MATH. SCAND. 93 (2003), 240–246

ON CHORDAL GRAPHS AND THEIR CHROMATIC POLYNOMIALS

GEIR AGNARSSON

Abstract

We derive a formula for the chromatic polynomial of a chordal or a triangulated graph in terms of its maximal cliques. As a corollary we obtain a way to write down an explicit formula for the chromatic polynomial for an arbitrary power of a graph which belongs to any given class of chordal graphs that are closed under taking powers.

1. Introduction

For a simple graph, recall the following definition.

DEFINITION 1.1. A *chord* of a cycle C is an edge which is not in C but has both its endvertices in C. A graph G is *chordal* if every cycle of length four or more in G has a chord in G.

In this article we derive a new form of the chromatic polynomial of a chordal graph and of a graph whose power is chordal, in an elementary way. Our form of the chromatic polynomial is in terms of the maximal cliques of the graph in question. This allows us, in a natural way, to present directly a formula for the chromatic polynomial of any power graph G^k of a graph G belonging to a class of chordal graphs which is closed under taking arbitrary powers. These classes include interval graphs and unit interval graphs [8], strongly chordal graphs [9], m-trapezoid graphs [1], and powers of trees. It is well known that any power of a tree is chordal [7], [6], so our result here in particular generalizes Theorem 5.3 in [2]. In fact, any power of a tree is strongly chordal [6]. A substantial amount of work has been done on chordal graphs and on these special important subclasses of them. For a brief overview of recent related results we refer to the introduction of [2].

Chromatic polynomials have been studied extensively. A recent and comprehensive bibliography, which contains 472 references on chromatic polynomials, is given in [5]. As mentioned there in the introduction, the intention was to make the bibliography as complete as possible.

Received December 27, 2001; in revised form July 15, 2002.

A considerable part of the articles published are about chromatic polynomials of some very specific graphs. Some other articles have appeared on chromatic polynomials of chordal graphs and subclasses of them. We mention two relevant articles: In [4], in which the chromatic polynomial of various types of chordal graphs is studied, it is shown that chordal graphs are chromatically equivalent to threshold and unit interval graphs. In [11] it is shown that a graph G without any subgraphs isomorphic to K_4, the complete graph on four vertices, is chordal if, and only if, its chromatic polynomial has the form $t(t-1)^m(t-2)^r$, where $m \geq 1$ and $r \geq 0$ are some integers. It is however well known that the chromatic polynomial of a chordal graph always takes the form $t(t-1)^{j_1}(t-2)^{j_2}\ldots(t-q+1)^{j_{q-1}}$, where q is the clique number of the graph, and the j_α's are integers [3]. Before we state our main results and proofs of them, we need to define our basic notation and recall some useful definitions.

The set $\{1, 2, 3, \ldots\}$ of natural numbers will be denoted by N. All graphs considered in this article are assumed to be simple unless otherwise stated. For a graph G and a vertex v of G, we denote by $N[v]$ the *closed neighborhood* of v in G, that is the set of all neighbors of v in G together with v itself. Likewise, we denote by $N(v)$ the *open neighborhood* of v in G, that is the set of all neighbors of v in G. For $k \in \mathsf{N}$, the *power graph* G^k is a graph with the same vertex set as G, but where every pair of vertices of distance k or less in G are connected by an edge in G^k. For $m \in \mathsf{N}$ we let $[m]$ denote the set $\{1, \ldots, m\}$, and $(t)_m = t(t-1)\ldots(t-m+1)$ be the *falling factorial* polynomial in t of degree m. Denote by $\chi(G)$ the chromatic number of the graph G. The chromatic polynomial of G will be denoted by $\chi_G(t)$. It describes the number of proper vertex colorings G has using at most $t \geq \chi(G)$ colors.

Recall that a graph G is chordal if, and only if, it has a *simplicial elimination ordering* of the vertices, $V(G) = \{v_1, \ldots, v_n\}$, such that for each vertex v_i the set $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ induces a clique in G, see [12, p. 226].

