d(c)d(ab) = (-1)^{\hat{\sigma}(c)(\hat{\sigma}(a)+\sigma(b))} d(a)bd(c) + (-1)^{\sigma(a)+\hat{\sigma}(c)(\sigma(a)+\hat{\sigma}(b))} ad(c)d(b).$$

After some reductions (13) follows from these.

One easily checks that (14) and (15) are mapped to zero in each of the cases $\sigma(a) = \sigma(b) = 0$, $\sigma(a) = \sigma(b) = 1$ and $\sigma(a) = \hat{\sigma}(b) = 1$. It also follows by small direct computations that (16)–(21) are mapped to zero.

4. The morphism Q and cohomology of $\bar{\Omega}(A)$

In this section we define an additive transformation $\tau : \bar{\Omega} \to \ell$ which corresponds to the T-transfer from $H^*(\Lambda X)$ to $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$. The map Q gives a morphism from $\ell(A)/(u)$ to the cycles in $\bar{\Omega}(A)$. Via this a map Φ similar to the Cartier map [3] is defined. It turns out that $\ell(A)/(u) \cong \ker(d)$ when Φ is an isomorphism. Parts of the material presented here correspond to section 8 in [2]. We let A denote an object in $\mathscr{F}$.

DEFINITION 4.1. Let $I_\delta(A) \subseteq \ell(A)$ denote the ideal $I_\delta(A) = (\delta(x) \mid x \in A)$.

PROPOSITION 4.2. *There is an* F_p*-linear map of degree* -1 *as follows*

$$\tau : \bar{\Omega}(A) \to \ell(A); \quad a_0 da_1 \dots da_n \mapsto \delta(a_0)\delta(a_1)\dots\delta(a_n), \quad a_0 \mapsto \delta(a_0)$$

where $a_i \in A$ *for each* i. *It has the following properties:*

$$\tau(Q(\alpha)\beta) = (-1)^{|\alpha|}\alpha\tau(\beta) \text{ for } \alpha \in \ell(A), \beta \in \bar{\Omega}(A), \quad Q \circ \tau = d, \quad \tau \circ Q = 0.$$

Note that $\tau \circ d = 0$ *and* $\mathrm{Im}(\tau) = I_\delta(A)$.

PROOF. We must show that τ is well defined. The relations arising from (1), (3) and (4) are respected since we have the same relations in $\ell(K)$ with d replaced by δ. We must verify that the following relation is respected:

$$a_0 da_1 \dots da_{i-1} d(a_i a_{i+1}) da_{i+2} \dots da_n = (-1)^{(k+\hat{\sigma}(a_i))\sigma(a_{i+1})} a_0 a_{i+1} da_1 \dots da_i da_{i+2} \dots da_n + (-1)^{(k+1)\sigma(a_i)} a_0 a_i da_1 \dots da_{i-1} da_{i+1} \dots da_n$$

where $k = |da_1 \dots da_{i-1}|$. It suffices to check that

$$xd(yz) = (-1)^{\hat{\sigma}(y)\sigma(z)} xzd(y) + (-1)^{\sigma(y)} xyd(z)$$

is respected. This follows by (13) after some work with the signs.

By definition we have $Q \circ \tau = 0$. By direct computations one sees that $\tau(Q(\alpha)\beta) = (-1)^{|\alpha|}\alpha\tau(\beta)$ when α equals $\phi(x)$, $q(x)$ or $\delta(x)$ and β equals $a_0 da_1 \dots da_n$ or a_0. The general case follows from this. In particular $\tau \circ Q = 0$ since $\tau(1) = 0$.

Definition 4.3. Let $\mathscr{L}(A) = \ell(A)/(u)$ and $\widetilde{\Omega}(A) = \mathscr{L}(A)/I_\delta(A)$. Explicitely, $\widetilde{\Omega}(A)$ is the free graded commutative F_p-algebra on generators $\phi(x)$, $q(x)$ for $x \in A$ of degrees $|\phi(x)| = p|x| - \sigma(x)(p-1)$, $|q(x)| = p|x| - 1 - \sigma(p-3)$ modulo the relations that ϕ and q are additive and

$$\phi(ab) = (1 - \sigma(a)\sigma(b))\phi(a)\phi(b), \tag{24}$$

$$q(ab) = \hat{\sigma}(b)q(a)\phi(b) + \hat{\sigma}(a)\phi(a)q(b), \tag{25}$$

$$\phi(\beta\lambda x) = q(x)^p, \tag{26}$$

$$q(\beta\lambda x) = 0. \tag{27}$$

Since $Q(I_\delta(A)) \subseteq d\bar{\Omega}(A)$ we may define an F_p-algebra map Φ by the following diagram where P denotes the canonical projection:

$$\begin{array}{ccc}
\mathscr{L}(A) & \xrightarrow{P} & \widetilde{\Omega}(A) \\
{\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\bar{\Omega}(A) & \longrightarrow & \bar{\Omega}(A)/d\bar{\Omega}(A)
\end{array}$$

Since $d \circ Q = 0$ we have in fact defined a morphism $\Phi : \widetilde{\Omega}(A) \to H^*(\bar{\Omega}(A))$.

Remark 4.4. Since $\tau \circ d = 0$ we can define τ as a map on $\bar{\Omega}(A)/d\bar{\Omega}(A)$. We have a commutative diagram as follows:

$$\begin{array}{ccccccc}
\widetilde{\Omega}(A) & \xrightarrow{\Phi} & \bar{\Omega}(A)/d\bar{\Omega}(A) & \xrightarrow{\tau} & \mathscr{L}(A) & \xrightarrow{P} & \widetilde{\Omega}(A) \\
& & \uparrow & & {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \Phi} \\
& & \bar{\Omega}(A) & \xrightarrow{d} & \bar{\Omega}(A) & \longrightarrow & \bar{\Omega}(A)/d\bar{\Omega}(A)
\end{array}$$

where the composite $\tau \circ \Phi$ vanishes and $\ker(P) = \operatorname{Im}(\tau)$.

Theorem 4.5. *Assume that the map* $\Phi : \widetilde{\Omega}(A) \to H^*(\bar{\Omega}(A))$ *is an isomorphism. Then so is* $Q : \mathscr{L}(A) \to \ker(d : \bar{\Omega}(A) \to \bar{\Omega}(A))$.

PROOF. The diagram is formally the same as the one above Theorem 8.5 of [2]. So the same diagram chase gives the result.

There is a filtration $\ell(A) \supseteq u\ell(A) \supseteq u^2\ell(A) \supseteq \dots$ with associated graded object $Gr_*\ell(A)$ given by $Gr_i\ell(A) = u^i\ell(A)/u^{i+1}\ell(A)$. Consider the following composite of surjective maps:

$$\ell(A) \xrightarrow{u^i\cdot} u^i\ell(A) \longrightarrow Gr_i\ell(A), \qquad i \geq 1$$

The ideal $I_\delta(A) + u\ell(A) \subseteq \ell(A)$ is send to zero so we get a surjective F_p-linear map $u^i\cdot : \widetilde{\Omega}(A) \to Gr_i\ell(A)$.

PROPOSITION 4.6. *For each $i \geq 1$ there is a unique F_p-linear map Φ_i such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{\Omega}(A) & \xrightarrow{u^i\cdot} & Gr_i\ell(A) \\ \Phi\big\downarrow & & \Phi_i\big\downarrow \\ H^*(\bar{\Omega}(A)) & \xrightarrow{u^i\otimes -} & u^i \otimes H^*(\bar{\Omega}(A)) \end{array}$$

If $\Phi : \widetilde{\Omega}(A) \to H^*(\bar{\Omega}(A))$ is an isomorphism then

$$Gr_*\ell(A) \cong \ker(d) \oplus (u \otimes \widetilde{\Omega}(A)) \oplus (u^2 \otimes \widetilde{\Omega}(A)) \oplus \cdots .$$

PROOF. The following elements generate the F_p-vector space $Gr_i\ell(A)$:

$$u^i\phi(x_1)\dots\phi(x_n)q(x_{n+1})\dots q(x_{n+m}) + u^{i+1}\ell(A) \tag{28}$$

where $n, m \geq 0$ and $x_j \in A$ for all j. (If n or m equals zero we have an empty product which equals 1 by definition.) We can describe the relations among these generators. Firstly they are additive in each variable x_j. Secondly there is a relation corresponding to each of the relations (24)–(27) for example

$$\begin{aligned} &u^i\phi(x_1)\dots\phi(x_t'x_t'')\dots\phi(x_n)q(x_{n+1})\dots q(x_{n+m}) \\ &\quad = (1-\sigma(x_t')\sigma(x_t''))u^i\phi(x_1)\dots\phi(x_t')\phi(x_t'')\dots\phi(x_n)q(x_{n+1})\dots q(x_{n+m}) \end{aligned}$$

modulo $u^{i+1}\ell(A)$. If the map Φ_i exists such that the diagram commutes it must send (28) to

$$u^i \otimes \Phi(\phi(x_1)\dots\phi(x_n)q(x_{n+1})\dots q(x_{n+m})).$$

But this formula gives a well defined map by the above identification of the relations among the generators.

The map $u^i \otimes -$ is an isomorphism so if Φ is also an isomorphism we see that $u^i\cdot$ is injective. By definition $u^i\cdot$ is always surjective so the result follows.

DEFINITION 4.7. Let $n\mathsf{F}_p$ denote the category of non-negatively graded F_p-vector spaces. Define the free functor $S_{\mathscr{F}} : n\mathsf{F}_p \to \mathscr{F}$ to be the left adjoint of the forgetful functor $\mathscr{F} \to n\mathsf{F}_p$.

REMARK 4.8. We have $S_{\mathscr{F}}(V \oplus W) = S_{\mathscr{F}}(V) \otimes S_{\mathscr{F}}(W)$. Furthermore there is an explicit description as follows

$$S_{\mathscr{F}}(V) = S_{\mathscr{A}lg}\Big(V \oplus \beta V^{*\geq 1} \oplus \bigoplus_{i\geq 1, \nu\in\{0,1\}} \beta^\nu \lambda^i \big(\beta V^{\mathrm{even},*\geq 2} \oplus V^{\mathrm{odd},*\geq 2}\big)\Big)$$

where $S_{\mathscr{A}lg}$ denotes the left adjoint of the forgetful functor $\mathscr{A}lg \to n\mathsf{F}_p$.

THEOREM 4.9. *The map $\Phi : \widetilde{\Omega}(A) \to H^*(\bar{\Omega}(A))$ is an isomorphism when A is a free object in $\mathscr{F}$.*

PROOF. By the results in the appendix section 10 it suffices to show that Φ is an isomorphism when $A = F_n = S_{\mathscr{F}}(V_n)$, $n \geq 0$ where V_n is the free F_p-vector space on one single generator x_n of degree n.

