MATH. SCAND. 93 (2003), 313–319

NUCLEAR AND INTEGRAL POLYNOMIALS ON $\mathsf{C}^{(I)}$, I UNCOUNTABLE

CHRISTOPHER BOYD

Abstract

We show that for I an uncountable index set and $n \geq 3$ the spaces of all n-homogeneous polynomials, all n-homogeneous integral polynomials and all n-homogeneous nuclear polynomials on $\mathsf{C}^{(I)}$ are all different. Using this result we then show that the class of locally Asplund spaces (see [10], [6] for definition) is not preserved under uncountable locally convex direct sums nor is separably determined.

1. Introduction

Given an uncountable index set I we consider the locally convex direct sum $\mathsf{C}^{(I)}$. Holomorphic functions on this space were first considered in [14] where it is shown that $\mathsf{C}^{(I)}$ is not holomorphically Mackey. In [5] we investigated holomorphic functions on $\mathsf{C}^{(I)}$ proving the three natural topologies, τ_0, τ_ω and τ_δ, coincided on $\mathscr{H}(\mathsf{C}^{(I)})$ if and only if I has cardinality less than the first measurable cardinal. Holomorphic functions on the Cartesian product C^I were considered by Barroso and Nachbin, [3], who proved that the compact open and Nachbin ported topologies differed on $\mathscr{P}({}^2\mathsf{C}^I)$ for I uncountable. In this paper we conclude our examination by showing that the space of n-homogeneous polynomials, n-homogeneous integral polynomials and n-homogeneous nuclear polynomials on $\mathsf{C}^{(I)}$ are all different when I is an uncountable set and $n \geq 3$.

Defant, [10], introduced and studied the concept of a space whose dual has the local Radon Nikodým property as a locally convex generalisation of the concept of Asplund Banach space. This property was renamed locally Asplund in [6] and further studied in [7], [8]. In the final section we use our results concerning polynomials on $\mathsf{C}^{(I)}$ to show that local Asplundness is neither preserved under locally convex direct sums nor is separably determined. We refer the reader to [15] for further information on homogeneous polynomials on locally convex spaces.

Received September 1, 2001; in revised form January 2, 2002.

2. Nuclear and Integral polynomials on $\mathbb{C}^{(I)}$

In his thesis, [1], Alencar says that an n-homogeneous polynomial P on a locally convex space E is (Pietsch-)*integral* if there is an absolutely convex closed neighbourhood of 0, U, and a finite regular measure μ on $(U^\circ, \sigma(E', E))$ so that

$$P(x) = \int_{U^\circ} \phi(x)^n \, d\mu(\phi)$$

for all $x \in E$. This definition generalises the concept of integral polynomial on a Banach space introduced by Dineen, [13]. The space of all n-homogeneous integral polynomials on E is denoted by $\mathcal{P}_I({}^nE)$. Clearly we have that $\mathcal{P}_I({}^nE) = \bigcup_{U\in\mathcal{U}} \mathcal{P}_I({}^n\hat{E}_U)$. We make $\mathcal{P}_I({}^n E)$ into a locally convex space by giving it the topology, τ_I, defined by the locally convex inductive limit

$$\text{ind}_{U\in\mathcal{U}}(\mathcal{P}_I({}^n\hat{E}_U), \|\,.\,\|_{(U,I)})$$

where

$$\|P\|_{U,I} = \inf\left\{\|\mu\|_{U^\circ} : P(x) = \int_{U^\circ} \phi(x)^n \, d\mu(\phi)\right\}$$

is the integral norm on $\mathcal{P}_I({}^n\hat{E}_U)$.

It can be shown, see [6], that the n-fold symmetric ϵ-tensor product, $\widehat{\bigotimes}_{s,n,\epsilon} E$, is an inductive predual of $(\mathcal{P}_I({}^nE), \tau_I)$, i.e. $\left(\widehat{\bigotimes}_{s,n,\epsilon} E\right)_i' = (\mathcal{P}_I({}^nE), \tau_I)$.

