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THE TRACIAL TOPOLOGICAL RANK OF EXTENSIONS OF C^*-ALGEBRAS

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Abstract

Let $0 \to \mathscr{J} \to \mathscr{A} \to \mathscr{A}/\mathscr{J} \to 0$ be a short exact sequence of separable C^*-algebras. We introduce the notion of tracially quasidiagonal extension. Suppose that $\mathscr{J}$ and $\mathscr{A}/J$ have tracial topological rank zero. We prove that if $(\mathscr{A}, \mathscr{J})$ is tracially quasidiagonal, then $\mathscr{A}$ has tracial topological rank zero.

1. Introduction

In the connection with classification of nuclear C^*-algebras the notion of tracial topological rank was introduced (see [9] and 2.1 below for the definition). It is a non-commutative version of covering dimension for topological spaces which plays a very important role in the classification of nuclear C^*-algebras. In this paper, we are interested in those C^*-algebras with tracial topological rank zero. Finite dimensional C^*-algebras are usually considered to have zero rank. AF-algebras are C^*-algebras which can be approximated pointwisely in norm by finite dimensional C^*-algebras. Therefore they are also considered to have zero rank. C^*-algebras with tracial topological rank zero are C^*-algebras that can be approximated pointwisely by finite dimensional C^*-algebras in "measure" (or in trace). A precise definition and further explanation will be given below (see 2.1 and 2.2 as well as 2.3). A unital simple C^*-algebra, with tracial topological rank zero has real rank zero, stable rank one and weakly unperforated K_0 and is quasidiagonal.

A classification of separable simple nuclear C^*-algebras with tracial topological rank zero which satisfy the Universal Coefficient Theorem was recently established in [13] (see also [12]). It is therefore important to know the tracial topological rank of certain C^*-algebras. All unital simple AH-algebras with slow dimension growth and with real rank zero have tracial topological rank zero (see [5] and also [15]). Many unital separable simple C^*-algebras with real rank zero, stable rank one, weakly unperforated K_0 are known to have tracial topological rank zero (cf. [14]). These results together with the

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recent result of Q. Lin and N. C. Phillips (see [19]) imply many simple crossed products have tracial topological rank zero. Kishimoto ([8]) shows that certain crossed products associated with non-commutative shifts have tracial topological rank zero. By results in [12] and [13], these C^*-algebras can be classified by their K-theory and therefore are isomorphic to some simple AH-algebras. Non-simple cases were also studied in [6], where it is shown, among other things, that the tracial topological rank of $C(X) \otimes \mathscr{A}$ is bounded by the sum of dimension of X and the tracial topological rank of $\mathscr{A}$. This further shows that tracial topological rank behaves similar to that of covering dimension for commutative C^*-algebras. In this paper, we will study the tracial topological rank of C^*-algebra, extensions.

Let $0 \to \mathscr{J} \to \mathscr{A} \to \mathscr{B} \to 0$ be a short exact sequence of C^*-algebras. In what follows, if $\mathscr{C}$ has tracial topological rank zero, we will write $\mathrm{TR}(\mathscr{C}) = 0$. Suppose that $\mathrm{TR}(\mathscr{J}) = \mathrm{TR}(\mathscr{B}) = 0$. The problem that we are interested in is when $\mathrm{TR}(\mathscr{A}) = 0$. If both $\mathscr{B}$ and $\mathscr{J}$ have real rank zero, i.e., $\mathrm{RR}(\mathscr{J}) = \mathrm{RR}(\mathscr{B}) = 0$, then it is proved in [23] and [2] that $\mathrm{RR}(\mathscr{A}) = 0$ if and only if every projection in $\mathscr{B}$ lifts to a projection in $\mathscr{A}$. This requires the index map from $K_0(\mathscr{B})$ to $K_1(\mathscr{J})$ to be zero. For the case of tracial topological rank zero, in order to have $\mathrm{TR}(\mathscr{A}) = 0$, certain finite dimensional C^*-subalgebras in $\mathscr{B}$ should be lifted to $\mathscr{A}$. If $\mathscr{A}$ is a quasidiagonal extension (see 2.13), then all finite dimensional C^*-subalgebras can be lifted. If $\mathscr{A}$ is an AF-algebra, then the extension is also quasidiagonal. From this, naturally, we first study quasidiagonal extensions. We show that if the extension is quasidiagonal, then $\mathrm{TR}(\mathscr{A}) = 0$. However C^*-algebras with tracial topological rank zero are not in general AF. In fact, it is "tracially AF". We discovered that condition that the extension is quasidiagonal is rather too strong for $\mathrm{TR}(\mathscr{A}) = 0$. This leads us to the notion of "tracially quasidiagonal" extensions (see 4.1). Indeed, we show that, under the condition that $\mathrm{TR}(e\mathscr{J}e) = 0$ for any projection $e \in \mathscr{A}$ and $\mathrm{TR}(\mathscr{B}) = 0$, $\mathrm{TR}(\mathscr{A}) = 0$ if and only if the extension is tracially quasidiagonal (Theorem 5.2). This result has been used in the subsequent papers [17] and [18].

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2. Preliminaries

Throughout the paper, we assume that $\mathscr{A}$ is a unital separable C^*-algebra and $\mathscr{J}$ is a closed ideal of $\mathscr{A}$ and $\pi\colon \mathscr{A} \to \mathscr{A}/\mathscr{J} = \mathscr{B}$ is the quotient map. So we have the following short exact sequence:

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{A} \xrightarrow{\pi} \mathscr{B} \longrightarrow 0. \tag{e-1}$$

(1) We also let $\mathscr{A}_+$ denote the set of positive elements in $\mathscr{A}$.

(2) For a C^*-subalgebra $\mathscr{A}_1$ of $\mathscr{A}$, we write $a \in_\epsilon \mathscr{A}_1$ if there is $b \in \mathscr{A}_1$ such that $\|a - b\| < \epsilon$.

(3) Let $\mathscr{A}$ be a C^*-algebra and $a, b \in \mathscr{A}_+$. We write $[a] \leq [b]$ if there is $x \in A$ such that $x^*x = a$ and $xx^* \in \mathrm{Her}(b) = \overline{bAb}$, the hereditary C^*-subalgebra generated by b. We write $n[a] \leq [b]$ if there are mutually orthogonal elements $c_1, \dots, c_n \in \mathrm{Her}(b)$ such that $[a] \leq [c_i]$, $i = 1, 2, \dots$.

For more information about the relation $[a] \leq [b]$, please see [4], [22], [10] and [9].

(4) Define a non-negative function $f_{\delta_2}^{\delta_1}$ by

$$f_{\delta_2}^{\delta_1}(t) = \begin{cases} 1 & t \geq \delta_1 \\ \dfrac{t - \delta_2}{\delta_1 - \delta_2} & \delta_2 < t < \delta_1 \\ 0 & t \leq \delta_2 \end{cases}.$$

(5) $\mathscr{I}^{(k)}$ denotes the class of all C^*-algebras which are unital hereditary C^*-subalgebra of C^*-algebras of the form $C(X) \otimes F$, where X is some k-dimensional finite CW complex and F is a finite dimensional C^*-algebra. These are typical C^*-algebras with rank k.

Definition 2.1 (cf. 3.1 in [9] and [6]). Let $\mathscr{A}$ be a unital separable C^*-algebra. We say that $\mathscr{A}$ is of tracial topological rank no more than k (denoted by $\mathrm{TR}(\mathscr{A}) \leq k$) if for any $\epsilon > 0$, any finite subset $\mathscr{F}$ in $\mathscr{A}$ containing $b \in \mathscr{A}_+$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$ and any integer $n >$ there exist a projection p and a C^*-subalgebra $\mathscr{B} \in \mathscr{I}^{(k)}$ of $\mathscr{A}$ with $1_{\mathscr{B}} = p$ such that

(1) $\|xp - px\| = \|[x, p]\| < \epsilon$,

(2) $pxp \in_\epsilon \mathscr{B}, \forall x \in \mathscr{F}$;

(3) $n\big[f_{\sigma_2}^{\sigma_1}((1 - p)b(1 - p))\big] \leq \big[f_{\sigma_4}^{\sigma_3}(pbp)\big]$.

If $\mathrm{TR}(\mathscr{A}) \leq k$ but $\mathrm{TR}(\mathscr{A}) \nleq k - 1$, we write $\mathrm{TR}(\mathscr{A}) = k$. If $\mathscr{A}$ has no unit, we define $\mathrm{TR}(\mathscr{A}) = \mathrm{TR}(\mathscr{A}^+)$, where $\mathscr{A}^+$ is the C^*-algebra obtained by unit 1 adjoined in $\mathscr{A}$.