We have $F_0 = \mathsf{F}_p[x_0]/(x_0^p - x_0)$ and $\bar{\Omega}(F_0) = F_0$ with zero differential such that $H^*(\bar{\Omega}(F_0)) = F_0$. On the other hand $\widetilde{\Omega}(F_0) \cong F_0$ with generator $\phi(x_0)$. So Φ is an isomorphism since $\Phi(\phi(x_0)) = x_0^p = x_0$.

Further, $F_1 = \Lambda(x_1) \otimes \mathsf{F}_p[\beta x_1]$ with $\lambda x_1 = x_1$. Since $(dx_1)^p = dx_1$ we can use the idempotents from Remark 4.11 below to get a splitting

$$\bar{\Omega}(F_1) = \bigoplus_{i\in\mathsf{F}_p} e_i \bar{\Omega}(F_1).$$

For each i we have $de_i = 0$ and $(dx_1)e_i = ie_i$. Also $d\beta x_1 = d\beta\lambda x_1 = 0$. Thus $d(x_1^\epsilon(\beta x_1)^r e_i) = \epsilon i(\beta x_1)^r e_i$. It follows that $H^*(e_i\bar{\Omega}(F_1)) = 0$ for $i \neq 0$ and $H^*(e_0\bar{\Omega}(F_1)) = F_1$ such that $H^*(\bar{\Omega}(F_1)) = F_1$. Since $\Phi(\phi(x_1)) = x_1e_0$ and $\Phi(q(x_1)) = \beta x_1$ we see that Φ is surjective. The relations $\phi(\beta x_1) = q(x_1)^p$ and $q(\beta x_1) = 0$ shows that $\phi(x_1)$ and $q(x_1)$ generate $\widetilde{\Omega}(K)$ so Φ is also injective.

Assume that n is even and $n \geq 2$. In the following we write $[-]$ for the functor which takes a set to the vector space it generates. We have

$$F_n = S_{\mathscr{A}lg}[x_n, \beta x_n, \lambda^i\beta x_n, \beta\lambda^i\beta x_n \mid i \geq 1]$$

and we find that $\bar{\Omega}(F_n) = F_n \otimes S_{\mathscr{A}lg}[dx_n, d\beta x_n]$. We change basis such that the differential becomes easier to describe:

$$\bar{\Omega}(F_n) = S_{\mathscr{A}lg}[x_n, dx_n] \otimes S_{\mathscr{A}lg}[\beta x_n, d\beta x_n] \otimes S_{\mathscr{A}lg}[\lambda^i \beta x_n - (d\lambda^{i-1}\beta x_n)^{p-1}\lambda^{i-1}\beta x_n, \beta\lambda^i\beta x_n \mid i \geq 1].$$

By the Künneth formula we find that $H^*(\bar{\Omega}(F_n))$ equals

$$S_{\mathscr{A}lg}[x_n^p, x_n^{p-1}dx_n] \otimes S_{\mathscr{A}lg}[\lambda^i \beta x_n - (d\lambda^{i-1}\beta x_n)^{p-1}\lambda^{i-1}\beta x_n, \beta\lambda^i\beta x_n \mid i \geq 1].$$

The algebra $\widetilde{\Omega}(F_n)$ is generated by the classes $\phi(x_n)$, $\phi(\lambda^i \beta x_n)$, $q(x_n)$ and $q(\lambda^i \beta x_n)$ where $i \geq 0$. We see that Φ maps these generators to the free generators for the cohomology of $\bar{\Omega}(F_n)$. Hence Φ is an isomorphism. The case where n is odd and $n \geq 3$ is similar.

LEMMA 4.10. *There is an isomorphism of rings as follows*

$$\alpha : \mathsf{F}_p[x]/(x^p - x) \to (\mathsf{F}_p)^p; \qquad x \mapsto (0, 1, 2, \ldots, p-1)$$

where $\mathsf{F}_p[x]$ *is the polynomial ring in one variable* x *of degree zero and* $(\mathsf{F}_p)^p$ *is the* p*-fold Cartesian product of* F_p *by itself.*

PROOF. Use the factorization $x^p - x = \prod_{n\in\mathsf{F}_p}(x-n)$ and the Chinese remainder theorem.

REMARK 4.11. Let $e_n = \alpha^{-1}(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 on the nth place for $n \in \mathsf{F}_p$. Clearly $e_n e_m = 0$ for $n \neq m$, $e_n^2 = e_n$ and $\sum e_n = 1$. Also $xe_n = ne_n$. Finding eigenvectors for $xf(x) = nf(x)$ and normalizing one gets the following:

$$e_0 = 1 - x^{p-1}, \quad e_m = -\sum_{i=1}^{p-1}\left(\frac{x}{m}\right)^i, \qquad m \neq 0.$$

5. Steenrod diagonal elements

In this section we use the functor R_+ of [6] to define a functor $R : \mathscr{K} \to \mathscr{K}$. We need R for a description of ℓ given in the next section. Let K denote an unstable $\mathscr{A}$-algebra and consider $\mathsf{F}_p[u]$ with $|u| = 2$ an object in $\mathscr{K}$ by the isomorphism $\mathsf{F}_p[u] \cong H^*(B\mathsf{T})$.

DEFINITION 5.1. For $x \in K$ and $\epsilon = 0, 1$ we define $St_\epsilon(x) \in \mathsf{F}_p[u] \otimes K$ by

$$St_\epsilon(x) = u^{-\epsilon\hat{\sigma}(x)} \sum_{i\geq 0}(-u^{p-1})^{[|x|/2]-i} \otimes \beta^\epsilon P^i x.$$

Note that the terms where the total exponent of u is negative has $\beta^\epsilon P^i x = 0$. Let $R(K) \subseteq \mathsf{F}_p[u] \otimes K$ be the sub-F_p-algebra generated by $u \otimes 1$ and $St_\epsilon(x)$ for all $x \in K$ and $\epsilon = 0, 1$.

THEOREM 5.2. *For each $\theta \in \mathscr{A}$ one has $\theta R(K) \subseteq R(K)$. Thus R is a functor $R : \mathscr{K} \to \mathscr{K}$. The explicit formulas are as follows where $n = [|x|/2]$ and $\epsilon = 0, 1$:*

$$P^i St_\epsilon(x) = \sum_t \binom{(p-1)(n-t) + \epsilon\sigma(x)}{i - pt} u^{(p-1)(i-pt)} St_\epsilon(P^t x)$$

$$- \epsilon(-1)^{\sigma(x)} \sum_t \binom{(p-1)(n-t) - 1 + \sigma(x)}{i - pt - 1}$$

$$\cdot\, u^{(p-1)(i-pt)-1+(2-p)\sigma(x)} St_0(\beta P^t x),$$

$$\beta St_\epsilon(x) = (1-\epsilon) u^{\hat{\sigma}(x)} St_1(x).$$

PROOF. The formula for the Bockstein operation follows directly by the definition of $St_\epsilon(x)$. We use results from [6] to prove the other formula. By [13] we have that $\mathsf{F}_p[u, u^{-1}]$ is an $\mathscr{A}$-algebra with $\beta = 0$ and

$$P^i u^j = \binom{j}{i} u^{j+i(p-1)}; \qquad i, j \in \mathsf{Z}; \quad i \geq 0.$$

Here the following extended definition of binomial coefficients is used where $r \in \mathsf{R}$ and $k \in \mathsf{Z}$.

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\ldots(r-k+1)}{k!}, & k > 0 \\ 1, & k = 0 \\ 0, & k < 0 \end{cases}$$

Let $\Delta = \Lambda(a) \otimes \mathsf{F}_p[b, b^{-1}]$ with $|a| = 2p-3$, $|b| = 2p-2$ be the $\mathscr{A}$-algebra introduced in [6] (2.6). That is $\beta a = b$ and

$$P^i(b^j) = (-1)^i \binom{(p-1)j}{i} b^{i+j},$$

$$P^i(ab^{j-1}) = (-1)^i \binom{(p-1)j - 1}{i} ab^{i+j-1}.$$

Note that we have changed the names of the generators. In [6] they were named u and v instead of a and b. We define an additive transfer map as follows:

$$\tau : \Delta \to \mathsf{F}_p[u, u^{-1}]; \qquad b^j \mapsto 0; \qquad ab^{j-1} \mapsto (-u^{p-1})^j u^{-1}.$$

Note that $|\tau| = -1$. A direct verification shows that τ is $\mathscr{A}$-linear.

A functor R_+ from the category of graded $\mathscr{A}$-modules to itself is constructed in [6]. In the case of an unstable $\mathscr{A}$-algebra K it comes with an $\mathscr{A}$-linear map $f : R_+K \to \Sigma\Delta \otimes K$ defined by [6] (3.1), (3.2). The composite

$$R_+K \xrightarrow{f} \sigma\Delta \otimes K \xrightarrow{\Sigma\tau\otimes 1} \Sigma\mathsf{F}_p[u, u^{-1}] \otimes K$$

is given by

$$sb^k \otimes x \mapsto -s\sum_j (-u^{p-1})^{k-j} u^{-1} \otimes \beta P^j x,$$

$$sab^{k-1} \otimes x \mapsto s\sum_j (-u^{p-1})^{k-j} u^{-1} \otimes P^j x.$$

Especially $sb^n \otimes x \mapsto -su^{\sigma(x)} St_1(x)$ and $sab^{n-1} \otimes x \mapsto su^{-1} St_0(x)$ where $n = [|x|/2]$. The formulas [6] (3.4), (3.5) for the $\mathscr{A}$-action on R_+M gives the following formulas for the $\mathscr{A}$-action on $u^{\sigma(x)} St_1(x)$ and $u^{-1} St_0(x)$:

$$\begin{aligned} P^i(u^{\sigma(x)} St_1(x)) = \sum_t \binom{(p-1)(n-t)}{i-pt} u^{(p-1)(i-pt)-\sigma(x)} St_1(P^t x) \\ - \sum_t (-1)^{\sigma(x)} \binom{(p-1)(n-t)-1}{i-pt-1} \\ \cdot u^{(p-1)(i-pt-\sigma(x))-1} St_0(\beta P^t x), \end{aligned}$$

$$P^i(u^{-1} St_0(x)) = \sum_t \binom{(p-1)(n-t)-1}{i-pt} u^{(p-1)(i-pt)-1} St_0(P^t x).$$

This proves the result directly for $\sigma(x) = 0$ and $\epsilon = 1$. By the Cartan formula applied to $uu^{-1} St_\epsilon(x)$ we have that $P^i St_\epsilon(x) = uP^i(u^{-1} St_\epsilon(x)) + u^p P^{i-1}(u^{-1} St_\epsilon(x))$. By combining this with the formulas above we get the result in the other cases.