An n-homogeneous polynomial P on a locally convex space E is said to be *nuclear* if there is an absolutely convex closed neighbourhood of 0, U, in E, $(\phi_k)_k$ bounded in $\hat{E}'_{U^\circ}$ and $(\lambda_k)_k \in \ell_1$ so that

$$P(x) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x)^n$$

for all x in E. The space of all n-homogeneous nuclear polynomials on E is denoted by $\mathcal{P}_N({}^nE)$. For A an absolutely convex subset of E we let Π_A be the semi-norm on $\mathcal{P}_N({}^nE)$ defined by

$$\Pi_A(P) = \inf\left\{\sum_{i=1}^{\infty} |\lambda_i| \|\phi_i\|_A^n : P = \sum_{i=1}^{\infty} \lambda_i \phi_i^n\right\}$$

Since $\mathcal{P}_N({}^nE) = \bigcup_{U\in\mathcal{U}} \mathcal{P}_N({}^n\hat{E}_U)$, we may define a topology Π_ω on $\mathcal{P}_N({}^nE)$ by

$$(\mathcal{P}_N({}^nE), \Pi_\omega) = \text{ind}_{U\in\mathcal{U}}(\mathcal{P}_N({}^n\hat{E}_U), \Pi_U).$$

In general we have $\mathcal{P}_N({}^nE) \subseteq \mathcal{P}_I({}^nE) \subseteq \mathcal{P}({}^nE)$ for any locally convex space E and any integer n.

Let us show that for $E = \mathbb{C}^{(I)}$ with I uncountable these inclusions are strict.

THEOREM 2.1. *Let I be a set and $n \geq 3$. Then $(\mathcal{P}_I({}^n\mathbb{C}^{(I)}), \tau_I) = (\mathcal{P}({}^n\mathbb{C}^{(I)}), \tau_\omega)$ if and only if I is countable.*

PROOF. If I is countable then $\mathbb{C}^{(I)}$ is a $\mathcal{D}FN$ space and hence by [15, Proposition 2.12] we have $\mathcal{P}_N({}^n\mathbb{C}^{(I)}) = \mathcal{P}({}^n\mathbb{C}^{(I)})$ for all n. Assume now that $(\mathcal{P}_I({}^n\mathbb{C}^{(I)}), \tau_I) = (\mathcal{P}({}^n\mathbb{C}^{(I)}), \tau_\omega)$. We have that $\big(\widehat{\bigotimes}_{s,n,\epsilon}\mathbb{C}^{(I)}\big)'_i = (\mathcal{P}_I({}^n\mathbb{C}^{(I)}), \tau_I)$ and the equicontinuous subsets of $\big(\widehat{\bigotimes}_{s,n,\epsilon}\mathbb{C}^{(I)}\big)'$ correspond to the locally bounded subsets of $\mathcal{P}_I({}^n\mathbb{C}^{(I)})$. Furthermore, $\big(\widehat{\bigotimes}_{s,n,\pi}\mathbb{C}^{(I)}\big)'_i = (\mathcal{P}({}^n\mathbb{C}^{(I)}), \tau_\omega)$ with the equicontinuous subsets of $\big(\widehat{\bigotimes}_{s,n,\pi}\mathbb{C}^{(I)}\big)'$ corresponding to the locally bounded subsets of $\mathcal{P}({}^n\mathbb{C}^{(I)})$. This gives us that $\bigotimes_{s,n,\epsilon}\mathbb{C}^{(I)} = \bigotimes_{s,n,\pi}\mathbb{C}^{(I)}$. Since $\mathbb{C}^{(I)}$ is stable, [2, Theorem 4.1] implies that $\bigotimes_{n,\epsilon}\mathbb{C}^{(I)} = \bigotimes_{n,\pi}\mathbb{C}^{(I)}$. If $n \geq 3$ then [19, Theorem] implies immediately that $\mathbb{C}^{(I)}$ is nuclear and hence I is countable.