REMARK 2.2. The condition (3) above means that $(1-p)b(1-p)$ is much "smaller" than pbp. Indeed if $\sigma > 0$ is given, the above definition implies that we may choose p so that $\tau(1-p) < \sigma$ for all tracial states τ. Thus condition (1), (2) and (3) say that the part of $\mathscr{F}$ which can not be approximated by $\mathscr{B}$ has arbitrarily small trace. If AF-algebras are viewed as C^*-algebras which can be approximated pointwisely in norm by finite dimensional C^*-algebras, then C^*-algebras with tracial topological rank zero may be viewed as C^*-algebras which can be approximated pointwisely in "measure" (or in trace). When $\mathscr{A}$ is simple, the condition (3) can be replaced by $[1-p] \le [a]$ for any given non-zero $a \in \mathscr{A}_+$ which simply says that $1-p$ is arbitrarily small. Comparing positive elements in a non-simple C^*-algebra becomes much more difficult. The complexity of (3) above are due to this fact.

REMARK 2.3. Real rank is also a non-commutative topological rank for C^*-algebras. S. Zhang showed ([25]) that every purely infinite simple C^*-algebra has real rank zero. It is perhaps not exactly what we expect to happen. C^*-algebras with tracial topological rank zero behave more like zero rank non-commutative space. For example, a simple C^*-algebra $\mathscr{A}$ with $\mathrm{TR}(\mathscr{A}) = 0$ has real rank zero as well as stable rank one. See also [6] for a discussion related to tensor products. K-theory of C^*-algebras with $\mathrm{TR}(\mathscr{A}) = k$ also behaves similarly to that of C^*-algebras in $\mathscr{I}^{(k)}$ (see [7]). More facts about tracial topological rank can be found in [9]–[13] and [6].

The following proposition is taken from [6].

PROPOSITION 2.4. *Let $\mathscr{A}$ be a unital separable C^*-algebra. Then following statements are equivalent:*

(a) $\mathrm{TR}(\mathscr{A}) \le k$;

(b) *for any $\epsilon > 0$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$ and any finite subset $\mathscr{F} \subset \mathscr{A}$ containing $b \in \mathscr{A}_+$, there exist a nonzero projection $p \in \mathscr{A}$ and a C^*-subalgebra $\mathscr{A}_0 \in \mathscr{I}^{(k)}$ of $\mathscr{A}$ with unit p such that the following conditions are satisfied:*

  (1) $\|xp - px\| < \epsilon$, $pxp \in_\epsilon \mathscr{A}_0, \forall x \in \mathscr{F}$;

  (2) $\big[f_{\sigma_2}^{\sigma_1}((1-p)b(1-p))\big] \le \big[f_{\sigma_4}^{\sigma_3}(pbp)\big]$;

(c) *for any $\epsilon > 0$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, any finite subset $\mathscr{F} \subset \mathscr{A}$ containing $b_1, \dots, b_n \in \mathscr{A}_+$ and any integer $n > 0$, there are a nonzero projection $p \in \mathscr{A}$ and a C^*-subalgebra $\mathscr{A}_0 \in \mathscr{I}^{(k)}$ of $\mathscr{A}$ with unit p such that the following conditions are satisfied:*

  (1′) $\|xp - px\| < \epsilon$, $pxp \in_\epsilon \mathscr{A}_0, \forall x \in \mathscr{F}$;

  (2′) $n\big[f_{\sigma_2}^{\sigma_1}((1-p)b_j(1-p))\big] \le \big[f_{\sigma_4}^{\sigma_3}(pb_jp)\big]$, $j = 1, \dots, n$.

For convenience, we list some known results as lemmas. Let $\mathscr{A}$ be a unital C^*-algebra. From [9] and [22], we have

LEMMA 2.5. *Let $a, b \in \mathscr{A}_+$ and p be a projection in $\mathscr{A}$.*

(1) *If $a \le \lambda b$ for some $\lambda > 0$, then $[a] \le [b]$;*

(2) $[a] = [a^2]$*;*

(3) *If there is $x \in \mathscr{A}$ such that $a = x^*x, b = xx^*$, then $[a] = [b]$ and $[f_{\sigma_1}^{\sigma_2}(a)] = [f_{\sigma_1}^{\sigma_2}(b)]$ for any $0 < \sigma_1 < \sigma_2 < \|a\|$;*

(4) *If $\|a - b\| < \delta_2$, then $[f_{\delta_2}^{\delta_1}(a)] \le [b]$ for any $0 < \delta_2 < \delta_1$;*

(5) *Suppose that $\|a\| \le 1, \|b\| \le 1$. Then for any $0 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < 1$, there is $\delta = \delta(\delta_3, \delta_4) > 0$ such that $\|a - b\| < \delta$ implies that $[f_{\delta_2}^{\delta_1}(a)] \le [f_{\delta_4}^{\delta_3}(b)]$;*

(6) *If $0 \le a \le b$, then $[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_4}^{\sigma_3}(b)]$ for $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$.*

COROLLARY 2.6. *Let $\mathscr{A}$ be a C^*-algebra and $a \in \mathscr{A}_+$. Then*

(1) $[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_4}^{\sigma_3}(a)]$ *for $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$;*

(2) $[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{d_2}^{d_1} \circ f_{\sigma_4}^{\sigma_3}(a)]$, $[f_{d_2}^{d_1} \circ f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_4}^{\sigma_3}(a)]$ *for any $0 < d_2 < d_1 < 1$.*

PROOF. (1) Since $f_{\sigma_2}^{\sigma_1}(t) \le f_{\sigma_4}^{\sigma_3}(t)$, the assertion comes from Lemma 2.5 (1).

(2) A simple computation shows that $f_{d_2}^{d_1} \circ f_{\sigma_4}^{\sigma_3} = f_{\sigma_6}^{\sigma_5}$, where $\sigma_5 = \sigma_4 + d_1(\sigma_3 - \sigma_4), \sigma_6 = \sigma_4 + d_2(\sigma_3 - \sigma_4)$. Noting that $\sigma_5 > \sigma_6$ and $\sigma_2 - \sigma_5 > (\sigma_3 - \sigma_4)(1 - d_1) > 0$, we have $[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_6}^{\sigma_5}(a)]$ by (1).

COROLLARY 2.7. *Let $a \in \mathscr{A}$ with $0 \le a \le 1$ and p be a nonzero projection in $\mathscr{A}$. Then for any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there is $\delta = \delta(\sigma_3, \sigma_4)$ such that $\|ap - pa\| < \delta$ implies*

$$\big[f_{\sigma_2}^{\sigma_1}(a)\big] \le \big[f_{\sigma_4}^{\sigma_3}(pap)\big] + \big[f_{\sigma_4}^{\sigma_3}((1-p)a(1-p))\big]$$

$$\big[f_{\sigma_2}^{\sigma_1}(pap)\big] + \big[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))\big] \le \big[f_{\sigma_4}^{\sigma_3}(a)\big].$$

PROOF. Let $\delta < \delta(\sigma_3, \sigma_4)/2$ be as in Lemma 2.5 (5). Note that

$$\|pap + (1-p)a(1-p) - a\| < \delta.$$

The assertions can be obtained from Lemma 2.5 (5).

The following is taken from [6]:

LEMMA 2.8. *For any* $0 < \sigma_4 < \sigma_3 < 1$ *and any integer m, there is* $\delta = \delta(\sigma_3, \sigma_4, m) > 0$ *such that for any* C^**-algebra* $\mathscr{A}$, *any* $b \in \mathscr{A}_+$ *with* $\|b\| \le 1$, *any mutually orthogonal projections* p_j, $j = 1, 2, \dots, m$, *with* $p = \sum_{j=1}^m p_j$ *and any* σ_1, σ_2 *with* $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, *if* $\|p_j b - b p_j\| \le \delta$, $1 \le j \le m$, *then*

$$\text{(e-2)} \quad \Big[f_{\sigma_2}^{\sigma_1}\Big(\sum_{j=1}^m (p_j b p_j)\Big)\Big] \le \big[f_{\sigma_4}^{\sigma_3}(pbp)\big]$$

$$\text{and} \qquad \big[f_{\sigma_2}^{\sigma_1}(pbp)\big] \le \Big[f_{\sigma_4}^{\sigma_3}\Big(\sum_{j=1}^m (p_j b p_j)\Big)\Big].$$

LEMMA 2.9 ([9], Theorem 5.8 and Theorem 5.3). *Let* $\mathscr{A}$ *be a unital separable* C^**-algebras with* $\mathrm{TR}(\mathscr{A}) \le k$. *Then* $\mathrm{TR}(\mathrm{M}_n(\mathscr{A})) \le k$ *for* $n \ge 1$ *and* $\mathrm{TR}(\mathscr{C}) \le k$ *for any unital hereditary* C^**-algebra* $\mathscr{C}$ *of* $\mathscr{A}$.