6. A pullback description of the functor ℓ

In this section we describe $\ell(K)$ as a pullback in the case where K is a free object in $\mathscr{K}$. We start by a result on cohomology of Eilenberg-MacLane spaces.

Recall that a sequence of integers $I = (\epsilon_1, s_1, \epsilon_2, s_2, \dots, \epsilon_k, s_k, \epsilon_{k+1})$ with $s_i \geq 0$ and $\epsilon_i \in \{0, 1\}$ is called admissible if $s_i \geq ps_{i+1} + \epsilon_{i+1}$ and $s_k \geq 1$ or if $k = 0$ when $I = (\epsilon)$. The degree of I is defined as $|I| = \sum \epsilon_j + \sum 2s_j(p-1)$ and the excess is defined recursively by $e((\epsilon, s), J) = 2s + \epsilon - |J|$. We use the following notation $P^I = \beta^{\epsilon_1} P^{s_1} \beta^{\epsilon_2} P^{s_2} \dots \beta^{\epsilon_k} P^{s_k} \beta^{\epsilon_{k+1}}$.

LEMMA 6.1. *The cohomology ring of the Eilenberg-MacLane space $K(\mathsf{F}_p, n)$ can be written in the following form when $n \geq 2$:*

$$H^*(K(\mathsf{F}_p, n)) = S_{\mathscr{F}}[P^I \iota_n \mid I \text{ is admissible}, e(I) \leq n-2, \epsilon_1 = 0].$$

Furthermore, $H^*(K(\mathsf{F}_p, 1)) = S_{\mathscr{F}}[\iota_1]$ and $H^*(K(\mathsf{F}_p, 0)) = S_{\mathscr{F}}[\iota_0]$.

PROOF. The cases $n = 0, 1$ are trivial. Assume that $n \geq 2$ and define the set

$$A(n) = \{ I \mid I \text{ is admisseble}, e(I) \leq n-1, |I| + n \text{ is odd} \}.$$

Remark that if $I \in A(n)$ then $(0, (|I|+n-1)/2, I) \in A(n)$. To see this write $I \in A(n)$ as $I = (\epsilon, s, I')$. Then $e(I) = 2s + \epsilon - |I'| \leq n-1$ or equivalently $2sp + 2\epsilon - |I| \leq n-1$ such that the sequence $(0, (|I|+n-1)/2, I)$ is admissible. Its excess is $n-1$ and its degree plus n is odd since $p-1$ is even.

By Cartan's computation (a special case of [9], Theorem 10.3) we have that $H^* B^n \mathsf{F}_p$ is the free graded commutative algebra on the set

$$B = \{ P^J \iota_n \mid J \text{ is admissible}, e(J) < n \text{ or } (e(J) = n \text{ and } \epsilon_1 = 1) \}.$$

Assume that $P^I \iota_n$ belongs to the set in the statement of the lemma. Then $P^I \iota_n$ and $\beta P^I \iota_n$ belongs to B. By the remark we see that if $|I| + n$ is even then $\beta^\epsilon \lambda^i \beta P^I \iota_n \in B$ and if $|I|+n$ is odd then $\beta^\epsilon \lambda^i P^I \iota_n \in B$ for $\epsilon = 0, 1$ and $i \geq 1$.

Conversely, assume that $P^J \iota_n \in B$. If $e(J) \leq n-2$ or $e(J) = n-1$ and $\epsilon_1 = 1$ it is clearly one of the generators described in the lemma. It suffices to handle the case $e(J) = n-1, \epsilon_1 = 0$ since the case $e(J) = n, \epsilon_1 = 1$ then follows. Write J as $J = (0, s, J')$ where $e(J) = 2s - |J|' = n-1$. Then $2s = n + |J'| - 1$ such that $P^J \iota_n = \lambda P^{J'} \iota_n$ and $e(J) \leq e(J')$. We can continue this process until the next ϵ equals one or the excess drops below $n-1$.

PROPOSITION 6.2. *For any object K in $\mathscr{K}$ there is natural morphism of F_p-algebras $\Delta : \ell(K) \to \mathsf{F}_p[u] \otimes K$ defined by*

$$\phi(x) \mapsto St_0(x), \qquad q(x) \mapsto St_1(x), \qquad \delta(x) \mapsto 0, \qquad u \mapsto u \otimes 1.$$

The image of this morphism is $\mathrm{Im}(\Delta) = R(K)$.

PROOF. We check that (10)–(21) are mapped to zero by the formulas defining Δ. Since $\delta(x)$ is mapped to zero this is trivial for all elements except (14), (15), (16) and (20).

By the Cartan formula and $\left[\frac{|ab|}{2}\right] = \left[\frac{|a|}{2}\right] + \left[\frac{|b|}{2}\right] + \sigma(a)\sigma(b)$ one verifies that

$$St_0(ab) = (-u^{p-1})^{\sigma(a)\sigma(b)} St_0(a) St_0(b),$$

$$St_1(ab) = (-u^{p-1})^{\sigma(a)\sigma(b)} (u^{\sigma(b)} St_1(a) St_0(b) + (-u)^{\sigma(a)} St_0(a) St_1(b))$$

such that (14) and (15) are mapped to zero. Lemma 2.7 implies that (16) and (20) are mapped to zero.

PROPOSITION 6.3. *If K is a free object in $\mathcal{K}$ then* $\ker(\Delta) = I_\delta(K)$.

PROOF. Assume that $K = S_{\mathcal{K}}(V)$ for a non negatively graded vector space V. We must show that $\bar{\Delta} : \ell(K)/I_\delta(K) \to \mathsf{F}_p[u] \otimes K$ is injective.

The algebra $\ell(K)/I_\delta(K)$ has generators $\phi(x)$, $q(x)$ for $x \in K$ and u. The relations are that ϕ and q are additive and that (14), (15), (16) and (20) equals zero. Let $\{v_s \mid s \in S\}$ denote a basis for V. By Lemma 6.1 we find that $K = S_{\mathcal{F}}(W)$ where W is the graded vector space with basis

$$B = \{\, P^I v_s \mid I \text{ admissible}, e(I) \leq |v_s| - 2, \epsilon_1 = 0, s \in S \,\}.$$

We see that the following elements are algebra generators for $\ell(K)/I_\delta(K)$ where $a \in B^0$, $b \in B^1$, $v \in B^{\mathrm{odd},*\geq 3}$, $w \in B^{\mathrm{even},*\geq 2}$ and $i \geq 0$:

$$u, \phi(a), \phi(b), q(\beta b),$$

$$\phi(\beta v), \phi(\lambda^i v), q(\beta v), q(\lambda^i v),$$

$$\phi(w), \phi(\lambda^i \beta w), q(w), q(\lambda^i \beta w).$$

We claim that these generators are mapped to algebraically independent elements in $\mathsf{F}_p[u] \otimes K$. By the formulas defining Δ we see that it suffices to check this claim in the case where V is one dimensional. So assume that $K = S_{\mathcal{K}}[\iota_n]$ where $|\iota_n| = n$.

For any n we have $u \mapsto u \otimes 1$. For $n = 0$ we have $\phi(\iota_0) \mapsto 1 \otimes \iota_0$ and for $n = 1$ we have $\phi(\iota_1) \mapsto 1 \otimes \iota_1$, $q(\iota_1) \mapsto 1 \otimes \beta\iota_1$ so in these two cases the claim holds.

Assume that $n \geq 2$. The algebra generators are mapped as follows modulo elements in the ideal $(u^{p-1} \otimes 1)$:

$$\begin{aligned} \phi(\beta v) &\mapsto 1 \otimes (\beta v)^p, & q(\beta v) &\mapsto -u^{p-2} \otimes \beta P^{(|v|-1)/2} \beta v, \\ \phi(\lambda^i v) &\mapsto 1 \otimes \lambda^{i+1} v, & q(\lambda^i v) &\mapsto 1 \otimes \beta \lambda^{i+1} v, \\ \phi(w) &\mapsto 1 \otimes w^p, & q(w) &\mapsto -u^{p-2} \otimes \beta P^{|w|/2-1} w, \\ \phi(\lambda^i \beta w) &\mapsto 1 \otimes \lambda^{i+1} \beta w, & q(\lambda^i \beta w) &\mapsto 1 \otimes \beta \lambda^{i+1} \beta w. \end{aligned}$$

If $|I| + n$ is odd we must look closer at $\beta P^{(|I|+n-1)/2} \beta P^I \iota_n$. Write I as $I = (0, s, I')$. We have $e(I) = 2s - |I'| \leq n - 2$ which implies that $(0, (|I| + n - 1)/2, 1, s, I')$ is admissible. Its excess equals $n - 2$ and we see that $P^{(|I|+n-1)/2} \beta P^I \iota_n \in B^{\text{even}}$.

If $|I| + n$ is even we must look at $\beta P^{(|I|+n-2)/2} P^I \iota_n$. As in the odd case we see that $P^{(|I|+n-2)/2} P^I \iota_n \in B^{\text{even}}$. However there is no β between the first two P-operations from the left.

We conclude that the claim holds for $n \geq 2$ which completes the proof.

In the following K denotes an object in $\mathscr{K}$. Before stating the main theorem we need some definitions and lemmas.