THEOREM 2.2. *Let I be a set. Then the n-homogeneous polynomial on $\mathbb{C}^{(I)}$, $P((x_i)_{i\in I}) = \sum_{i\in I} x_i^n$, is integral.*

PROOF. First we observe that the set $D^I = \{(x_i)_{i\in I} : |x_i| \leq 1 \text{ all } i\}$ is bounded in $\mathbb{C}^I = (\mathbb{C}^{(I)})'_b$. As $\mathbb{C}^{(I)}$ is barrelled D^I is an equicontinuous subset of $(\mathbb{C}^{(I)})'$. Furthermore, it follows from [17, Proposition 3.14.3] that $\sigma(\mathbb{C}^I, \mathbb{C}^{(I)})$ induces the product topology on D^I. For each $i \in I$ we let μ_i be the Radon measure on $D = \{z : |z| \leq 1\}$ such that

$$\int_D z^k \, d\mu_i(z) = \begin{cases} 1 & \text{if } k = n; \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in I$ let π_i denote the natural projection from D^I onto the i^{th} coordinate. By [4, p. 112] (see also [23]) there is a (unique) Radon measure μ on D^I such that $\mu \circ \pi_i = \mu_i$ for each i in I. Since

$$\int_{D^I} \phi(x)^n \, d\mu(\phi) = \sum_{(i_1,\ldots,i_n)\in I^n} x_{i_1}\ldots x_{i_n} \int_{D^I} \phi_{i_1}\ldots\phi_{i_n} \, d\mu(\phi) = \sum_{i\in I} x_i^n$$

for all $x \in \mathbb{C}^{(I)}$, $P(x) = \sum_{i\in I} x_i^n$ is integral.

Given a set I consider the Cartesian product $\mathbb{C}^I$. It is shown in [16, Theorem 2.3.7], [22] and [21] that $\mathbb{C}^I$ is separable if and only if I has cardinality less than or equal to c. However, [20] (see also [9, Example 2.5.7]) show that $\mathbb{C}^{(I)}$ with the topology induced from $\mathbb{C}^I$ is separable if and only I is countable.

LEMMA 2.3. *Let I be uncountable and $n \geq 2$. Then the n-homogeneous polynomial on $\mathbb{C}^{(I)}$ $P((x_i)_{i\in I}) = \sum_{i\in I} x_i^n$ is not nuclear.*

PROOF. For $i \in I$ let e_i be the element of $\mathsf{C}^{(I)}$ or C^I which is 1 in the i^{th} coordinate and 0 in each other coordinate. Since $\check{P}(e_i)^{n-1} = ne_i$ we see that $\check{P}(\mathsf{C}^{(I)})^{n-1}$ is $\mathsf{C}^{(I)}$ with the topology induced from C^I. Let us now suppose that P is nuclear. Then we can find a sequence $(\phi_k)_{k=1}^{\infty}$ in C^I so that $\sum_{k=1}^{\infty} \|\phi_k\|_U^n < \infty$ for some neighbourhood of zero, U, in $\mathsf{C}^{(I)}$ and $P(x) = \sum_{k=1}^{\infty} \phi_k(x)^n$ for all $x \in \mathsf{C}^{(I)}$. For each $x \in \mathsf{C}^{(I)}$, $\check{P}(x)^{n-1} \in \mathsf{C}^I$, and is given by $\check{P}(x)^{n-1} = n \sum_{k=1}^{\infty} \phi_k(x)^{n-1}\phi_k$. Since this series converges in C^I we see that $\check{P}(\mathsf{C}^{(I)})^{n-1}$ is a separable subspace of C^I. However, as we noted before the statement of the Lemma, $\mathsf{C}^{(I)}$ with the topology induced from C^I is not separable and our assumption that P is nuclear is not true.

COROLLARY 2.4. *Let I be uncountable and $n \geq 3$. Then $\mathscr{P}_N(\mathsf{C}^{(I)}) \neq \mathscr{P}_I(\mathsf{C}^{(I)}) \neq \mathscr{P}(\mathsf{C}^{(I)})$.*

Finally we observe that since every bounded subset of $\mathsf{C}^{(I)}$ is finite dimensional every homogeneous polynomial on $\mathsf{C}^{(I)}$ is weakly continuous on bounded sets.