COROLLARY 2.10. *Let* $\mathscr{A}$ *be a unital* C^**-algebra with* $\mathrm{TR}(\mathscr{A}) \le k$ *and* $\mathscr{J}$ *be a closed ideal of* $\mathscr{A}$. *If* $\mathscr{J}$ *admits an approximate unit consisting of projections, then* $\mathrm{TR}(\mathscr{J}) \le k$.

The following lemma is a standard argument of function approximation (see for example Lemma 2.5.11 of [16]).

LEMMA 2.11. *Let* $a \in \mathscr{A}_+$ *with* $0 \le a \le 1$ *and* $f(t)$ *be a continuous function on* $[0, 1]$ *with* $f(0) = 0$. *Then for any* $\epsilon > 0$, *there is* $\delta = \delta(f, \epsilon) > 0$ *such that for any nonzero projection* $p \in \mathscr{A}$ *with* $\|ap - pa\| < \delta$ *one has that* $\|f(pap) - pf(a)p\| < \epsilon$ *and* $\|f(a)p - pf(a)\| < \epsilon$.

LEMMA 2.12. *Let* $0 < \sigma_4 < \sigma_3 < \delta_4 < \cdots < \delta_1 < \sigma_2 < \sigma_1 < 1$ *and n be a positive integer. There is* $\delta > 0$ *satisfying the following: Suppose that* $\mathscr{A}$ *is a* C^**-algebra* $a, b, x_i \in \mathscr{A}$ $(i = 1, \dots, n)$ *with* $0 \le a \le 1$ *such that* $x_i^* x_i = f_{\delta_2}^{\delta_1}(a)$ *and* $x_i x_i^* \in \mathrm{Her}(f_{\delta_4}^{\delta_3}(b))$, *and* $x_i x_i^*$ $(i = 1, \dots, n)$ *are mutually orthogonal. If there is a projection* $p \in \mathscr{A}$ *such that* $\|py - yp\| < \delta$ *for* $y \in \{a, b, x_i, x_i^*, i = 1, \dots, n\}$, *then*

$$n\big[f_{\sigma_2}^{\sigma_1}(pap)\big] \le \big[f_{\sigma_4}^{\sigma_3}(pbp)\big].$$

PROOF. Set $d_i = \frac{1}{i}$, $i = 1, \dots, 8$. By Lemma 2.5 (5), there is $\eta_i > 0$ such that, for any $x, y \in A_+$ with $\|x\| \le 1$ and $\|y\| \le 1$, if $\|x - y\| < \eta_i$ then $[f_{d_{i+1}}^{d_i}(x)] \le [f_{d_{i+3}}^{d_{i+2}}(y)]$, $i = 1, 2, 3, 4, 5$. Let $\eta_0 = \frac{1}{8}\min\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$. By Lemma 2.11, there is $\eta_1' > 0$ such that $\|[x, p]\| < \eta_1'$ implies

$$\big\| f_{\delta_2}^{\delta_1}(pxp) - p f_{\delta_2}^{\delta_1}(x) p \big\| < \eta_0 \qquad \text{and} \qquad \big\| \big[p, f_{\delta_2}^{\delta_1}(x)\big] \big\| < \eta_0.$$

Also there is $\eta_2' > 0$ such that $\|[p, x]\| < \eta_2'$ implies

$$\left\| f_{\sigma_4}^{\sigma_3}(pxp) - pf_{\sigma_4}^{\sigma_3}(x)p \right\| < \eta_0 \qquad \text{and} \qquad \left\| \left[p, f_{\sigma_4}^{\sigma_3}(x)\right] \right\| < \eta_0.$$

Let $\delta = \frac{1}{2n} \min\{\eta_0, \eta_1', \eta_2'\}$. Now suppose p is a projection satisfying $\|[p, y]\| < \delta$ for all $y \in \{a, b, x_i, x_i^*, i = 1, 2, \ldots, n\}$. Then

$$\begin{aligned}
n\left[f_{\sigma_2}^{\sigma_1}(pap)\right] &\leq n\left[f_{d_2}^{d_1} \circ f_{\delta_2}^{\delta_1}(pap)\right] && \text{(by Corollary 2.6 (2))} \\
&\leq n\left[f_{d_4}^{d_3}\left(pf_{\delta_2}^{\delta_1}(a)p\right)\right] && \text{(by Lemma 2.5 (5))} \\
&= \sum_{i=1}^{n} \left[f_{d_4}^{d_3}(px_i^* x_i p)\right] && (x_i^* x_i = f_{\delta_2}^{\delta_1}(a)) \\
&= \sum_{i=1}^{n} \left[f_{d_4}^{d_3}(x_i p x_i^*)\right]. && \text{(by Lemma 2.5 (3))}
\end{aligned}$$

Note that $\left\| \sum_{i=1}^n (x_i p x_i^*) - p\left(\sum_{i=1}^n x_i x_i^*\right) p \right\| < 2n\delta < \eta_3$. Thus, by Lemma 2.5 (5),

$$\sum_{i=1}^{n} \left[f_{d_4}^{d_3}(x_i p x_i^*)\right] = \left[f_{d_4}^{d_3}\left(\sum_{i=1}^{n} (x_i p x_i^*)\right)\right] \leq \left[f_{d_6}^{d_5}\, p\left(\sum_{i=1}^{n} x_i x_i^*\right) p\right].$$

Since $\sum_{i=1}^n x_i x_i^* \in \overline{f_{\delta_4}^{\delta_3}(b) A f_{\delta_4}^{\delta_3}(b)}$ and $f_{\sigma_4}^{\sigma_3}(b) f_{\delta_4}^{\delta_3}(b) = f_{\delta_4}^{\delta_3}(b)$, we have

$$\left[f_{d_6}^{d_5}\, p\left(\sum_{i=1}^{n} x_i x_i^*\right) p\right] = \left[f_{d_6}^{d_5}\left(pf_{\sigma_4}^{\sigma_3}(b)\left(\sum_{i=1}^{n} x_i x_i^*\right) f_{\sigma_4}^{\sigma_3}(b) p\right)\right].$$

Note that

$$\left\| \left[p, f_{\sigma_4}^{\sigma_3}(b)\right] \right\| < \eta_0$$

and

$$\left\| pf_{\sigma_4}^{\sigma_3}(b)p - f_{\sigma_4}^{\sigma_3}(pbp) \right\| < \eta_0.$$

This implies that $\|pf_{\sigma_4}^{\sigma_3}(b) - f_{\sigma_4}^{\sigma_3}(pbp)\| < 2\eta_0$. Note also that $\{x_i x_i^*\}$ are mutually orthogonal. Thus $\left\| \sum_{i=1}^n x_i x_i^* \right\| \leq 1$. Therefore

$$\left\| pf_{\sigma_4}^{\sigma_3}(b)\left(\sum_{i=1}^{n} x_i x_i^*\right) f_{\sigma_4}^{\sigma_3}(b) p - f_{\sigma_4}^{\sigma_3}(pbp)\left(\sum_{i=1}^{n} x_i x_i^*\right) f_{\sigma_4}^{\sigma_3}(pbp) \right\| < 8\eta_0 < \eta_5.$$

Finally

$$\Big[f_{d_6}^{d_5}\Big(pf_{\sigma_4}^{\sigma_3}(b)\Big(\sum_{i=1}^{n} x_i x_i^*\Big) f_{\sigma_4}^{\sigma_3}(b)p\Big)\Big]$$

$$\le \Big[f_{d_8}^{d_7}\Big(f_{\sigma_4}^{\sigma_3}(pbp)\Big(\sum_{i=1}^{n} x_i x_i^*\Big) f_{\sigma_4}^{\sigma_3}(pbp)\Big)\Big] \qquad \text{(by Lemma 2.5 (5))}$$

$$\le \Big[f_{\sigma_4}^{\sigma_3}(pbp)\Big(\sum_{i=1}^{n} x_i x_i^*\Big) f_{\sigma_4}^{\sigma_3}(pbp)\Big] \le \big[f_{\sigma_4}^{\sigma_3}(pbp)\big].$$

Thus we obtain $n[f_{\sigma_2}^{\sigma_1}(pap)] \le [f_{\sigma_4}^{\sigma_3}(pbp)]$.