2. The Main Results

In this section we derive our main results. We start with the following useful fact from [10, Theorem 3]:

LEMMA 2.1. *For a graph G with subgraphs H and K, such that $G = H \cup K$ and $H \cap K$ is a clique, we have*

$$\chi_G(t) = \frac{\chi_H(t)\chi_K(t)}{\chi_{H\cap K}(t)}.$$

This can now be generalized as follows:

COROLLARY 2.2. *For a graph, which is a union of cliques $G = Q_1 \cup \cdots \cup Q_m$, where $Q_{k+1} \cap (Q_1 \cup \cdots \cup Q_k)$ is a clique for each $k \in \{1, \ldots, m-1\}$, let $q_S = |\bigcap_{k \in S} Q_k|$ and $(t)_S = (t)_{q_S}$ for each $S \subseteq [m]$. Then*

$$\chi_G(t) = \prod_{S \subseteq [m]} (t)_S^{(-1)^{|S|-1}}.$$

PROOF. We use induction on m. If $G = Q_1$ is a clique then the formula is clearly correct.

For $m > 1$ we have by Lemma 2.1 that

$$\chi_G(t) = \frac{\chi_{G'}(t)\chi_{Q_m}(t)}{\chi_{G' \cap Q_m}(t)} \tag{1}$$

where $G' = Q_1 \cup \cdots \cup Q_{m-1}$. By the induction hypothesis we have

$$\chi_{G'}(t) = \prod_{S' \subseteq [m-1]} (t)_{S'}^{(-1)^{|S'|-1}}. \tag{2}$$

We now have $G' \cap Q_m = (Q_1 \cap Q_m) \cup \cdots \cup (Q_{m-1} \cap Q_m)$ and moreover

$$(Q_{k+1} \cap Q_m) \cap [(Q_1 \cap Q_m) \cup \cdots \cup (Q_k \cap Q_m)] = [Q_{k+1} \cap (Q_1 \cup \cdots \cup Q_k)] \cap Q_m,$$

which is an intersection of two cliques, and hence a clique itself. Therefore by induction hypothesis we have

$$\chi_{G' \cap Q_m}(t) = \prod_{\emptyset \neq S'' \subseteq [m-1]} (t)_{S'' \cup \{m\}}^{(-1)^{|S''|-1}}. \tag{3}$$

Putting (2) and (3) in (1), bearing in mind that Q_m is a clique of size $q_{\{m\}}$, we finally get

$$\begin{aligned}
\chi_G(t) &= \frac{\prod_{S' \subseteq [m-1]} (t)_{S'}^{(-1)^{|S'|-1}} \cdot (t)_{\{m\}}}{\prod_{\emptyset \neq S'' \subseteq [m-1]} (t)_{S'' \cup \{m\}}^{(-1)^{|S''|-1}}} \\
&= \prod_{S' \subseteq [m-1]} (t)_{S'}^{(-1)^{|S'|-1}} \cdot \prod_{\emptyset \neq S'' \subseteq [m-1]} (t)_{S'' \cup \{m\}}^{(-1)^{|S'' \cup \{m\}|-1}} \cdot (t)_{\{m\}} \\
&= \prod_{S \subseteq [m]} (t)_S^{(-1)^{|S|-1}},
\end{aligned}$$

proving our corollary.

The next lemma provides a proof of the fact that our assumption in Corollary 2.2 is valid for every chordal graph. It is a direct consequence of the fact that a chordal graph has a simplicial elimination ordering. At each step we establish a clique among the previous neighbors, but only some of these cliques are maximal.

LEMMA 2.3. *For a chordal graph G let $\mathcal{Q}_G$ be the set of all the distinct maximal cliques of G. We have $G = \bigcup_{Q \in \mathcal{Q}_G} Q$ and if $|\mathcal{Q}_G| = m$ then there is a labeling*

$$\mathcal{Q}_G = \{Q_1, \dots, Q_m\}$$

such that $Q_{k+1} \cap (Q_1 \cup \cdots \cup Q_k)$ is a clique for every $k \in \{1, \dots m-1\}$.

PROOF. We use induction on $n = |V(G)|$. For $n = 1$ the statement is clearly true.

Assume G to be a chordal graph and $n \geq 2$. Let $V(G) = \{v_1, \dots, v_n\}$ be a simplicial elimination ordering, and let $\mathcal{Q}_G = \{Q_1, \dots, Q_m\}$ be a labeling where Q_m is a maximal clique in G containing the vertex v_n. Since $G(N[v_n])$ is a maximal clique containing v_n, we must have $Q_m = G(N[v_n])$, and hence Q_m is the unique maximal clique containing v_n. Since $Q_1, \dots, Q_{m-1}$ are all maximal cliques in G which do not contain v_n, then they are also distinct maximal cliques in the chordal graph $G \setminus \{v_n\}$. Now, $Q'_m = Q_m \cap (G \setminus \{v_n\}) = G(N(v_n))$ is also a clique in G; this clique is clearly not maximal in G. However, Q'_m is either maximal clique in $G \setminus \{v_n\}$ or not.