3. Application to Locally Asplund spaces

In modern Banach space theory the Radon-Nikodým property plays a central role. There are many equivalent ways in which this concept can be introduced. For example, a Banach space has the Radon-Nikodým property if and only if every closed nonempty convex bounded set is the closed convex hull of its strongly exposed points. Alternatively, a Banach space E has the Radon-Nikodým property if and only if every integral operator with values in E is nuclear. The Radon-Nikodým property is dual to another Banach space property – Asplundness. Again there are many equivalent definitions of an Asplund Banach space. For example, a Banach space E is Asplund if and only if every separable subspace of E has a separable dual or equivalently if every integral operator on E is nuclear. We refer the reader to [12] for more details. In [10] Defant defined what is meant for a locally convex space to have dual with the locally Radon-Nikodým property. A Banach space has dual with the local Radon-Nikodým property if and only if it is Asplund. In [7] this property was renamed locally Asplund. Locally Asplund spaces have good stability properties. It is shown in [10] that the class of locally Asplund spaces is closed under the formation of subspaces, quotients, arbitrary projective limits and countable locally convex inductive limits. In [7] it is shown that this class is also closed under Schwartz ϵ-products. In this section we show that the locally convex space $\mathsf{C}^{(I)}$, with I uncountable is not locally Asplund. This will prove that the class of locally Asplund locally convex spaces is not closed under the formation of uncountable direct sums and, contrary to the Banach space case, is not separably determined.

Given E and F locally convex spaces we let $L\llbracket E; F'\rrbracket$ denote the space of all linear maps from E into F' transforming some neighbourhood of zero into an equicontinuous set. Therefore $T \in L\llbracket E; F'\rrbracket$ if and only if there exists an absolutely convex closed neighbourhood of zero V in F such that T factors continuously through the Banach space F'_{V°.

Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. An operator $T\colon L^1(\mu) \to X$ is said to be *representable*, [12], if there is a Bochner-integrable $f \in L^\infty(\mu; X)$ such that

$$T\phi = \int \phi f \, d\mu$$

for all $\phi \in L^1(\mu)$.

Given a locally convex space E an operator $T \in L\llbracket L^1(\mu); E'\rrbracket$ is said to be *locally representable* if there is a neighbourhood of zero V in E and a representable operator $\hat{T} \in L(L^1(\mu); E'_{V^\circ})$ such that the following diagram commutes

$$\begin{tikzcd}
L^1(\mu) \arrow[rr, "T"] \arrow[dr, "\hat{T}"'] & & E' \\
& \hat{E}'_{V^\circ} \arrow[ur, hook] &
\end{tikzcd}$$

Defant, [10], says that a locally convex space E has a dual with the local Radon-Nikodým property if for every finite measure space (Ω, Σ, μ) all operators in $L\llbracket L^1(\mu); E'\rrbracket$ are locally representable. As in [7] we rename this property and from this point on say that E is *locally Asplund*. It is shown in [10] that a locally convex space E is locally Asplund if and only if for every absolutely convex neighbourhood U of 0 in E and every positive Radon measure ν on $(U^\circ, \sigma(E', E))$ there is V an absolutely convex neighbourhood of 0 in E, $V \subseteq U$, such that the embedding $(U^\circ, \sigma(E', E)) \hookrightarrow \llbracket V^\circ \rrbracket$ is ν-measurable.

Defant [10] proceeds to give many reformalizations of the concept of a locally Asplund spaces. Among these is that a locally convex space E is locally Asplund if and only if given any locally convex space F every integral bilinear form on $E \times F$ is nuclear. In [6] the author proved that if E is a locally Asplund locally convex space for any positive integer n the locally convex space $(\mathcal{P}_I({}^nE), \tau_I)$ is isomorphic to the locally convex space $(\mathcal{P}_N({}^nE), \Pi_\omega)$.

PROPOSITION 3.1. *If I is an uncountable set then $\mathbb{C}^{(I)}$ is not locally Asplund.*

PROOF. Apply [6, Theorem 3], Theorem 2.2 and Lemma 2.3.

In contrast to the Banach space case we get:

COROLLARY 3.2. *Local Asplundness is not separably determined.*

PROOF. Every separable subspace of $\mathbb{C}^{(I)}$ is either isomorphic to $\mathbb{C}^{(N)}$ or $\mathbb{C}^n$ which is a $\mathscr{D}FN$ space and hence is locally Asplund.

We can also obtain the following result.