DEFINITION 2.13. Let $\mathscr{A}$ and $\mathscr{J}$ be as in (e-1). Recall that the pair $(\mathscr{A}, \mathscr{J})$ is said to be quasidiagonal if there is an approximate unit $\{r_n\}_1^\infty$ of $\mathscr{J}$ consisting of projections such that $\lim_{n\to\infty} \|r_n x - x r_n\| = 0$ for all $x \in \mathscr{A}$.

The following proposition follows immediately from the definition.

PROPOSITION 2.14. *Let $(\mathscr{A}, \mathscr{J})$ be quasidiagonal. Then for any projection $e \in \mathscr{A}$, $(e\mathscr{A}e, e\mathscr{J}e)$ is quasidiagonal.*

Thus, combining Lemma 2.14 and Lemma 2.10, we have

COROLLARY 2.15. *Let $(\mathscr{A}, \mathscr{J})$ be quasidiagonal such that* $\mathrm{TR}(\mathscr{J}) \le k$. *Then for any projection $p \in \mathscr{A}$,* $\mathrm{TR}(p\mathscr{J}p) \le k$.

LEMMA 2.16. *Let $\mathscr{A}$ be a σ-unital but non-unital C^*-algebra. If* $\mathrm{TR}(\mathscr{A}) = 0$, *then $\mathscr{A}$ admits an approximate unit consisting of projections.*

PROOF. Let h be a strictly positive element in $\mathscr{A}$. We first prove that for any $\epsilon > 0$ there is a projection $e \in \mathscr{A}$ such that $\|(1-e)h\| < \epsilon$.

Since $\mathrm{TR}(\mathscr{A}^+) = 0$, there is a nonzero projection $p \in \mathscr{A}^+$ and a finite dimensional C^*-algebra $\mathscr{A}_1$ of $\mathscr{A}^+$ with unit p such that

(1) $\|hp - ph\| < \epsilon/3$,

(2) $php \in_{\epsilon/3} \mathscr{A}_1$ and

(3) $[1-p] \le [p]$. Let ρ be the canonical homomorphism of $\mathscr{A}^+$ onto $\mathbb{C}$ (the complex field).

Then either $\rho(p) = 0$ or $\rho(1-p) = 0$. From (3) above, we have $[\rho(1-p)] \le [\rho(p)]$, so that $\rho(1-p) = 0$, i.e., $1-p \in \mathscr{A}$. Put $\mathscr{A}_0 = \mathscr{A} \cap \mathscr{A}_1$. Then $\mathscr{A}_0$ is an finite dimensional C^*-algebra with the unit q. Set $e = 1-p+q$ $(qp = q)$.

Choose $a = a_1 + \lambda \in \mathscr{A}_1$ with $a_1 \in \mathscr{A}$ such that $\|php - a\| < \epsilon/3$. Then $|\lambda| < \epsilon/3$ and

$$\begin{aligned}\|(1-e)h\| &= \|(1-q)(ph - php) + (1-q)(php - a) + (1-q)a\| \\ &\leq \|ph - hp\| + \|php - a\| + |\lambda| < \epsilon.\end{aligned}$$

Now put $\epsilon_n = n^{-1}$. Then by the above argument, there is a projection $e_1 \in \mathscr{A}$ such that $\|(1-e_1)h\| < \epsilon_1$. Since $\mathrm{TR}((1-e_1)\mathscr{A}^+(1-e_1)) = 0$, by Lemma 2.9, we obtain a projection $e_2 \in (1-e_1)\mathscr{A}(1-e_1)$ such that

$$\|(1-e_2)(1-e_1)h^2(1-e_1)\| < \epsilon_2$$

which implies that

$$\|(1-e_1-e_2)h\|^2 = \|(1-e_2)(1-e_1)h^2(1-e_1)(1-e_2)\| < \epsilon_2^2.$$

Set $E_n = \sum_{i=1}^n e_i$, $n = 1, 2$. Using the same proof as in the [2, Proposition 2.9], we can obtain an increase sequence $\{E_n\}_1^\infty$ in $\mathscr{A}$ consisting of projections such that $\|(1-E_n)h\| < \epsilon_n$ for $n \geq 1$, that is, $\{E_n\}_1^\infty$ is an approximate unit for $\mathscr{A}$.

3. Quasidiagonal extensions

The following is well known (see Lemma 2.11 (2) and (3) of [24] and [3, Lemma 9.8, Lemma 9.9].

LEMMA 3.1. *Let $(\mathscr{J}, \mathscr{A}, \pi)$ be as in* (e-1). *Suppose that $(\mathscr{A}, \mathscr{J})$ is quasidiagonal. Then for any finite dimensional C^*-subalgebra $\mathscr{C} \subset \mathscr{B}$, there is a C^*-subalgebra $\mathscr{A}_0 \subset \mathscr{A}$ which is isomorphic to $\mathscr{C}$ such that $\pi(\mathscr{A}_0) = \mathscr{C}$. Moreover, we can find an approximate unit $\{r_n\}_1^\infty$ of $\mathscr{J}$ consisting of projections such that $r_n x = x r_n$, $\forall x \in \mathscr{A}_0, n \geq 1$.*

LEMMA 3.2. *Let $(\mathscr{J}, \mathscr{A}, \pi)$ be as in* (e-1) *and $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$. Suppose that $(\mathscr{A}, \mathscr{J})$ is quasidiagonal. If $p \in \mathscr{A}$ is a projection, $a \in (1-p)\mathscr{A}(1-p)$ and $b \in p\mathscr{A}p$ are nonzero positive elements with $0 \leq a, b \leq 1$ such that*

$$n\big[f_{\delta_2}^{\delta_1}(\pi(a))\big] \leq \big[f_{\delta_4}^{\delta_3}(\pi(b))\big]$$

then there exists a projection $r \in (1-p)\mathscr{J}(1-p)$ such that

$$n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big] \leq \big[f_{\sigma_4}^{\sigma_3}(b)\big]$$

where n is a positive integer. Moreover, for any finite subset $\mathscr{F} \subset \mathscr{A}$ and $\epsilon > 0$, if $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F}$, then we can require that $\|rx - xr\| < 3\epsilon$ for all $x \in \mathscr{F} \cup \{a\}$.

PROOF. Let $d_i = \frac{1}{i+1}$, $i = 1, 2, \dots, 18$. Choose a sufficiently small $\eta > 0$ depending on δ_i, σ_i and d_i. There is $v_i \in \mathscr{B}$ such that $v_i v_i^* = \pi(f_{\delta_2}^{\delta_1}(a))$ and $\{v_i^* v_i, i = 1, \dots, n\}$ are mutually orthogonal elements of $\mathrm{Her}(\pi(f_{\delta_4}^{\delta_3}(b)))$. Choose $u_i \in \mathscr{A}$ such that $\pi(u_i) = v_i$. Then

$$c_i = u_i u_i^* - f_{\delta_2}^{\delta_1}((1-p)a(1-p)) \in \mathscr{J}, \qquad i = 1, \dots, n.$$

There are mutually orthogonal elements $\{y_i : i = 1, \dots, n\} \subset \mathrm{Her}(f_{\delta_4}^{\delta_3}(b))$ such that $\pi(y_i) = u_i^* u_i$, $(i = 1, \dots, n)$. Let $e_i = u_i^* u_i - y_i \in \mathscr{J}$, $(i = 1, \dots, n)$. Since $(\mathscr{A}, \mathscr{J})$ is quasidiagonal, there is a projection $g \in \mathscr{J}$ such that

$$\|c_i(1-g)\| < \delta, \qquad \|e_i(1-g)\| < \delta, \quad i = 1, \dots, n$$

and for all $x \in \mathscr{F} \cup \left\{ f_{\sigma_4}^{\sigma_3}(b)p, a, b, \right\} \cup \left\{ u_i, u_i^*, u_i u_i^*, u_i^* u_i, y_i, y_i^{1/2}, i = 1, \dots, n, \right\}$, $\|xg - gx\| < \delta$.