If Q'_m is a maximal clique in $G \setminus \{v_n\}$ then

$$G \setminus \{v_n\} = Q_1 \cup \cdots \cup Q_{m-1} \cup Q'_m$$

is a distinct union of all the maximal cliques in $G \setminus \{v_n\}$. By the induction hypothesis we can assume $Q_{k+1} \cap (Q_1 \cap \cdots \cap Q_k)$ is a clique for all $i \in \{1, \dots, m-2\}$, and also that $Q'_m \cap (Q_1 \cup \cdots \cup Q_{m-1})$ is a clique. Since v_n is not contained in any of $Q_1, \dots, Q_{m-1}$, we have

$$Q_m \cap (Q_1 \cup \cdots \cup Q_{m-1}) = Q'_m \cap (Q_1 \cup \cdots \cup Q_{m-1})$$

which is therefore a clique in G. Since $G = Q_1 \cup \cdots \cup Q_m$, we have proven the theorem in this case.

Assume now that Q'_m is not a maximal clique in $G \setminus \{v_n\}$. Then Q'_m must be contained in some maximal clique Q'' of $G \setminus \{v_n\}$. Since Q'' is a clique in G, then Q'' must be contained in one of the maximal cliques $Q_1, \dots, Q_m$ of G. If $Q'' \subseteq Q_m$ we have $Q'_m \subset Q'' \subseteq Q_m$ and hence $Q'' = Q_m$, contradicting the fact that Q'' is a maximal clique in $G \setminus \{v_n\}$ which does not contain v_n.

Therefore, Q'' is contained in one of the maximal cliques $Q_1, \dots, Q_{m-1}$. Hence $Q_1, \dots, Q_{m-1}$ is the complete list of maximal cliques of $G \setminus \{v_n\}$ and

$$G \setminus \{v_n\} = Q_1 \cup \cdots \cup Q_{m-1} \cup Q'_m = Q_1 \cup \cdots \cup Q_{m-1}.$$

Again by induction hypothesis we can assume the labeling to be such that $Q_{k+1} \cap (Q_1 \cup \cdots \cup Q_k)$ is a clique for each $i \in \{1, \dots, m-2\}$. But now we have in addition

$$Q_m \cap (Q_1 \cup \cdots \cup Q_{m-1}) = Q_m \cap (G \setminus \{v_n\}) = G(N(v_n))$$

which is indeed a clique in G, and we have the theorem in this case also.

THEOREM 2.4. *For a chordal graph G with maximal cliques $Q_1, \dots, Q_m$ let $(t)_S$ be as in Corollary 2.2 for each $S \subseteq [m]$. Then*

$$\chi_G(t) = \prod_{S \subseteq [m]} (t)_S^{(-1)^{|S|-1}}.$$

PROOF. By Lemma 2.3 there is a permutation $\sigma : [m] \to [m]$ such that $Q_{\sigma(k+1)} \cap (Q_{\sigma(1)} \cup \cdots \cup Q_{\sigma(k)})$ is a clique for each $k \in \{1, \dots, m-1\}$. Since σ is bijective it yields a bijection $\tilde{\sigma} : \mathbf{P}([m]) \to \mathbf{P}([m])$ by $\tilde{\sigma}(S) = \{\sigma(k) : k \in S\}$ for each $S \subseteq [m]$. By Corollary 2.2 we therefore have

$$\chi_G(t) = \prod_{S \subseteq [m]} (t)_{\tilde{\sigma}(S)}^{(-1)^{|\tilde{\sigma}(S)|-1}} = \prod_{S \subseteq [m]} (t)_S^{(-1)^{|S|-1}},$$

proving our theorem.

REMARK 2.5. It is well known that a given simplicial elimination ordering of the vertices $\{v_1, \dots, v_n\}$ of a chordal graph G yields the following form for the chromatic polynomial of G,

$$\chi_G(t) = \prod_{i=1}^{n} (t - d(i)),$$

where $d(i) = |N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$. This is a direct consequence of the product rule for counting the number of ways one can color the first vertex v_1, then the second vertex v_2, and so on, finally coloring the last vertex v_n, see [12, p. 224]. This formula however depends on the given simplicial elimination ordering.