PROPOSITION 3.3. *If I is an uncountable set then the bilinear form $B: \mathbb{C}^{(I)} \times \mathbb{C}^{(I)} \to \mathbb{C}$, $B((x_i)_{i\in I}, (y_i)_{i\in I}) = \sum_{i\in I} x_i y_i$ is integral but not nuclear.*

Given locally convex spaces E and F we say that $z \in E\widehat{\bigotimes}_\pi F$ has a *series representation* if there is a sequence $(\lambda_n)_n \in \ell_1$ and bounded sequence $(x_n)_n$ and $(y_n)_n$ in E and F respectively so that $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$. From [11, §4 Remark] we get:

PROPOSITION 3.4. *There is an $\mathscr{L}_\infty$ Banach space F and $z \in \mathbb{C}^I \widehat{\bigotimes}_\pi F$ which does not admit a series representation.*

ACKNOWLEDGEMENT. On the 11th of July 2001 I presented the result in this talk at a seminar at the University of Valencia. I would to thank those present who pointed out to me that my proof of Lemma 2.3 was much more subtle that I had originally appreciated.

REFERENCES

1. Alencar, R., *Aplicações nucleares e integrais e a propriedade de Radon-Nikodým*, Ph.D. Thesis, Universidade de Saõ Paulo, (1982).
2. Ansemil, J. M. and Floret, K., *The symmetric tensor product of a direct sum of locally convex spaces*, Studia Math. 129 (3), (1998), 285–295.
3. Barroso, J. A. and Nachbin, L., *Some topological properties of spaces of holomorphic mappings in infinitely many variables*, Advances in Holomorphy, (J. A. Barroso, ed.), North-Holland Math. Stud. 34 (1979), 67–91.
4. Bourbaki, N., *Intégration, Chapitre V*, Hermann, Paris (1956).
5. Boyd, C., *Holomorphic mappings on $\mathbb{C}^{(I)}$, I uncountable*, Results Math. 36 (1999), 21–25.
6. Boyd, C., *Duality and reflexivity of spaces of approximable polynomials on locally convex spaces*, Monatsh. Math. 130 (2000), 177–188.
7. Boyd, C., Dineen, S. and Rueda, P., *Locally Asplund spaces of holomorphic functions*, Michigan Math. J. 50 (2002), 493–506.
8. Boyd, C., Dineen, S. and Rueda, P., *Locally Asplund preduals of spaces of holomorphic functions*, Acta. Math. Hungar. 96 (1-2) (2002), 105–116.
9. Pérez Carreras, P. and Bonet, J., *Barrelled Locally Convex spaces*, North-Holland Mathematics Stud. 131 (1987).
10. Defant, A., *The local Radon Nikodým property for duals of locally convex spaces*, Bull. Soc. Roy. Sci. Liège 53 (1984), 233–246.
11. Defant, A., *Topological tensor product and the approximation property of locally convex spaces*, Bull. Soc. Roy. Sci. Liège 58 (1989), 29–51.
12. Diestel, J. and Uhl, J. J., *Vector Measures*, Amer. Math. Soc. Math. Surveys 15 (1977).
13. Dineen, S., *Holomorphic types on a Banach space*, Studia Math. 39 (1971), 241–288.

14. Dineen, S., *Holomorphic functions on inductive limits of* $\mathbf{C}^N$, Proc. Roy. Irish. Acad. Sect. A 86 (2) (1986), 143–146.
15. Dineen, S., *Complex Analysis on Infinite Dimensional Spaces*, Monographs Math. (1999).
16. Engelking, R., *Outline of General Topology*, North Holland Publishing Company, Amsterdam (1968).
17. Horváth, J., *Topological Vector Spaces and Distributions, Vol. 1*, Addison-Wesley, Massachusetts (1966).
18. Jarchow, H., *Locally Convex Spaces*, B. G. Teubner, Stuttgart (1981).
19. John, K., *Tensor products of several spaces and nuclearity*, Math. Ann. 269 (1984), 333–356.
20. Lohman, R. H. and Steiles, W. J., *On separability in linear topological spaces*, Proc. Amer. Math. Soc. 42 (1974), 236–237.
21. Marczewski, E, *Séparabilité et multipication cartésienne des espaces topologiques*, Fund. Math. 34 (1947), 127–143.
22. Pondiczery, E. S., *Power problems in abstract spaces*, Duke Math. J. 11 (1944), 835–837.
23. Prohkorov, U. V., *Convergence of random measures and limit theorems in probability theory*, Theory Prob. Appl. I (1956), 157–214.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE DUBLIN
BELFIELD
DUBLIN 4
IRELAND