It follows from $\|(1-p)g - g(1-p)\| < \delta$ that there exists projections $r \in (1-p)\mathscr{J}(1-p)$, $\bar{r} \in p\mathscr{J}p$ such that $\|(1-p)g(1-p) - r\| < 2\delta$ and $\|p(1-g)p - \bar{r}\| < 2\delta$. Furthermore if we choose $\delta < \epsilon/5$, then $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F} \cup \{a\}$ implies

$$\begin{aligned}
\|xr - rx\| &\le \|xr - x(1-p)g(1-p)\| \\
&\quad + \|x(1-p)g(1-p) - (1-p)g(1-p)x\| \\
&\quad + \|(1-p)g(1-p)x - rx\| \\
&< 2\delta + (2\epsilon + \delta) + 2\delta \\
&< 3\epsilon.
\end{aligned}$$

Also we have

$$\begin{aligned}
&\|(1-p-r) + \bar{r} - (1-g)\| \\
&\quad \le \|1 - p - (1-p)g(1-p) + p(1-g)p \\
&\qquad\qquad - ((1-p)(1-g)(1-p) + p(1-g)p)\| \\
&\qquad + \|(1-p)(1-g)(1-p) - p(1-g)p - (1-g)\| \\
&\quad < 4\delta.
\end{aligned}$$

Choose δ so small that (by applying Lemma 2.5 (5), Lemma 2.11 and Lemma 2.9)

$$\text{(e-3)} \qquad \left[f_{d_2}^{d_1} \circ f_{\delta_4}^{\delta_3}((1-g)a(1-g)) \right] \le \left[f_{d_4}^{d_3}((1-g) f_{\delta_2}^{\delta_1}(a)(1-g)) \right].$$

and

(e-4)

$$\left[f_{d_{10}}^{d_9}((1-g) f_{\delta_4}^{\delta_3}(b) d f_{\delta_4}^{\delta_3}(b)(1-g)) \right] \le \left[f_{\delta_4}^{\delta_3}(f_{\delta_4}^{\delta_3}(b)(1-g) d (1-g) f_{\delta_4}^{\delta_3}(b)) \right].$$

Therefore (with sufficiently small δ),

$$
\begin{aligned}
&n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big]\\
&\quad\le n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big]+n\big[f_{\sigma_2}^{\sigma_1}(\bar{r}a\bar{r}\big]\\
&\quad\le n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)+\bar{r})a((1-p-r)+\bar{r}))\big] \quad \text{(by Corollary 2.7)}\\
&\quad\le n\big[f_{d_2}^{d_1}\circ f_{\delta_2}^{\delta_1}((1-g)a((1-g))\big] \quad \text{(by Corollary 2.6 and Lemma 2.5 (5))}\\
&\quad\le n\big[f_{d_4}^{d_3}((1-g)f_{\delta_2}^{\delta_1}(a)(1-g))\big] \quad \text{(by (e-3))}
\end{aligned}
$$

Since $\|(1-g)(f_{\delta_2}^{\delta_1}(a)-u_iu_i^*)(1-g)\|=\|(1-g)c_i(1-q)\|<\delta, i=1,\dots,n$, with sufficiently small δ, we have

$$
\begin{aligned}
&n\big[f_{d_4}^{d_3}((1-g)f_{\delta_2}^{\delta_1}(a)(1-g))\big]\\
&\qquad\le\sum_{i=1}^{n}\big[f_{d_6}^{d_5}((1-g)u_iu_i^*(1-g))\big] \qquad \text{(by Lemma 2.5 (5))}\\
&\qquad=\sum_{i=1}^{n}\big[f_{d_6}^{d_5}(u_i^*(1-g)u_i)\big] \qquad \text{(by Lemma 2.5 (3))}\\
&\qquad\le\sum_{i=1}^{n}\big[f_{d_8}^{d_7}((1-g)u_i^*u_i(1-g))\big] \qquad \text{(by Lemma 2.5 (5))}
\end{aligned}
$$

Also since $\|(1-g)(u_i^*u_i-y_i)(1-g)\|=\|(1-g)e_i(1-g)\|<\delta, i=1,\dots,n$, with sufficiently small δ,

$$
\begin{aligned}
&\sum_{i=1}^{n}\big[f_{d_8}^{d_7}((1-g)u_i^*u_i(1-g))\big]\\
&\qquad\le\sum_{i=1}^{n}\big[f_{d_{10}}^{d_9}((1-g)y_i(1-g))\big] \qquad \text{(by Lemma 2.5 (5))}\\
&\qquad\le\sum_{i=1}^{n}\big[f_{d_{12}}^{d_{11}}\big(y_i^{1/2}(1-g)y_i^{1/2}\big)\big] \qquad \text{(by Lemma 2.5 (5))}\\
&\qquad=\Big[f_{d_{12}}^{d_{11}}\Big(\sum_{i=1}^{n}\big(y_i^{1/2}(1-g)y_i^{1/2}\big)\Big)\Big] \qquad (y_iy_j=0 \text{ if } i\ne j)\\
&\qquad\le\Big[f_{d_{14}}^{d_{13}}\Big((1-g)\Big(\sum_{i=1}^{n}y_i\Big)(1-g)\Big)\Big] \qquad \text{(by Lemma 2.5 (5))}
\end{aligned}
$$

Since $y_i \in \mathrm{Her}\big(f_{\delta_4}^{\delta_3}(b)\big)$ and $f_{\delta_4}^{\delta_3}(b) f_{\sigma_4}^{\sigma_3}(b) = f_{\delta_4}^{\delta_3}(b)$, we have

$$\sum_{i=1}^{n} y_i = \big(f_{\sigma_4}^{\sigma_3}(b)\big)^{1/2}\Big(\sum_{i=1}^{n} y_i\Big)\big(f_{\sigma_4}^{\sigma_3}(b)\big)^{1/2}, \qquad i = 1, \ldots, n.$$

Let $y = \sum_{i=1}^{n} y_i$, then

$$\begin{aligned}
&\big[f_{d_{14}}^{d_{13}}((1-g)y(1-g))\big]\\
&\quad = \big[f_{d_{14}}^{d_{13}}((1-g)(f_{\sigma_4}^{\sigma_3}(b))^{1/2} y (f_{\sigma_4}^{\sigma_3}(b))^{1/2}(1-g))\big]\\
&\quad \leq \big[f_{d_{16}}^{d_{15}}\big((f_{\sigma_4}^{\sigma_3}(b))^{1/2})(1-g)y(1-g)(f_{\sigma_4}^{\sigma_3}(b))^{1/2}\big)\big] \quad \text{(by Lemma 2.5 (5))}\\
&\quad \leq \big[f_{d_{18}}^{d^{17}} \circ f_{\sigma_4}^{\sigma_3}(b)\big] \quad \text{(by Lemma 2.5 (6))}\\
&\quad \leq \big[f_{\sigma_3}^{\sigma_4}(b)\big].
\end{aligned}$$

COROLLARY 3.3. *Let* $(\mathscr{J}, \mathscr{A}, \pi)$ *be as in* (e-1) *and* $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$. *Suppose that* $(\mathscr{A}, \mathscr{J})$ *is quasidiagonal. If* $a \in \mathscr{A}$ *with* $0 \leq a \leq 1$ *and*

$$n\big[f_{\delta_2}^{\delta_1}((1-\pi(p))\pi(a)(1-\pi(p)))\big] \leq \big[f_{\delta_4}^{\delta_3}(\pi(p)\pi(a)\pi(p))\big]$$

for some projection $p \in \mathscr{A}$, *then there exists a projection* $r \in (1-p)\mathscr{J}(1-p)$ *such that*

$$n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big] \leq \big[f_{\sigma_4}^{\sigma_3}(pap)\big],$$

where n *is an integer. Moreover, for any finite subset* $\mathscr{F} \subset \mathscr{A}$ *and* $\epsilon > 0$, *if* $\|px - xp\| < \epsilon$ *for all* $x \in \mathscr{F} \cup \{a\}$ *we can require that* $\|rx - xr\| < 3\epsilon$ *for all* $x \in \mathscr{F} \cup \{a\}$.

Now we prove following theorem:

THEOREM 3.4. *Let*

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{A} \xrightarrow{\pi} \mathscr{B} \longrightarrow 0$$

be a short exact sequence of C^**-algebras with* $\mathscr{A}$ *unital and* $(\mathscr{A}, \mathscr{J})$ *quasidiagonal. Suppose that for any* $\epsilon > 0$, *any finite subset* $\mathscr{F} \subset \mathscr{A}$ *containing a nonzero positive element* a, *any integer* $n > 0$ *and any* $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$, *there exits a projection* $\bar{q} \in \mathscr{B}$ *and a finite dimensional* C^**-subalgebra* $\overline{F} \subset \mathscr{B}$ *with* $\bar{q} = 1_{\overline{F}}$ *such that*

(i) $\|\bar{q}\pi(x) - \pi(x)\bar{q}\| < \epsilon$, $\bar{q}\pi(x)\bar{q} \in_\epsilon \bar{F}$ *for all* $x \in \mathscr{F}$ *and*

(ii) $n\big[f_{\delta_2}^{\delta_1}((1-\bar{q})\pi(a)(1-\bar{q}))\big] \le \big[f_{\delta_4}^{\delta_3}(\bar{q}\pi(a)\bar{q})\big]$.