LEMMA 6.4. *Let $a_1, \ldots, a_p \in K$ be elements of odd degree and define the following element in $I_\delta(K)$:*

$$D(a_1, \ldots, a_p) = \sum_{i=2}^{p} \delta(a_1 a_i) \delta(a_2) \ldots \widehat{\delta(a_i)} \ldots \delta(a_p).$$

where the hat means that the factor is left out. Then for any permutation $\tau \in \Sigma_p$ one has $D(a_1, \ldots, a_p) = D(a_{\tau(1)}, \ldots, a_{\tau(p)})$. The element is mapped as follows under the map $Q : \ell(K) \to \bar{\Omega}(K)$:

$$D(a_1, \ldots, a_p) \mapsto \sum_{i=1}^{p} a_i da_1 \ldots \widehat{da_i} \ldots da_p.$$

PROOF. We first show the invariance under permutation. Since the degree of $\delta(a_i)$ is even $D(a_1, \ldots, a_p)$ is invariant under permutations fixing a_1. Thus it suffices to show that $D(a_1, a_2, \ldots, a_p) = D(a_2, \ldots, a_p, a_1)$. We prove the following more general formula for $n \geq 3$:

$$\begin{aligned} &\sum_{i=2}^{n} \delta(a_1 a_i) \delta(a_2) \ldots \widehat{\delta(a_i)} \ldots \delta(a_n) \\ &= \sum_{j=3}^{n} \delta(a_2 a_j) \delta(a_1) \delta(a_3) \ldots \widehat{\delta(a_j)} \ldots \delta(a_n) - (n-1) \delta(a_2 a_1) \delta(a_3) \ldots \delta(a_n). \end{aligned}$$

The proof is by induction on n. For $n = 3$ we have

$$\begin{aligned}\delta(a_1a_2)\delta(a_3) + \delta(a_1a_3)\delta(a_2) &= \delta(a_1a_2)\delta(a_3) - \delta(a_3a_1)\delta(a_2)\\ &= 2\delta(a_1a_2)\delta(a_3) + \delta(a_2a_3)\delta(a_1)\\ &= -2\delta(a_2a_1)\delta(a_3) + \delta(a_2a_3)\delta(a_1)\end{aligned}$$

where we used (13) at the second equality sign. Assume that the formula holds for $n - 1$. Then we have

$$\begin{aligned}&\sum_{i=2}^{n} \delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_n)\\ &= \left(\sum_{i=2}^{n-1} \delta(a_1a_i)\delta(a_2)\dots\widehat{\delta(a_i)}\dots\delta(a_{n-1})\right)\delta(a_n) + \delta(a_1a_n)\delta(a_2)\dots\delta(a_{n-1})\\ &= \left(\sum_{j=3}^{n-1} \delta(a_2a_j)\delta(a_1)\delta(a_3)\dots\widehat{\delta(a_j)}\dots\delta(a_{n-1})\right)\delta(a_n)\\ &\qquad - (n-2)\delta(a_2a_1)\delta(a_3)\dots\delta(a_{n-1})\delta(a_n) + \delta(a_1a_n)\delta(a_2)\dots\delta(a_{n-1}).\end{aligned}$$

Since $\delta(a_1a_n)\delta(a_2) + \delta(a_2a_1)\delta(a_n) = \delta(a_2a_n)\delta(a_1)$ by relation (13) the sum of the last two terms above equals

$$-(n-1)\delta(a_2a_1)\delta(a_3)\dots\delta(a_n) + \delta(a_2a_n)\delta(a_1)\dots\delta(a_{n-1})$$

and we recover the formula for n.

We use that $d(a_1a_i) = a_i da_1 - a_1 da_i$ to compute the image under Q:

$$\begin{aligned}D(a_1,\dots,a_p) \mapsto &\sum_{i=2}^{p} d(a_1a_i)da_2\dots\widehat{da_i}\dots da_p\\ &= \sum_{i=2}^{p} a_i da_1\dots\widehat{da_i}\dots da_p - (p-1)a_1da_2\dots da_p.\end{aligned}$$

DEFINITION 6.5. For any non negative integer n we let $B(n)$ denote the following set:

$$B(n) = \{(\beta_1,\dots,\beta_p) \in \mathsf{Z}^p \mid \forall i : \beta_i \geq 0, \beta_1 + \dots + \beta_p = n, \exists i, j : \beta_i \neq \beta_j\}.$$

The cyclic group on p elements C_p act on $B(n)$ by cyclic permutation of coordinates. For $x \in K$ we define the following elements in $I_\delta(K)$:

$$D_0^n(x) = -\sigma(x) \sum D(P^{\beta_1}(x), P^{\beta_2}(x), \dots, P^{\beta_p}(x)),$$

$$D_1^n(x) = \hat{\sigma}(x) \sum \delta(P^{\beta_1}(x) P^{\beta_2}(x) \dots P^{\beta_p}(x))$$

where both sums are taken over $\beta \in B(n)/C_p$. Note that $D_0^n(x)$ is well defined by Lemma 6.4

LEMMA 6.6. *For any $x \in K$ the following formulas hold in $\bar{\Omega}(K)$:*

$$P^i \circ Q(\phi(x)) = Q(\phi(P^{i/p}x) + D_0^i(x)), \tag{29}$$

$$P^i \circ Q(q(x)) = Q(q(P^{i/p}x) + D_1^i(x)) \tag{30}$$

where by convention $P^t = 0$ when t is a rational number which is not a non negative integer.

PROOF. We first prove (29). Recall that $Q(\phi(x)) = x^p + \lambda x - x(dx)^{p-1}$. We have $P^i \lambda x = \lambda P^{i/p} x$ by Lemma 2.7 and also $P^i(x^p) = (P^{i/p}x)^p$ so it suffices to prove the following for $|x|$ odd:

$$P^i(x(dx)^{p-1}) = (P^{i/p}x)(dP^{i/p}x)^{p-1} - Q(D_0^i(x)).$$

By the Cartan formula we have

$$P^i(x(dx)^{p-1}) = \sum P^{\beta_1}(x) dP^{\beta_2}(x) \dots dP^{\beta_p}(x)$$

where we sum over the tuples $(\beta_1, \dots, \beta_p)$ with $\sum \beta_j = i$. The cyclic group C_p acts on the set of such tuples and an orbit has length 1 or p. Arranging the terms according to this the result follows by the definition of $D_0^i(x)$ and Lemma 6.4.

For the proof of (30) recall that $Q(q(x)) = x^{p-1}dx + \beta\lambda x$. We have $P^i(\beta\lambda x) = \beta\lambda(P^{i/p}x) + (\beta P^{(i-1)/p}x)^p$ by Lemma 2.7 so when $|x|$ is odd we are done. For $|x|$ even we must show that

$$P^i(x^{p-1}dx) = (P^{i/p}x)^{p-1} dP^{i/p}x + Q(D_1^i(x)).$$

This follows by the Cartan formula and a similar argument on orbits as the above.

THEOREM 6.7. *For any object K in $\mathscr{K}$ there is an $\mathscr{A}$-module structure on $\ell(K)$ such that ℓ becomes a functor $\ell : \mathscr{K} \to \mathscr{K}$. The explicit formulas for the* *action are as follows where $x \in K$, $n = [|x|/2]$ and $i \geq 0$. Firstly, the action on $\phi(x)$ is given by:*

$$P^i\phi(x) = D_0^i(x) + \sum_t \binom{(p-1)(n-t)}{i-pt} u^{(p-1)(i-pt)}\phi(P^t x),$$

$$\beta\phi(x) = u^{\hat{\sigma}(x)}(q(x) - \delta(x)^{p-2}\delta(x\beta x)).$$

Secondly, the action on $q(x)$ is given by:

$$\begin{aligned} P^i q(x) = D_1^i(x) + \sum_t &\binom{(p-1)(n-t)+\sigma(x)}{i-pt} u^{(p-1)(i-pt)} q(P^t x) \\ &- (-1)^{\sigma(x)} \sum_t \binom{(p-1)(n-t)-1+\sigma(x)}{i-pt-1} \\ &\cdot u^{(p-1)(i-pt)-1+(2-p)\sigma(x)}\phi(\beta P^t x), \end{aligned}$$

$$\beta q(x) = -\delta(x^{p-1}\beta x).$$

Thirdly, the actions on $\delta(x)$ and u are as follows:

$$P^i\delta(x) = \delta(P^i x), \qquad \beta\delta(x) = -\delta(\beta x), \qquad P^1 u = u^p, \qquad \beta u = 0.$$

Furthermore the maps Q and Δ becomes $\mathscr{A}$-linear and there is a commutative diagram in $\mathscr{K}$ as follows:

$$\begin{array}{ccc} \ell(K) & \xrightarrow{\Delta} & R(K) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle p_1} \\ \ker(d) & \xrightarrow{p_2} & K \end{array}$$

where the morphisms p_1 and p_2 are given by $p_1(u) = 0$, $p_1(x) = x$, $p_2(dx) = 0$, $p_2(x) = x$ for $x \in K$. Finally, if K is a free object in $\mathscr{K}$ then the diagram is a pullback square.

PROOF. By the definition of Δ and Q there is a commutative diagram as stated in the category of F_p-algebras. We first prove that this diagram is a pullback when K is a free object in $\mathscr{K}$.

By Lemma 6.1 and Theorem 4.9 the map Φ is an isomorphism. So by Theorem 4.5 the kernel of Q is the ideal $(u) \subseteq \ell(K)$. The kernel of p_1 is the ideal $(u \otimes 1) \subseteq R(K)$ so it suffices to show that the restriction of the map Δ to these kernels $\Delta| : (u) \to (u \otimes 1)$ is an isomorphism. It is surjective since Δ is surjective and $\Delta(u) = u \otimes 1$. By Proposition 6.3 we have $\ker(\Delta) = I_\delta(K)$

such that $\ker(\Delta|) = (u) \cap I_\delta(K)$. Let $x \in (u) \cap I_\delta(K)$. We can write $x = uz$ for some $z \in \ell(K)$. Since $0 = \Delta(uz) = u\Delta(z)$ we have $\Delta(z) = 0$ so $z \in I_\delta(K)$ and $x = uz = 0$. Thus $(u) \cap I_\delta(K) = 0$ and $\Delta|$ is injective.

When K is a free object the pullback defines an $\mathscr{A}$-module structure on $\ell(K)$. By Theorem 5.2 and Lemma 6.6 we see that the stated formulas describe this $\mathscr{A}$-action. A standard naturality argument now proves the statements for general objects K in $\mathscr{K}$.

7. Homotopy orbits of T-spaces

In this section we list some results on homotopy orbits of T-spaces. They are all similar to results for $p = 2$ considered in [2] and we often refer to the proofs given there. In the entire section Y denotes a T-space. We write C_n for the cyclic group of order n. We let u of degree $|u| = 2$ and v of degree $|v| = 1$ denote algebra generators as follows: $H^*\mathsf{T} = \Lambda(v)$, $H^*B\mathsf{T} = \mathsf{F}_p[u]$ and $H^*BC_{p^n} = \Lambda(v) \otimes \mathsf{F}_p[u]$.

PROPOSITION 7.1. *The fibration $Y \to E\mathsf{T} \times_{\mathsf{T}} Y \to B\mathsf{T}$ has the following Leray-Serre spectral sequence:*

$$E_2^{**} = H^*(B\mathsf{T}) \otimes H^*(Y) \Rightarrow H^*(E\mathsf{T} \times_{\mathsf{T}} Y).$$

The differential in the E_2-term is given by

$$d_2 : H^*(Y) \to uH^*(Y); \qquad d_2(y) = ud(y)$$

where d is the differential associated to the T-action (see Proposition 2.3).

PROOF. Similar to the proof of [2] Proposition 3.3.

DEFINITION 7.2. Let $E_\infty Y = E\mathsf{T} \times_{\mathsf{T}} Y$ and define

$$E_n Y = E\mathsf{T} \times_{C_{p^n}} Y \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

For nonnegative integers n and m with $m > n$ define the maps

$$q_m^n : H^*E_mY \to H^*E_nY, \qquad \tau_n^m : H^*E_nY \to H^*E_mY$$

by letting q_m^n be the map induced by the quotient map and τ_n^m be the transfer map. Also define $q_\infty^n : H^*E_\infty Y \to H^*E_nY$ as the map induced by the quotient.