For our last result, we need the definition of a k-ball of a graph.

DEFINITION 2.6. For a graph G and $k \in \mathbb{N}$, we define a *k-ball* as a set $B \subseteq V(G)$, such that every two vertices of B are of distance k or less from each other in G.

Assume now that we have a graph G and a number $k \in \mathbb{N}$, such that G^k is chordal. Clearly a k-ball in G becomes a clique in G^k and vice versa, a clique in G^k is a k-ball in G. Thus, there is a 1-1 correspondence between k-balls of G and cliques in G^k. Just as for chordal graphs, if $B_1, \ldots, B_m$ is the complete list of all the maximal k-balls of a graph G, which is such that G^k is chordal, we let $b_S = |\bigcap_{i\in S} B_i|$ and likewise $(t)_S = (t)_{b_S}$ for each $S \subseteq [m]$. With this in mind, we get from Theorem 2.4 that we can directly write down the chromatic polynomial of G^k.

THEOREM 2.7. *Let G be a graph and $k \in \mathbb{N}$ such that G^k is chordal. If $B_1, \ldots, B_m$ is the complete list of all the maximal k-balls of G then*

$$\chi_{G^k}(t) = \prod_{S\subseteq[m]} (t)_S^{(-1)^{|S|-1}}.$$

In particular, since T^k is a chordal graph for any tree T and $k \in \mathbb{N}$, we see that Theorem 2.7 here above generalizes Theorem 5.3 in [2].

COROLLARY 2.8. *Let $\mathcal{G}$ be a class of chordal graphs which is closed under taking arbitrary powers. Keeping the notation as in Theorem 2.7 we have for any $G \in \mathcal{G}$ and any $k \in \mathbb{N}$ that*

$$\chi_{G^k}(t) = \prod_{S\subseteq[m]} (t)_S^{(-1)^{|S|-1}}.$$

ACKNOWLEDGMENTS. Some of the early drafts of this work was done while the author still was a post-doctorate fellow at the Science Institute, University of Iceland in Reykjavík, Iceland, and a Visiting Scholar at Los Alamos National Laboratory in Los Alamos, New Mexico. The author thanks Dr. Madhav Marathe for his hospitality.

REFERENCES

1. Agnarsson, Geir, *On powers of some intersection graphs*, Congr. Numer. 151 (2001), 97–109.
2. Agnarsson, Geir, Greenlaw, Raymond, and Halldórsson, Magnús M., *On powers of chordal graphs and their colorings*, Congr. Numer. 144 (2000), 41–65.
3. Braun, K., Kretz, M., Walter, B., and Walter, M., *Die chromatischen Polynome Unterring frei Graphen*, Manuscripta Math. 14 (1974), 223–234.
4. Chandrasekharan, N., Laskar, R., and Veni Madhaven, C. E., *Chromatic polynomials of chordal graphs*, Congr. Numer. 61 (1988), 133–142.

5. Chia, G. L., *A bibliography on chromatic polynomials*, Discrete Math. 172 (1997), 175–191.
6. Corneil, D. G., and Kearney, P. E., *Tree powers*, J. Algorithms 29 (1998), 111–131.
7. Lin, Y.-L., and Skiena, S., *Algorithms for square roots of graphs*, SIAM J. Discrete Math. 8 (1995), 99–118.
8. Raychaudhuri, A., *On powers of interval and unit interval graphs*, Congr. Numer. 59 (1987), 235–242.
9. Raychaudhuri, A., *On powers of strongly chordal and circular graphs*, Ars Combin. 34 (1992), 147–160.
10. Read, R. C., *An introduction to chromatic polynomials*, J. Combin. Theory 4 (1968), 52–71.
11. Vaderlind, Paul, *Chromaticity of triangulated graphs*, J. Graph Theory 12 (1988), 245–248.
12. West, Douglas B., *Introduction to Graph Theory*, Prentice-Hall Inc., Upper Saddle River, New Jersey, second edition, 2001.

DEPARTMENT OF MATHEMATICAL SCIENCES
GEORGE MASON UNIVERSITY, MS 3F2
4400 UNIVERSITY DRIVE
FAIRFAX VA 22030
USA
E-mail: geir@math.gmu.edu