Then there is a finite dimensional C^*-subalgebra $F \subset \mathscr{A}$ with $1_F = p$, $\pi(p) = \bar{q}$ and $\pi(F) = \overline{F}$ and a projection $r \in (1-p)\mathscr{J}(1-p)$ such that

(1) $\|(p+r)x - x(p+r)\| < 32\epsilon$ *and* $\|rx - xr\| < 24\epsilon$ *for* $x \in \mathscr{F}$,

(2) $(p+r)x(p+r) \in_{18\epsilon} F + r\mathscr{J}r$ *for* $x \in \mathscr{F}$ *and*

(3) $n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big] \le \big[f_{\sigma_4}^{\sigma_3}((p+r)a(p+r))\big]$.

Moreover, if $\mathrm{TR}(\mathscr{J}) \le k$, $\mathrm{TR}(\mathscr{B}) = 0$, *then* $\mathrm{TR}(\mathscr{A}) \le k$.

PROOF. Let $\delta_4 > d_4 > d_3 > \sigma_3, \sigma_2 > d_1 > d_2 > \delta_1$. By Lemma 3.1, we can find a finite dimensional C^*-subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ with $1_{F_0} = p_0$, and an approximate unit $\{r_n\}_1^\infty$ consisting of projections in $\mathscr{J}$ such that $\pi(p_0) = q$, $\pi(\mathscr{A}_0) = \overline{F}$ and r_n commute with $\mathscr{A}_0$, $n = 1, 2, \ldots$.

Condition (i) implies that there are $a_x, b_x \in \mathscr{J}$, $c_x \in \mathscr{A}_0$ such that

$$\|xp_0 - p_0x - a_x\| < \epsilon, \quad \|p_0xp_0 - b_x - c_x\| < \epsilon, \qquad \forall x \in \mathscr{F}.$$

Choose n_0 such that

$$\|(1-r_{n_0})a_x\| < \epsilon, \quad \|a_x(1-r_{n_0})\| < \epsilon, \\ \|(1-r_{n_0})b_x\| < \epsilon \quad \text{and} \quad \|b_x(1-r_{n_0})\| < \epsilon$$

for all $x \in \mathscr{F}$. Set $p = p_0(1-r_{n_0})$, $F_1 = p\mathscr{A}_0p$. Then $\pi(p) = q$, $\pi(F_1) = \overline{F}$ and F_1 is finite-dimensional with unit p. We have $pxp \in_{2\epsilon} F_1$, $\|xp - p_0xp\| < 2\epsilon$ for all $x \in \mathscr{F}$. Moreover, for any $x \in \mathscr{F}$,

$$\begin{aligned} \|xp - pxp_0\| &\le \|xp - p_0xp\| + \|b_x(1-r_{n_0}) - (1-r_{n_0})b_x\| \\ &\quad + \|(p_0xp_0 - b_x - c_x)(1-r_{n_0}) \\ &\qquad - (1-r_{n_0})(p_0xp_0 - b_x - c_x)\| < 6\epsilon, \\ \|xp - px\| &= \|xp - pxp_0 + p(xp_0 - p_0x - a_x) + pa_x\| < 8\epsilon \end{aligned}$$

Applying 3.3 to condition (ii), we can find a nonzero projection $r \in (1-p)\mathscr{J}(1-p)$ such that

$$n\big[f_{\sigma_2}^{\sigma_1}((1-p-r)a(1-p-r))\big] \le \big[f_{d_2}^{d_1}(pap)\big] \tag{e-5}$$

and $\|xr - rx\| < 24\epsilon$, $\forall x \in \mathscr{F}$. Therefore, we have

$$\|x(p+r) - (p+r)x\| < 32\epsilon, \quad (p+r)x(p+r) \in_{18\epsilon} F_1 + r\mathscr{J}r, \quad \forall x \in \mathscr{F}$$

and

$$n\big[f_{\delta_2}^{\delta_1}((1-p-r)a(1-p-r))\big] \le \big[f_{\delta_4}^{\delta_3}(pap)\big] \le \big[f_{\sigma_4}^{\sigma_3}((p+r)a(p+r))\big] \tag{e-6}$$

by (e-5) and Corollary 2.7.

Now assume that $\mathrm{TR}(\mathscr{B}) = 0$ and $\mathrm{TR}(\mathscr{J}) \leq k$. Then there is a finite dimensional C^*-subalgebra $\overline{F}$ of $\mathscr{B}$ with $1_{\overline{F}} = \overline{q}$ such that

(a) $\|\pi(x)\overline{q} - \overline{q}\pi(x)\| < \epsilon$,

(b) $\overline{q}\pi(x)\overline{q} \in_\epsilon \overline{F}$ for all $x \in \mathscr{F}$ and

(c) $\big[f_{\delta_2}^{\delta_1}((1-\bar{q})\pi(a)(1-\bar{q}))\big] \leq \big[f_{\delta_4}^{\delta_3}(\bar{q}\pi(a)\bar{q})\big]$.

Thus by the above argument, there are a finite dimensional C^*-subalgebra F of $\mathscr{A}$ with $1_F = p, \pi(p) = \bar{q}, \pi(F) = \bar{F}$ and a projection $r \in (1-p)\mathscr{J}(1-p)$ such that

$$\text{(e-7)}\quad \|px - xp\| < 8\epsilon, \quad \|rx - xr\| < 24\epsilon, \quad pxp \in_{2\epsilon} F,$$
$$\text{for all } x \in \mathscr{F} \text{ and } n\big[f_{d_2}^{d_1}((1-p-r)a(1-p-r))\big] \leq \big[f_{d_4}^{d_3}(pap)\big].$$

By Lemma 2.9, $\mathrm{TR}(\mathscr{J}) \leq k$ implies that $\mathrm{TR}(r\mathscr{J}r) \leq k$. So that there is a nonzero projection $e \leq r$ and there is a C^*-subalgebra $\mathscr{J}_0$ of $r\mathscr{J}r$ with unit e which belongs to $\mathscr{I}^{(k)}$ such that

$$\text{(e-8)}\quad \|erxr - rxre\| < \epsilon, \quad exe \in_\epsilon \mathscr{J}_0,$$
$$\text{for all } x \in \mathscr{F} \text{ and } \big[f_{d_2}^{d_1}((r-e)a(r-e))\big] \leq \big[f_{d_4}^{d_3}(eae)\big].$$

Set $p_1 = p + e$ and $\mathscr{A}_\epsilon = F + \mathscr{J}_0$. Then $\mathscr{A}_\epsilon \in \mathscr{I}^{(k)}$ with the unit p_1. We also have $\|p_1x - xp_1\| < 41\epsilon$ and $p_1xp_1 \in_{19\epsilon} \mathscr{A}_\epsilon$, $\forall x \in \mathscr{F}$. Furthermore, with sufficiently ϵ,

$$\begin{aligned}
&\big[f_{\sigma_2}^{\sigma_1}((1-p_1)a(1-p_1))\big] \\
&\quad \leq \big[f_{d_2}^{d_1}((1-p-r)a(1-p-r))\big] \\
&\qquad + \big[f_{d_2}^{d_1}((r-e)a(r-e))\big] && \text{(by Lemma 2.5)} \\
&\quad \leq \big[f_{d_4}^{d_3}(pap)\big] + \big[f_{d_4}^{d_3}(eae)\big] && \text{(by (e-7) and (e-8))} \\
&\quad = \big[f_{d_4}^{d_3}(pap + eae)\big] \leq \big[f_{\sigma_3}^{\sigma_4}(p_1ap_1)\big] && \text{(by Lemma 2.5).}
\end{aligned}$$

4. Tracially quasidiagonal extensions of C^*-algebras

DEFINITION 4.1. Let $(\mathscr{J}, \mathscr{A}, \pi)$ be as in §1. We say that $(\mathscr{A}, \mathscr{J})$ is *tracially quasidiagonal* if for any $\epsilon > 0$, any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$ and any subset $\mathscr{F} = \{x_1, \ldots, x_k, b\} \subset \mathscr{A}$ with $b \in \mathscr{A}_+$, there exist a projection $p \in \mathscr{A}$ and a C^*-subalgebra $\mathscr{C}$ of $\mathscr{A}$ with $1_{\mathscr{C}} = p$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F}$;

(2) $\big[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))\big] \le \big[f_{\sigma_3}^{\sigma_4}(pbp)\big]$;

(3) $pxp \in_\epsilon \mathscr{C}$ for all $x \in \mathscr{F}$ and

(4) $C \cap \mathscr{J} = p\mathscr{J}p$ and $(C, p\mathscr{J}p)$ is quasidiagonal.