The following theorem is inspired by a result of Tom Goodwillie which can be found in [8] p. 279.

THEOREM 7.3. *There is a commutative diagram as follows for any $m \geq 1$:*

$$\begin{array}{ccc} E_m Y & \xrightarrow{Q} & E_\infty Y \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle pr_1}\downarrow \\ BC_{p^m} & \xrightarrow{Bj} & B\mathsf{T} \end{array} \tag{31}$$

Here Q denotes the quotient map and $j : C_{p^m} \hookrightarrow \mathsf{T}$ the inclusion. The diagram gives rise to an isomorphism.

$$\Theta : H^*(BC_{p^m}) \otimes_{H^*(B\mathsf{T})} H^*(E_\infty Y) \cong H^*(E_m Y); \qquad x \otimes y \mapsto pr_1^*(x) q_\infty^m(y)$$

The transfer map $\tau_m^{m+1} : H^* E_m Y \to H^* E_{m+1} Y$ is zero on elements of the form $\Theta(1 \otimes y)$ and the identity on elements of the form $\Theta(v \otimes y)$. We get an isomorphism

$$\operatorname{colim} H^* E_m Y = v H^* E_\infty Y \cong \widetilde{H}^*(\Sigma(E_\infty Y)_+).$$

PROOF. Similar to the proof of [2] Theorem 4.2.

We use the above theorem to give a convenient definition of the T-transfer:

DEFINITION 7.4. For non negative n the T-transfer $\tau_n^\infty : H^* E_n Y \to H^* E_\infty Y$ is defined as the following composite:

$$H^* E_n Y \longrightarrow \operatorname{colim} H^* E_m Y \xrightarrow{v^{-1}} H^* E_\infty Y.$$

The colimit is taken over the transfer maps τ_m^{m+1}. Note that $|\tau_n^\infty| = -1$.

PROPOSITION 7.5. *Frobenius reciprocity holds for any $n \geq 0$:*

$$\tau_n^\infty(q_\infty^n(x) y) = (-1)^{|x|} x \tau_n^\infty(y).$$

Furthermore the following composition formulas hold.

$$\tau_0^\infty \circ q_\infty^0 = 0, \qquad q_\infty^0 \circ \tau_0^\infty = d.$$

PROOF. Similar to the proof of [2] Proposition 4.6, 4.7 and 4.8.

PROPOSITION 7.6. *There is always an inclusion* $\operatorname{Im}(q_\infty^0) \subseteq \ker(d)$. *If we have equality* $\operatorname{Im}(q_\infty^0) = \ker(d)$ *then the Leray-Serre spectral sequence of the fibration $Y \to E\mathsf{T} \times_{\mathsf{T}} Y \to B\mathsf{T}$ collapses at the E_3-term.*

PROOF. By Proposition 7.5 we have $d \circ q_\infty^0 = q_\infty^0 \circ \tau_0^\infty \circ q_\infty^0 = 0$. The collapse statement follows by Proposition 7.1.

DEFINITION 7.7. Put $\zeta_p = exp(2\pi i/p)$ and define the map

$$f'_Y : \mathsf{T} \times Y \to E\mathsf{T} \times Y^p; \qquad (z, y) \mapsto (ze, zy, \zeta_p zy, \zeta_p^2 zy, \dots, \zeta_p^{p-1} zy).$$

We let C_p act on the space to the left by $\zeta_p \cdot (z, y) = (\zeta_p z, y)$ and on the space to the right by $\zeta_p \cdot (e, y_1, \dots, y_p) = (\zeta_p e, y_2, \dots, y_p, y_1)$. Then the above map is C_p-equivariant. Passing to the quotients we get a map

$$f_Y : \mathsf{T}/C_p \times Y \to E\mathsf{T} \times_{C_p} Y^p.$$

Note that this map is natural in Y with respect to C_p-equivariant maps.

Recall the followings facts on the order p cyclic construction [10], [9] and [12]. For any space X with homology of finite type there is a natural isomorphism

$$H^*(E\mathsf{T} \times_{C_p} X^p) \cong H^*(C_p; H^*(X)^{\otimes p})$$

where C_p acts on $H^*(X)^{\otimes p}$ by cyclic permutation with the usual sign convention. For a homogeneous element $y \in H^*X$ the C_p invariant $y^{\otimes p}$ defines an element $1 \otimes y^{\otimes p}$ in the zeroth cohomology group of C_p. Let $N = 1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1}$ be the norm element in the group ring $\mathsf{F}_p[C_p]$. If $x_1, \dots, x_p \in H^*X$ are homogeneous elements, which are not all equal, then the invariant $Nx_1 \otimes \cdots \otimes x_p$ also defines an element $1 \otimes Nx_1 \otimes \cdots \otimes x_p$ in the zeroth cohomology group of C_p.

THEOREM 7.8. *The following formula holds where $\delta_{i,j}$ denotes the Kronecker delta: $f_Y^*(1 \otimes y^{\otimes p}) = 1 \otimes y^p + v \otimes y^{p-1}dy + \delta_{p,3} v \otimes \beta\lambda y$.*

PROOF. We write Y_0 for the space Y with trivial T-action. We first prove the theorem in the special case $Y = Y_0$. Here the differential is zero. There is a factorization

$$f_{Y_0} : \mathsf{T}/C_p \times Y_0 \xrightarrow{i \times 1} E\mathsf{T}/C_p \times Y_0 \xrightarrow{\times \Delta} E\mathsf{T} \times_{C_p} Y_0^p.$$

By this and the formula for the Steenrod diagonal, [12] p. 119 & Errata, the result follows.

Next we prove the following formula for a general T-space:

$$f_Y^*(1 \otimes Nx_1 \otimes \cdots \otimes x_p) = v \otimes d(x_1 \dots x_p). \tag{32}$$

There is a commutative diagram as follows:

$$\begin{array}{ccc} H^*(\mathsf{T}/C_p \times Y) & \xleftarrow{f_Y^*} & H^*(E\mathsf{T} \times_{C_p} Y^p) \\ \uparrow{\scriptstyle \tau_0^1 \otimes 1} & & \uparrow{\scriptstyle \tau_0^1} \\ H^*(\mathsf{T} \times Y) & \xleftarrow{f_Y'^*} & H^*(E\mathsf{T} \times Y^p) \end{array}$$

The lower horizontal map is given by

$$f_Y'^*(1 \otimes x_1 \otimes \cdots \otimes x_p) = \prod_{i=1}^{p}(1 \otimes x_i + v \otimes dx_i)$$

as seen by the factorization

$$f_Y' : \mathsf{T} \times Y \xrightarrow{\Delta_2} (\mathsf{T} \times Y)^2 \xrightarrow{pr_1 \times \Delta_p} \mathsf{T} \times (\mathsf{T} \times Y)^p$$
$$\xrightarrow{i \times \eta^p} E\mathsf{T} \times Y^p \xrightarrow{1 \times 1 \times \zeta_p \times \cdots \times \zeta_p^{p-1}} E\mathsf{T} \times Y^p.$$

The norm class is hit by the transfer and by finding the coefficient to v in the above formula (32) follows.

Finally we prove the Theorem for a general T-space Y. Because of the degrees $f_{\mathsf{T}}^*(1 \otimes v^{\otimes p}) = 0$. The two projection maps $pr_1 : \mathsf{T} \times Y_0 \to \mathsf{T}$ and $pr_2 : \mathsf{T} \times Y_0 \to Y_0$ are T-equivariant. Thus we can use naturality together with the case $Y = Y_0$ and the above equation to find the equations below

$$f_{\mathsf{T}\times Y_0}^*(1 \otimes (1 \otimes y)^{\otimes p}) = 1 \otimes 1 \otimes y^p + \delta_{p,3} v \otimes 1 \otimes \beta\lambda y,$$

$$f_{\mathsf{T}\times Y_0}^*(1 \otimes (v \otimes 1)^{\otimes p}) = f_{\mathsf{T}\times Y_0}^*(1 \otimes (v \otimes dy)^{\otimes p}) = 0.$$

The action map $\eta : \mathsf{T} \times Y_0 \to Y$ is also an T-equivariant map, hence by naturality we have a commutative diagram

$$\begin{array}{ccc}
\mathsf{T}/C_p \times (\mathsf{T} \times Y_0) & \xrightarrow{f_{\mathsf{T}\times Y_0}} & E\mathsf{T} \times_{C_p} (\mathsf{T} \times Y_0)^p \\
\downarrow{\scriptstyle 1\times\eta} & & \downarrow{\scriptstyle 1\times\eta^p} \\
\mathsf{T}/C_p \times Y & \xrightarrow{f_Y} & E\mathsf{T} \times_{C_p} Y^p
\end{array}$$

We compute the pull back of the class $1 \otimes y^{\otimes p}$ to the cohomology of the upper left corner. First we find

$$\begin{aligned}
(1 \times \eta^p)^*(1 \otimes y^{\otimes p}) &= 1 \otimes (1 \otimes y + v \otimes dy)^{\otimes p} \\
&= 1 \otimes (1 \otimes y)^{\otimes p} + 1 \otimes (v \otimes dy)^{\otimes p} \\
&\quad + \sum_{i=1}^{p-1} 1 \otimes N(1 \otimes y)^{\otimes i} \otimes (v \otimes dy)^{\otimes (p-i)}.
\end{aligned}$$

By (32) we can compute $f_{\mathsf{T}\times Y_0}^*$ applied to the norm element terms. Only the

$i = p-1$ term contributes.

$$\begin{aligned} f^*_{\mathsf{T}\times Y_0}(1\otimes N(1\otimes y)^{\otimes(p-1)}\otimes(v\otimes dy)) &= v\otimes d_{\mathsf{T}\times Y_0}(v\otimes y^{p-1}dy)\\ &= v\otimes(d_{\mathsf{T}}(v)\otimes y^{p-1}dy\\ &\qquad + v\otimes d_{Y_0}(y^{p-1}dy))\\ &= v\otimes 1\otimes y^{p-1}dy \end{aligned}$$

Altogether we have

$$\begin{aligned} (1\otimes\eta^*)\circ f^*_Y(1\otimes y^{\otimes p}) &= f^*_{\mathsf{T}\times Y_0}\circ(1\times\eta^p)^*(1\otimes y^{\otimes p})\\ &= f^*_{\mathsf{T}\times Y_0}(1\otimes(1\otimes y)^{\otimes p}) + v\otimes 1\otimes y^{p-1}dy. \end{aligned}$$

Let $\gamma : Y \to \mathsf{T}\times Y$ be the map given by $y \mapsto (1, y)$. We have $\gamma^*\circ\eta^* = 1$. By applying $1\otimes\gamma^*$ on both sides of the above equation the results follows.