Obviously, if $\mathscr{A}$ is quasidiagonal, then $\mathscr{A}$ is tracially quasidiagonal.

DEFINITION 4.2. Let $\mathscr{A}$ be a unital C^*-algebra. We say that $\mathscr{A}$ satisfies the property $(\mathrm{P_k})$ if the following holds: for any $\epsilon > 0$, any integer n, any finite subset $\mathscr{F} \subset A$ containing a nonzero element $a \in \mathscr{A}_+$ and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$ there exist a projection and a C^*-subalgebra $\mathscr{C}$ of $\mathscr{A}$ with $1_{\mathscr{C}} = p$ and with $\mathrm{TR}(\mathscr{C}) \le k$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F}$,

(2) $n\big[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))\big] \le 2\big[f_{\sigma_4}^{\sigma_3}(pap)\big]$ and

(3) $pxp \in_\epsilon \mathscr{C}$ for all $x \in \mathscr{F}$.

LEMMA 4.3. *Let $\mathscr{A}$ be a unital C^*-algebra with the property* $(\mathrm{P_k})$. *Then for any $\epsilon > 0$, any integer $n > 0$, any finite subset $\mathscr{F} \subset \mathscr{A}$ containing nonzero element $a \in \mathscr{A}_+$ and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exists a C^*-subalgebra $\mathscr{B} \in \mathscr{I}^{(k)}$ of $\mathscr{A}$ with $1_{\mathscr{B}} = p$ such that*

(1) *$\|px - xp\| < \epsilon$, $pxp \in_\epsilon \mathscr{B}$, for all $x \in \mathscr{F}$;*

(2) *$n\big[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))\big] \le 3\big[f_{\sigma_4}^{\sigma_3}(pap)\big]$.*

PROOF. For any $\epsilon > 0$ and any $0 < \sigma_4 < \sigma_3 < d_{10} < d_9 < \cdots < d_2 < d_1 < \sigma_2 < \sigma_1 < 1$, there is a C^*-subalgebra $\mathscr{C}$ of $\mathscr{A}$ with $1_{\mathscr{C}} = p_1$ and with $\mathrm{TR}(\mathscr{C}) \le k$ such that

(1) $\|p_1x - xp_1\| < \epsilon/3$, $p_1xp_1 \in_{\epsilon/3} \mathscr{C}$, for all $x \in \mathscr{F}$;

(2) $n\big[f_{d_2}^{d_1}((1-p_1)a(1-p_1))\big] \le 2\big[f_{d_4}^{d_3}(p_1ap_1)\big]$.

For any $x \in \mathscr{F}$, choose $\hat{x} \in \mathscr{C}$ such that $\|p_1xp_1 - \hat{x}\| < \epsilon/3$. Since $\mathrm{TR}(\mathscr{C}) \le k$, it follows that there is a C^*-subalgebra $\mathscr{B}$ of $\mathscr{C}$ with $1_{\mathscr{B}} = p \le p_1$ and $\mathscr{B} \in \mathscr{I}^{(k)}$ such that

(3) $\|p\hat{x} - \hat{x}p\| < \epsilon/3$, $p\hat{x}p \in_{\epsilon/3} \mathscr{B}$, for all $x \in \mathscr{F}$;

(4) $(n+2)\big[f_{d_8}^{d_7}((p_1-p)\hat{a}(p_1-p))\big] \le \big[f_{d_{10}}^{d_9}(p\hat{a}p)\big]$.

From condition (1) and (3), it is easy to check that $\|xp - px\| < \epsilon$ and $pxp \in_\epsilon \mathscr{B}$, for all $x \in \mathscr{F}$. By Lemma 2.5 (5),

$$\big[f_{d_2}^{d_1}((p_1-p)a(p_1-p))\big] \le \big[f_{d_8}^{d_7}((p_1-p)\hat{a}(p_1-p))\big]$$
$$\text{and}\quad \big[f_{d_4}^{d_3}(p_1ap_1)\big] \le \big[f_{d_6}^{d_5}(\hat{a})\big]$$

with ϵ sufficiently small. Therefore, by applying Lemma 2.5 to Condition (2) and (4), we have, with sufficiently small ϵ,

$$\begin{aligned}
n\big[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))\big]
&\le n\big[f_{d_2}^{d_1}((1-p_1)a(1-p_1))\big] + n\big[f_{d_2}^{d_1}((p_1-p)a(p_1-p))\big]\\
&\le 2\big[f_{d_4}^{d_3}(p_1ap_1)\big] + n\big[f_{d_2}^{d_1}((p_1-p)a(p_1-p))\big]\\
&\le 2\big[f_{d_6}^{d_5}(\hat{a})\big] + n\big[f_{d_8}^{d_7}((p_1-p)\hat{a}(p_1-p))\big]\\
&\le (n+2)\big[f_{d_8}^{d_7}((p_1-p)\hat{a}(p_1-p))\big] + \big[f_{d_8}^{d_7}(p\hat{a}p)\big]\\
&\le 2\big[f_{d_{10}}^{d_9}(p\hat{a}p)\big] + \big[f_{d_{10}}^{d_9}(p\hat{a}p)\big]\\
&\le 3\big[f_{\sigma_4}^{\sigma_3}(pap)\big].
\end{aligned}$$

LEMMA 4.4. *Suppose that $\mathscr{B}$ is a finite dimensional C^*-algebra. Then for any $a, b \in \mathscr{B}_+$, any integer k and any $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$,*

$$k\big[f_{\delta_2}^{\delta_1}(a)\big] \le k\big[f_{\delta_4}^{\delta_3}(b)\big] \qquad \textit{implies} \qquad \big[f_{\sigma_2}^{\sigma_1}(a)\big] \le \big[f_{\sigma_4}^{\sigma_3}(b)\big].$$

PROOF. There exists $w \in M_k(\mathscr{B})$ such that

$$w^*w = \operatorname{diag}\big(\overbrace{f_{\delta_2}^{\delta_1}(a), \dots, f_{\delta_2}^{\delta_1}(a)}^{k}\big)$$
$$\text{and} \quad ww^* \in \operatorname{Her}\big(\operatorname{diag}\big(\overbrace{f_{\delta_4}^{\delta_3}(b), \dots, f_{\delta_4}^{\delta_3}(b)}^{k}\big)\big).$$

Let σ_3', σ_4' be such that $\sigma_4 < \sigma_3 < \sigma_4' < \sigma_3' < \delta_4 < \delta_3$. Since $f_{\sigma_4'}^{\sigma_3'}(b) f_{\delta_4}^{\delta_3}(b) = f_{\delta_4}^{\delta_3}(b)$, we have

$$ww^* \le \operatorname{diag}\big(f_{\sigma_4'}^{\sigma_3'}(b), \dots, f_{\sigma_4'}^{\sigma_3'}(b)\big).$$

If Tr is the standard trace defined on $\mathscr{B}$, then

$$\begin{aligned}
\operatorname{rank}\operatorname{diag}\big(f_{\sigma_2}^{\sigma_1}(a), \dots, f_{\sigma_2}^{\sigma_1}(a)\big) &\le \operatorname{Tr}\big[\operatorname{diag}(f_{\delta_2}^{\delta_1}(a), \dots, f_{\delta_2}^{\delta_1}(a))\big]\\
&= \operatorname{Tr}(w^*w) = \operatorname{Tr}(ww^*)\\
&\le \operatorname{Tr}\operatorname{diag}\big(f_{\sigma_4'}^{\sigma_3'}(b), \dots, f_{\sigma_4'}^{\sigma_3'}(b)\big)\\
&\le \operatorname{rank}\operatorname{diag}\big(f_{\sigma_4}^{\sigma_3}(b), \dots, f_{\sigma_4}^{\sigma_3}(b)\big).
\end{aligned}$$

It follows that $\operatorname{rank}(f_{\sigma_2}^{\sigma_1}(a)) \le \operatorname{rank}(f_{\sigma_4}^{\sigma_3}(b))$ and $[f_{\sigma_2}^{\sigma_1}(a)] \le [f_{\sigma_4}^{\sigma_3}(b)]$.