8. Construction of certain classes in string cohomology

In this section X denotes a connected space. We shall construct certain classes in string cohomology of X from classes in ordinary cohomology of X.

DEFINITION 8.1. Put $\zeta_p = \exp(2\pi i/p)$ and define evaluation maps as follows:

$$\begin{aligned} &ev_0 : \Lambda X \to X; && \gamma\mapsto\gamma(1),\\ &ev_1 : E\mathsf{T}\times_{C_p}\Lambda X\to E\mathsf{T}\times_{C_p} X^p; && [e,\gamma]\mapsto[e,\gamma(1),\gamma(\zeta_p),\dots,\gamma(\zeta_p^{p-1})]. \end{aligned}$$

DEFINITION 8.2. The classes $f(x), g(x), \delta(x)\in H^*(E\mathsf{T}\times_{\mathsf{T}}\Lambda X)$ for $x\in H^*X$ are defined by

$$f(x)=\tau_1^\infty\circ ev_1^*(v\otimes x^{\otimes p}),\quad g(x)=\tau_1^\infty\circ ev_1^*(1\otimes x^{\otimes p}),\quad \delta(x)=\tau_0^\infty\circ ev_0^*(x).$$

The class u is defined by $u = pr_1^*(u)$ where $pr_1 : E\mathsf{T}\times_{\mathsf{T}}\Lambda X\to B\mathsf{T}$ is the projection on the first factor.

THEOREM 8.3. *Let $i_0 : X\hookrightarrow \Lambda X$ denote the constant loop inclusion and let i_∞ be the corresponding map of T-homotopy orbits. There is a commutative diagram as follows*

$$\begin{array}{ccc} H^*(E\mathsf{T}\times_{\mathsf{T}}\Lambda X) & \xrightarrow{i_\infty^*} & H^*(B\mathsf{T}\times X)\\ \Big\downarrow{\scriptstyle q_\infty^0} & & \Big\downarrow\\ H^*(\Lambda X) & \xrightarrow{i_0^*} & H^*(X) \end{array}\tag{33}$$

and an inclusion $\mathrm{Im}(q_\infty^0) \subseteq \ker(d : H^*(\Lambda X) \to H^*(\Lambda X))$. *The constructed classes are mapped as follows under* i_∞^*.

$$i_\infty^*(f(x)) = \hat{\sigma}(x)St_0(x) + \sigma(x)(-1)^m m!u^m St_0(x),$$
$$i_\infty^*(g(x)) = \hat{\sigma}(x)St_1(x) + \sigma(x)(-1)^m m!u^{m-1} St_1(x),$$
$$i_\infty^*(\delta(x)) = 0 \qquad \textit{and} \qquad i_\infty^*(u) = u \otimes 1.$$

Here $m = (p-1)/2$. *Under* q_∞^0 *the images of the classes are as follows.*

$$q_\infty^0(f(x)) = \hat{\sigma}(x)e(x^p),$$
$$q_\infty^0(g(x)) = \hat{\sigma}(x)e(x^{p-1}dx) + \sigma(x)\delta_{p,3}e(\beta\lambda x),$$
$$q_\infty^0(\delta(x)) = e(dx) \qquad \textit{and} \qquad q_\infty^0(u) = 0.$$

Here $\delta_{p,3} = 1$ *for* $p = 3$ *and zero otherwise.*

Proof. A commutative diagram of spaces gives the diagram (33) and Proposition 7.5 gives the stated inclusion.

We check that the formulas for i_∞^* are valid. Since i_∞ sits over the identity on $B\mathsf{T}$ we have $i_\infty^*(u) = u \otimes 1$. There is a commutative diagram as follows where $\Delta_p : X \to X^p$ is the diagonal and i_1 is the map of C_p-homotopy orbits induced by i_0.

$$\begin{array}{ccccc}
H^*(X) & \xrightarrow{ev_0^*} & H^*(\Lambda X) & & \\
\uparrow{\scriptstyle \Delta_p^*} & & \uparrow{\scriptstyle id} & & \\
H^*(X^p) & \longrightarrow & H^*(\Lambda X) & & \\
\downarrow{\scriptstyle Tr_0^1} & & \downarrow{\scriptstyle \tau_0^1} & & \\
H^*(E\mathsf{T} \times_{C_p} X^p) & \xrightarrow{ev_1^*} & H^*(E\mathsf{T} \times_{C_p} \Lambda X) & \xrightarrow{i_1^*} & H^*(BC_p \times X) \\
 & & \downarrow{\scriptstyle \tau_1^\infty} & & \downarrow{\scriptstyle \tau_1^\infty \otimes 1} \\
 & & H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X) & \xrightarrow{i_\infty^*} & H^*(B\mathsf{T} \times X)
\end{array}$$

The horizontal map with no label is the induced in cohomology of the map $\gamma \mapsto (\gamma(1), \gamma(\zeta_p), \dots, \gamma(\zeta_p^{p-1}))$. A homotopy commutative square of spaces shows that the upper square commutes and it is obvious that the other two are commutative.

The composite $ev_1 \circ i_1$ is the diagonal Δ_1. Its induced in cohomology is the

Steenrod diagonal Δ_1^* given by the following ([12] p. 119 & Errata):

$$\nu(q)\Delta_1^*(1\otimes x^{\otimes p}) = \sum_i (-1)^i u^{m(q-2i)} \otimes P^i x + \sum_i (-1)^i v u^{m(q-2i)-1} \otimes \beta P^i x$$

where $q = |x|$ and $\nu(q) = (m!)^q(-1)^{m(q^2+q)/2}$. From this formula and the lower part of the diagram we see that

$$\nu(q) i_\infty^*(f(x)) = \sum_i (-1)^i u^{m(q-2i)} \otimes P^i x = (-1)^{[q/2]} u^{\sigma(x)m} St_0(x),$$

$$\nu(q) i_\infty^*(g(x)) = \sum_i (-1)^i u^{m(q-2i)-1} \otimes \beta P^i x = (-1)^{[q/2]} u^{\sigma(x)(m-1)} St_1(x).$$

By [12] Lemma 6.3 one has $(m!)^2 = (-1)^{m+1} \bmod p$ and from this one sees that $\nu(q)^{-1}(-1)^{[q/2]} = 1$ for q even and $\nu(q)^{-1}(-1)^{[q/2]} = (-1)^m m!$ for q odd. Hence we have verified the formulas for $i_\infty^*(f(x))$ and $i_\infty^*(g(x))$.

By the left part of the diagram we see that

$$\delta(x) = \tau_1^\infty \circ ev_1^* \circ Tr_0^1(x \otimes 1 \otimes \cdots \otimes 1).$$

The composite $\Delta_1^* \circ Tr_0^1$ is zero by [12] Lemma 4.1 so $i_\infty^*(\delta(x)) = 0$.

We now check the formulas for q_∞^0. It follows directly from Proposition 7.5 that $\delta(x)$ is mapped as stated and clearly u is mapped to zero. For the classes $f(x)$ and $g(x)$ we proceed as follows.

Let Y be a T-space and let e be a point in $E\mathsf{T}$. There is a T-equivariant map $\theta_0 : \mathsf{T} \times Y_0 \to E\mathsf{T} \times Y$ given by $(z, y) \mapsto (ze, zy)$ where Y_0 means Y with trivial T-action. Let θ_1 be the associated map of C_p-orbits ie. $\theta_1 = \theta_0/C_p$. There is a commutative diagram

$$\begin{array}{ccc} H^*(E\mathsf{T} \times_{C_p} Y) & \xrightarrow{\theta_1^*} & H^*(\mathsf{T}/C_p) \otimes H^*Y \\ \downarrow{\scriptstyle \tau_1^\infty} & & \downarrow{\scriptstyle \tau_1^\infty \otimes 1} \\ H^*(E\mathsf{T} \times_{\mathsf{T}} Y) & \xrightarrow{q_\infty^0} & H^*Y \end{array}$$

where $\tau_1^\infty : H^*(\mathsf{T}/C_p) \to \mathsf{F}_p$ is given by $1 \mapsto 0$ and $v \mapsto 1$. This is proved in as similar way as [2] Proposition 4.6. When $Y = \Lambda X$ we have

$$q_\infty^0 \circ \tau_1^\infty \circ ev_1^* = (\tau_1^\infty \otimes 1) \circ \theta_1^* \circ ev_1^* = (\tau_1^\infty \otimes 1) \circ (ev_1 \circ \theta_1)^*.$$

Note that $ev_1 \circ \theta_1$ equals the composite

$$\mathsf{T}/C_p \times \Lambda X \xrightarrow{f_{\Lambda X}} E\mathsf{T} \times_{C_p} (\Lambda X)^p \xrightarrow{1 \times ev_0^p} E\mathsf{T} \times_{C_p} X^p$$

where $f_{\Lambda X}$ is the map from Definition 7.7. Thus we have

$$q_\infty^0 \circ \tau_1^\infty \circ ev_1^* = (\tau_1^\infty \otimes 1) \circ f_{\Lambda X}^* \circ (1 \times ev_0^p)^*.$$

From this and Theorem 7.8 we get the stated results.

PROPOSITION 8.4. *The following diagram is a pullback square:*

$$\begin{array}{ccc} H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda B\mathsf{F}_p) & \xrightarrow{i_\infty^*} & \mathsf{F}_p[u] \otimes H^*B\mathsf{F}_p \\ {\scriptstyle q_\infty^0}\downarrow & & \downarrow \\ \ker(d) & \xrightarrow{i_0^*} & H^*B\mathsf{F}_p. \end{array}$$

PROOF. Define the action map $f_n : \mathsf{Z} \times \mathsf{F}_p \to \mathsf{F}_p$ by $(r, [s]) \mapsto [nr + s]$ for $n \in \mathsf{F}_p$. We let $B\mathsf{F}_p(n)$ denote $B\mathsf{F}_p$ equipped with T-action Bf_n and write $d_{(n)}$ for the corresponding action differential on $H^*B\mathsf{F}_p(n)$. So we have $H^*B\mathsf{F}_p(n) = \Lambda(v_n) \otimes \mathsf{F}_p[\beta v_n]$ where $|v_n| = 1$.

We claim that $d_{(n)}(v_n) = n$ and $d_{(n)}(\beta v_n) = 0$. Firstly, $(Bf_n)^*(v_n) = 1 \otimes v_n + nv \otimes 1$ as one sees from $H_1(Bf_n) = \pi_1(Bf_n) = f_n$ by taking duals. Secondly, $\lambda v_n = v_n$ so $d_{(n)}(\beta v_n) = 0$.

From [1] Lemma 7.11 we have $\Lambda B\mathsf{F}_p \simeq \sqcup B\mathsf{F}_p$ where the disjoint union is taken over $n \in \mathsf{F}_p$. Define maps as follows for $n \in \mathsf{F}_p$:

$$j_n : B\mathsf{F}_p(n) \to \Lambda B\mathsf{F}_p; \qquad x \mapsto (z \mapsto Bf_n(z, x)).$$

These are T-equivariant maps. Let $(\Lambda B\mathsf{F}_p)(n)$ denote the component of $\Lambda B\mathsf{F}_p$ containing the image of j_n. Then the restriction $j_n|$ of j_n to $(\Lambda B\mathsf{F}_p)(n)$ is T-equivariant and a homotopy equivalence. Especially the induced in cohomology $(j_n|)^*$ is an isomorphism of differential graded algebras. Thus $(\Lambda B\mathsf{F}_p)(n) \neq (\Lambda B\mathsf{F}_p)(m)$ for $n \neq m$ since the differentials on their cohomology rings are different. Hence $\sqcup j_n : \sqcup B\mathsf{F}_p(n) \to \Lambda B\mathsf{F}_p$ is T-equivariant and a homotopy equivalence. It follows that the induced map of T-homotopy orbits $(\sqcup j_n)_\infty$ is a weak homotopy equivalence.

The diagram in the statement is via $(\sqcup j_n)_\infty$ equivalent to the following diagram:

$$\begin{array}{ccc} \bigoplus H^*(E\mathsf{T} \times_{\mathsf{T}} B\mathsf{F}_p(n)) & \xrightarrow{pr_0} & H^*(B\mathsf{T} \times B\mathsf{F}_p) \\ {\scriptstyle \oplus Q_{(n)}^*}\downarrow & & \downarrow \\ \bigoplus \ker(d_{(n)}) & \longrightarrow & H^*(B\mathsf{F}_p) \end{array}$$

where $Q_{(n)} : E\mathsf{T} \times B\mathsf{F}_p(n) \to E\mathsf{T} \times_{\mathsf{T}} B\mathsf{F}_p(n)$ denotes the quotient map and pr_0 is the projection on the direct summand with $n = 0$.

We have $H^*(E\mathsf{T} \times_{\mathsf{T}} B\mathsf{F}_p(n)) \cong \ker(d_{(n)})$ for $n \neq 0$ since here the Leray-Serre spectral sequence has $E_3^{i,*} = 0$ for $i \geq 1$. It follows that the diagram is a pullback.

As indicated by Theorem 8.3 above it turns out that when $|x|$ is odd then both $f(x)$ and $g(x)$ can be written as a product of some power of u with another class. This was not the case for $p = 2$ as described in [2]. We construct new classes to get around this difficulty.

THEOREM 8.5. *Let $x \in H^*X$ be a cohomology class of odd degree. Then there exist classes $\phi(x), q(x) \in H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$ with $|\phi(x)| = p(|x|-1)+1$ and $|q(x)| = p(|x|-1)+2$ such that*

$$i_\infty^*(\phi(x)) = St_0(x), \qquad q_\infty^0(\phi(x)) = \lambda x - x(dx)^{p-1},$$
$$i_\infty^*(q(x)) = St_1(x), \qquad q_\infty^0(q(x)) = \beta\lambda x.$$

PROOF. It suffices to prove the theorem when $X = K(\mathsf{F}_p, n)$ for odd $n \geq 1$. The general case then follows by defining $\phi(x) = (1 \times_{\mathsf{T}} \Lambda h)^* \phi(\iota_n)$ and $q(x) = (1 \times_{\mathsf{T}} \Lambda h)^* q(\iota_n)$ where $n = |x|$ and $h : X \to K(\mathsf{F}_p, n)$ has $h^*(\iota_n) = x$. So assume that $X = K(\mathsf{F}_p, n)$.

For $n = 1$ we have $St_0(\iota_1) = 1 \otimes \iota_1$ and $St_1(\iota_1) = 1 \otimes \beta\iota_1$ so here the result follows from Proposition 8.4.

Assume that $n = 2r+1$ where $r \geq 1$. By Proposition 7.1, Theorem 2.9 and Theorem 4.9 the E_3-term of the Leray-Serre spectral sequence for the fibration $\Lambda X \to E\mathsf{T} \times_{\mathsf{T}} \Lambda X \to B\mathsf{T}$ has the following form:

$$E_3 \cong \mathrm{Im}(d) \oplus (\mathsf{F}_p[u] \otimes \widetilde{\Omega}(K))$$

where $K = H^*X$. Here u has bidegree $(2, 0)$ and an element y in $\mathrm{Im}(d)$ or $\widetilde{\Omega}(K)$ has bidegree $(0, |y|)$. Define $s : B\mathsf{T} \to E\mathsf{T} \times_{\mathsf{T}} \Lambda X$ such that $pr_1 \circ s = id$ by choosing a constant loop. By s^* we see that the horizontal line $(*, 0)$ survives to E_∞.

Up to dimension $2rp + 2p - 1$ the only horizontal lines $(*, m)$, $m \geq 0$ which are non trivial for $* > 0$ are $(*, 0)$, $(*, 2rp + 1)$, $(*, 2rp + 2)$ and $(*, 2rp + 2p - 1)$ corresponding to powers of u times the classes 1, $\phi(\iota_n)$, $q(\iota_n)$ and $q(\beta\iota_n)$ in $\widetilde{\Omega}(K)$ respectively. Hence we can define $\phi(\iota_n)$, $q(\iota_n)$ in $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$ by

$$q_0^\infty(\phi(\iota_n)) = \lambda\iota_n - \iota_n(d\iota_n)^{p-1}, \quad q_0^\infty(q(\iota_n)) = \beta\lambda\iota_n \quad \text{and} \quad s^*(q(\iota_n)) = 0.$$

Since $|f(\iota_n)| = 2rp + p$ and $|g(\iota_n)| = 2rp + p - 1$ we see that $f(\iota_n) = C_1 u^m \phi(\iota_n)$ and $g(\iota_n) = C_2 u^{m-1} q(\iota_n)$ where $C_1, C_2 \in \mathsf{F}_p$ and $m = (p-1)/2$

as before. By Theorem 8.3 we conclude that

$$C_1 u^m i_\infty^*(\phi(\iota_n)) = (-1)^m m! u^m St_0(\iota_n),$$
$$C_2 u^{m-1} i_\infty^*(q(\iota_n)) = (-1)^m m! u^{m-1} St_1(\iota_n)$$

and the result follows.

DEFINITION 8.6. For $x \in H^*X$ of even degree we simply define $\phi(x) = f(x)$ and $q(x) = g(x)$.

9. String cohomology and the functor ℓ

In this section we prove the main result of this paper:

THEOREM 9.1. *Let X be a connected space with H_*X of finite type. Then there is a morphism of unstable $\mathcal{A}$-algebras*

$$\psi : \ell(H^*X) \to H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$$

which sends $\phi(x)$, $q(x)$, $\delta(x)$ for $x \in H^*X$ and u to the constructed classes with the same names. The morphism is natural in X. If both of the maps

$$e : \bar{\Omega}(H^*X) \to H^*(\Lambda X), \qquad \Phi : \widetilde{\Omega}(H^*X) \to H^*(\bar{\Omega}(H^*X))$$

are isomorphisms then so is ψ. In particular ψ is an isomorphism when H^*X is a free object in $\mathcal{K}$.

PROOF. Assume that both e and Φ are isomorphisms and put $K = H^*X$. By Theorem 4.5 we have that $\mathrm{Im}(Q) = \ker(d)$. From the results in Section 8 we see that $\mathrm{Im}(Q) \subseteq \mathrm{Im}(q_\infty^0)$ so $\ker(d) \subseteq \mathrm{Im}(q_\infty^0)$. It now follows from Proposition 7.6 that $\ker(d) = \mathrm{Im}(q_\infty^0)$ and that the Leray-Serre spectral sequence associated to the fibration $\Lambda X \to E\mathsf{T} \times_{\mathsf{T}} \Lambda X \to B\mathsf{T}$ collapses at the E_3-term:

$$E_\infty = E_3 \cong \ker(d) \oplus (u \otimes \widetilde{\Omega}(K)) \oplus (u^2 \otimes \widetilde{\Omega}(K)) \oplus \cdots \tag{34}$$

By Proposition 4.6 the filtration of $\ell(K)$ by powers of the ideal (u) also has (34) as associated graded object. If we fix a degree the filtrations are finite and we conclude that $\ell(K)$ and $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$ have the same dimension in each degree. Hence it suffices to show that the map ψ in the statement is a well defined morphism which is surjective.

The constructed classes are algebra generators for $H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X)$ by the collapse, and the formulas for their images under i_∞^* given in Section 8 show that $\mathrm{Im}(i_\infty^*) = R(K)$. Hence we have a commutative diagram as follows:

$$\begin{array}{ccc} H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X) & \xrightarrow{i_\infty^*} & R(K) \\ {\scriptstyle q_\infty^0}\downarrow & & \downarrow{\scriptstyle p_1} \\ \ker(d) & \xrightarrow{p_2} & K \end{array}$$

The kernel of p_1 is the ideal $(u \otimes 1)$ and $i_\infty^*(u) = u \otimes 1$. Since $u \in \ker(q_\infty^0)$ and i_∞^* is surjective we conclude that the restriction $i_\infty^*| : \ker(q_\infty^0) \to \ker(p_1)$ is surjective. Hence we have a surjection into the pullback.

We now restrict to the case where H^*X is a free object in $\mathcal{K}$. Here e is an isomorphism by Proposition 2.9 and Φ is an isomorphism by Theorem 4.9 and Lemma 6.1.

The above surjection into the pullback together with Theorem 6.7 gives us a surjective morphism $\psi' : H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X) \to \ell(K)$ which is then an isomorphism. By definition it has inverse ψ.

By the fact that $K(\mathsf{F}_p, n)$ classifies degree n cohomology and naturality of the constructed classes, we can now conclude that the defining relations for $\ell(K)$ are universal for the constructed classes. Hence ψ is a well defined morphism in general.

In the case where e and Φ are isomorphisms, the collapse ensures that ψ is surjective and hence an isomorphism.

Corollary 9.2. *Let X be a connected space with H_*X of finite type. If H^*X is a polynomial algebra on a set of even dimensional generators then ψ is an isomorphism.*

Proof. If K is zero in odd degrees then $\bar{\Omega}(K)$ is the ordinary de Rham complex $\Omega(K|\mathsf{F}_p)$. Furthermore, $\widetilde{\Omega}(K)$ is the de Rham complex $\Omega(\bar{K}|\mathsf{F}_p)$ where $\bar{K}$ is the algebra defined by $\bar{K}^{np} = K^n$ and $\bar{K}^m = 0$ for $m \neq 0 \bmod p$. The map Φ is the Cartier map.

The Eilenberg-Moore spectral sequence for $H^*(\Lambda X)$ has Hochschild homology of H^*X as its E_2-term and it collapses since the algebra generators sit in $E_2^{0,*}$ and $E_2^{-1,*}$. By the Hochschild-Konstant-Rosenberger theorem Hochschild homology is isomorphic to the de Rham complex and one concludes that e is an isomorphism. The Cartier map Φ is also an isomorphism.

Remark 9.3. We have a commutative diagram which describes the ideas of our approximations:

$$\begin{array}{ccccc} \bar{\Omega}(H^*X) & \xrightarrow{\tau} & \ell(H^*X) & \xrightarrow{Q} & \bar{\Omega}(H^*X) \\ {\scriptstyle e}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle e}\downarrow \\ H^*(\Lambda X) & \xrightarrow{\tau_0^\infty} & H^*(E\mathsf{T} \times_{\mathsf{T}} \Lambda X) & \xrightarrow{q_\infty^0} & H^*(\Lambda X). \end{array}$$

10. Appendix: Limits and colimits in $\mathscr{F}$

PROPOSITION 10.1. *The category $\mathscr{F}$ has all finite coproducts. The coproduct $A \otimes A'$ of two objects A, A' in $\mathscr{F}$ is the tensor product of the underlying F_p-algebras equipped with maps $\lambda * \lambda'$ and $\beta * \beta'$ on $A \otimes A'$ defined by*

$$\lambda * \lambda'(x \otimes y) = \lambda(x) \otimes y^p + x^p \otimes \lambda'(y)$$

$$\beta * \beta'(x \otimes y) = \beta(x) \otimes y + (-1)^{|x|} x \otimes \beta'(y)$$

PROOF. By direct computations one verifies that $A \otimes A'$ is indeed an object in $\mathscr{F}$. It is then easy to see that $A \otimes A'$ is the categorical coproduct where the canonical inclusions $i : A \to A \otimes A'$ and $j : A' \to A \otimes A'$ are defined by $i(x) = x \otimes 1$ and $j(y) = 1 \otimes y$.

PROPOSITION 10.2. *The category $\mathscr{F}$ is complete and cocomplete ie. all small limits and colimits exist in $\mathscr{F}$.*

PROOF. Similar to the proof for $p = 2$ given in [11].

PROPOSITION 10.3. *The functor $\ell : \mathscr{F} \to \mathscr{A}lg$ commutes with filtered colimits. The functors $\bar{\Omega}, \tilde{\Omega} : \mathscr{F} \to \mathscr{A}lg$ commute with all colimits.*

PROOF. By standard arguments $\bar{\Omega}$, $\tilde{\Omega}$ and ℓ commute with filtered colimits. Thus it suffices to show that $\bar{\Omega}$ and $\tilde{\Omega}$ commute with finite coproducts and coequalizers of pairs of maps [7].

For $A \in \mathscr{A}lg$ we let $D(A)$ be the free graded commutative and unital A-algebra on generators dx for $x \in A$ of degree $|dx| = |x| - 1$ modulo the ideal generated by the elements $d(x+y)-dx-dy$ and $d(xy)-d(x)y-(-1)^{|x|}xd(y)$ for $x, y \in A$. The functor $D : \mathscr{A}lg \to \mathscr{A}lg$ is left adjoint to the functor $\mathscr{A}lg \to \mathscr{A}lg$; $A \mapsto \Lambda(v) \otimes A$.

The functor D commutes with colimits since it is a left adjoint. In particular the canonical morphism $h : D(A) \otimes D(B) \to D(A \otimes B)$ is an isomorphism. Let k denote its inverse. Using the factorization $a \otimes b = (a \otimes 1)(1 \otimes b)$ we find

$$k(a \otimes b) = a \otimes b, \qquad k(d(a \otimes b)) = d(a) \otimes b + a \otimes d(b).$$

Now assume that A and B are objects in $\mathscr{F}$. We have a quotient map $D(A) \to \bar{\Omega}(A)$ inducing an isomorphism $D(A)/((da)^p - d(\lambda a), d\beta\lambda a | a \in A) \cong \bar{\Omega}(A)$. Consider the composite map

$$D(A \otimes B) \xrightarrow{k} D(A) \otimes D(B) \longrightarrow \bar{\Omega}(A) \otimes \bar{\Omega}(B).$$

Elements of the form $(d(a \otimes b))^p + d(\lambda(a \otimes b))$ or $d(\beta\lambda(a \otimes b))$ are mapped to zero under this composite by the above formulas for k. So we can factor through $\bar{\Omega}(A \otimes B)$ and get an inverse to the map $\bar{\Omega}(A) \otimes \bar{\Omega}(B) \to \bar{\Omega}(A \otimes B)$.

By a similar proof one sees that $\bar{\Omega}$ commutes with coequalizers of pair of maps. Thus $\bar{\Omega}$ commutes with all colimits.

We now show that $\widetilde{\Omega}$ commutes with finite coproducts. The canonical map $h : \widetilde{\Omega}(A) \otimes \widetilde{\Omega}(B) \to \widetilde{\Omega}(A \otimes B)$ is given by $h(x \otimes y) = \widetilde{\Omega}(i)(x)\widetilde{\Omega}(j)(y)$ where i and j are the inclusions of A and B in the coproduct $A \otimes B$.

By the factorization $a \otimes b = i(a)j(b)$, $a \in A$, $b \in B$ the following equations hold in $\widetilde{\Omega}(A \otimes B)$:

$$\phi(a \otimes b) = (1 - \sigma(a)\sigma(b))\phi(i(a))\phi(j(b))$$

$$q(a \otimes b) = \hat{\sigma}(b)q(i(a))\phi(j(b)) + \hat{\sigma}(a)\phi(i(a))q(j(b))$$

In order to get an inverse to h we define the following morphism:

$$k : \widetilde{\Omega}(A \otimes B) \to \widetilde{\Omega}(A) \otimes \widetilde{\Omega}(B);$$

$$\phi(a \otimes b) \mapsto (1 - \sigma(a)\sigma(b))\phi(a) \otimes \phi(b),$$

$$q(a \otimes b) \mapsto \hat{\sigma}(b)q(a) \otimes \phi(b) + \hat{\sigma}(a)\phi(a)q(b).$$

We must check that k is well defined ie. that the relations (24)–(27) are respected. It suffices to consider the following special form of relation (24):

$$\phi((x \otimes y)(z \otimes w)) = (1 - \sigma(x \otimes y)\sigma(z \otimes w))\phi(x \otimes y)(z \otimes w).$$

We apply k on the left hand side. Since $(x \otimes y)(z \otimes w) = (-1)^{\sigma(y)\sigma(z)}xz \otimes yw$ we get the element $(-1)^{\sigma(y)\sigma(z)}\phi(x)\phi(z) \otimes \phi(y)\phi(w)$ times the constant α below. When applying k to the right hand side we get the same element times the constant β below

$$\alpha = (1 - \sigma(xz)\sigma(yw))(1 - \sigma(x)\sigma(z))(1 - \sigma(y)\sigma(w)),$$

$$\beta = (1 - \sigma(x \otimes y)\sigma(z \otimes w))(1 - \sigma(x)\sigma(y))(1 - \sigma(z)\sigma(w)).$$

Thus it suffices to check that $\alpha = \beta$. If $\sigma(y) = \sigma(z) = 0$ then $\alpha = \beta = 1 - \sigma(x)\sigma(w)$. If one of $\sigma(y)$, $\sigma(z)$ equals one and the other equals zero then $\alpha = \beta = \hat{\sigma}(w)\hat{\sigma}(x)$. If $\sigma(y) = \sigma(z) = 1$ then $\alpha = \beta = 0$. Hence the relation (24) is respected by k. A similar argument shows that the relation (25) is respected by k.

By additivity and symmetry it suffices to check that k respects the following special form of relation (26): $\phi(\beta\lambda(a \otimes b)) = q(a \otimes b)^p$ where $\sigma(a) = 0$ and $\sigma(b) = 1$. Since $\beta\lambda(a \otimes b) = a^p \otimes \beta\lambda b$ we see that k applied to the left hand side equals $\phi(a^p) \otimes \phi(\beta\lambda b) = \phi(a)^p \otimes q(b)^p$. Since k applied to the right hand side equals $(\phi(a) \otimes q(b))^p = \phi(a)^p \otimes q(b)^p$ the relation is respected.

By additivity and symmetry it suffices to check that k respects the following special form of relation (27): $q(\beta\lambda(a \otimes b)) = 0$ where $\sigma(a) = 0$ and $\sigma(b) = 1$. We find $k(q(\beta\lambda(a \otimes b))) = \phi(a^p) \otimes q(\beta\lambda b) = 0$ so the relation is respected. We have shown that k is well defined. We have $h \circ k = id$ and also $k \circ h = id$ as one sees by evaluating on algebra generators. Hence k is an isomorphism and $\widetilde{\Omega}$ commutes with finite products.

Finally we verify that $\widetilde{\Omega}$ commutes with coequalizers of pairs of maps. For $f, g : A \rightrightarrows B$ in $\mathscr{F}$ we have $\text{coeq}(f, g) = B/(f(a) - g(a) | a \in A)$ and

$$\text{coeq}(\widetilde{\Omega}(f), \widetilde{\Omega}(g)) = \widetilde{\Omega}(B)/(\widetilde{\Omega}(f)(x) - \widetilde{\Omega}(g)(x) | x \in \widetilde{\Omega}(A)).$$

The canonical morphism $h : \text{coeq}(\widetilde{\Omega}(f), \widetilde{\Omega}(g)) \to \widetilde{\Omega}(\text{coeq}(f, g))$ is given by $h[\phi(b)] = \phi([b])$ and $h[q(b)] = q([b])$. We check that there is a well defined map k in the opposite direction with $k(\phi([b])) = [\phi(b)]$ and $k(q([b])) = [q(b)]$.

It suffices to verify that if y is an element in the ideal $(f(a) - g(a) | a \in A)$ then $\phi(y)$ and $q(y)$ lies in the ideal $(\widetilde{\Omega}(f)(x) - \widetilde{\Omega}(g)(x) | x \in \widetilde{\Omega}(A))$. Writing $x = (f(a) - g(a))z$ for some $a \in A$ and $z \in B$ this follows directly by the relations (24) and (25). By definition k is the inverse to h.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF AARHUS
BUILDING 530
DK-8000 ÅRHUS C
DENMARK
E-mail: ottosen@imf.au.dk