LEMMA 4.5. *Let $\mathscr{A}$ be a C^*-algebra satisfying the property* (P_0). *Then for any integer $n > 0$, any $a, b \in \mathscr{A}_+$, any $0 < \sigma_4 < \sigma_3 < \delta_4 < \delta_3 < \delta_2 < \delta_1 < \sigma_2 < \sigma_1 < 1$,*

$$9(n+1)\big[f_{\delta_2}^{\delta_1}(a)\big] \le 3\big[f_{\delta_4}^{\delta_3}(b)\big] \qquad \textit{implies} \qquad n\big[f_{\sigma_2}^{\sigma_1}(a)\big] \le \big[f_{\sigma_4}^{\sigma_3}(b)\big].$$

PROOF. There is $w \in M_{9(n+1)}(\mathscr{A})$ such that

$$w^*w = \mathrm{diag}\big(\overbrace{f_{\delta_2}^{\delta_1}(a), \dots, f_{\delta_2}^{\delta_1}(a)}^{9(n+1)}\big)$$
$$\text{and} \quad ww^* \in \mathrm{Her}\big(\mathrm{diag}\big(f_{\delta_4}^{\delta_3}(b), f_{\delta_4}^{\delta_3}(b), f_{\delta_4}^{\delta_3}(b), 0, \dots, 0\big)\big).$$

Let $b_0 = \mathrm{diag}(f_{\delta_4'}^{\delta_3'}(b), f_{\delta_4'}^{\delta_3'}(b), f_{\delta_4'}^{\delta_3'}(b), 0, \dots, 0)$, where $\sigma_3 < \delta_4' < \delta_3' < \delta_4$. Then

$$ww^* = b_0^{1/2} ww^* b_0^{1/2} \le b_0.$$

Set $d_i = \frac{1}{i}$, $i = 1, \dots, 24$. Write $w = (w_{ij})_{9(n+1)\times 9(n+1)}$ and set

$$\mathscr{F} = \{w_{ij} : 1 \le i, j \le 9(n+1)\} \cup \{a, b\}.$$

By Lemma 4.3, there exist a projection p and a finite dimensional C^*-subalgebra $\mathscr{B}$ with $1_{\mathscr{B}} = p$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F}$, $pxp \in_\epsilon \mathscr{B}$ for all $x \in \mathscr{F}$ and

(2) $n\big[f_{d_4}^{d_3}\big((1-p) f_{\delta_2}^{\delta_1}(a)(1-p)\big)\big] \le 3\big[f_{d_6}^{d_5}\big(p f_{\delta_2}^{\delta_1}(a) p\big)\big]$.

From (1), we have $\hat{a}, \hat{b} \in \mathscr{B}$ and $\hat{w}_{ij} \in \mathscr{B}$ such that

$$\|pap - \hat{a}\| < \epsilon, \qquad \|pbp - \hat{b}\| < \epsilon, \qquad \|pw_{ij}p - \hat{w}_{ij}\| < \epsilon.$$

Let $\hat{w} = (\hat{w}_{ij})$ and $\hat{b}_0 = \mathrm{diag}(f_{\delta_4'}^{\delta_3'}(\hat{b}), f_{\delta_4'}^{\delta_3'}(\hat{b}), f_{\delta_4'}^{\delta_3'}(\hat{b}), 0, \dots, 0)$. Note

$$\begin{aligned}
&\|\,\mathrm{diag}(f_{\delta_2}^{\delta_1}(\hat{a}), \dots, f_{\delta_2}^{\delta_1}(\hat{a})) - \hat{w}^*\hat{w}\|\\
&\le \|\,\mathrm{diag}(f_{\delta_2}^{\delta_1}(\hat{a}), \dots, f_{\delta_2}^{\delta_1}(\hat{a})) - \mathrm{diag}(pf_{\delta_2}^{\delta_1}(a)p, \dots, pf_{\delta_2}^{\delta_1}(a)p)\|\\
&+ \|\,\mathrm{diag}(pf_{\delta_2}^{\delta_1}(a)p, \dots, pf_{\delta_2}^{\delta_1}(a)p) - \mathrm{diag}(p, \dots, p)w^*w\,\mathrm{diag}(p, \dots, p)\|\\
&+ \|\,\mathrm{diag}(p, \dots, p)w^*w\,\mathrm{diag}(p, \dots, p) - \hat{w}^*\hat{w}\|
\end{aligned}$$

and

$$\begin{aligned}\big\|\hat{w}\hat{w}^* - \hat{b}_0^{1/2}\hat{w}\hat{w}^*\hat{b}_0^{1/2}\big\| \\ &\le \|\hat{w}\hat{w}^* - \operatorname{diag}(p,\dots,p)ww^*\operatorname{diag}(p,\cdots,p)\| \\ &\quad + \big\|\operatorname{diag}(p,\dots,p)ww^*\operatorname{diag}(p,\dots,p) \\ &\qquad - \operatorname{diag}(p,\dots,p)b_0^{1/2}ww^*b_0^{1/2}\operatorname{diag}(p,\dots,p)\big\| \\ &\quad + \big\|\operatorname{diag}(p,\dots,p)b_0^{1/2}ww^*b_0^{1/2}\operatorname{diag}(p,\dots,p) - \hat{b}_0^{1/2}\hat{w}\hat{w}^*\hat{b}_0^{1/2}\big\|.\end{aligned}$$

So by Lemma 2.11, for sufficiently small ϵ,

$$\big\|\operatorname{diag}\big(f_{\delta_2}^{\delta_1}(\hat{a}),\dots,f_{\delta_2}^{\delta_1}(\hat{a})\big) - \hat{w}^*\hat{w}\big\| < \delta(d_{11},d_{12}) \quad \text{and} \quad \big\|\hat{w}\hat{w}^* - \hat{b}_0^{1/2}\hat{w}\hat{w}^*\hat{b}_0^{1/2}\big\| < \delta(d_{13},d_{14}),$$

where $\delta(d_{11},d_{12})$ and $\delta(d_{13},d_{14})$ are as in (5) of Lemma 2.5. Therefore, in $M_{9(n+1)}(\mathscr{B})$, we have

$$\begin{aligned}[f_{d_{10}}^{d_9}(\operatorname{diag}(f_{\delta_2}^{\delta_1}(\hat{a}),\dots,f_{\delta_2}^{\delta_1}(\hat{a}))] &\le [f_{d_{12}}^{d_{11}}(\hat{w}^*\hat{w})] && \text{(by Lemma 2.5 (5))} \\ &= [f_{d_{12}}^{d_{11}}(\hat{w}\hat{w}^*)] && \text{(by Lemma 2.5 (3))} \\ &\le [f_{d_{14}}^{d_{13}}(\hat{b}_0^{1/2}\hat{w}\hat{w}^*\hat{b}_0^{1/2})] && \text{(by Lemma 2.5 (5))} \\ &\le [f_{d_{16}}^{d_{15}}(\hat{b}_0)] && \text{(by Lemma 2.5 (6)),}\end{aligned}$$

that is $9(n+1)[f_{d_{10}}^{d_9}\circ f_{\delta_2}^{\delta_1}(\hat{a})] \le 3[f_{d_{16}}^{d_{15}}\circ(f_{\delta_4'}^{\delta_3'}(\hat{b})]$ in $M_{9(n+1)}(\mathscr{B})$.

For sufficiently small ϵ, (in $M_{9(n+1)}(\mathscr{B})$) we have

$$\begin{aligned}3\big[f_{d_{16}}^{d_{15}}\circ(f_{\delta_4'}^{\delta_3'}(\hat{b})\big] &\le 3\big[f_{d_{18}}^{d_{17}}(pf_{\delta_4'}^{\delta_3'}(b)p)\big] && \text{(by Lemma 2.5 (5))} \\ &\le 3\big[f_{d_{20}}^{d_{19}}\circ f_{\delta_4'}^{\delta_3'}(\hat{b})\big].\end{aligned}$$

Thus we obtain $9(n+1)[f_{d_{10}}^{d_9}\circ f_{\delta_2}^{\delta_1}(\hat{a})] \le 3[f_{d_{20}}^{d_{19}}\circ f_{\delta_4'}^{\delta_3'}(\hat{b})]$ in $M_{9(n+1)}(\mathscr{B})$. It follows from Lemma 4.4,

$$3(n+1)\big[f_{d_8}^{d_7}\circ f_{\delta_2}^{\delta_1}(\hat{a})\big] \le \big[f_{d_{20}}^{d_{19}}\circ f_{\delta_4'}^{\delta_3'}(\hat{b})\big]. \tag{e-9}$$

Finally, we have

$$\begin{aligned}
&n[f_{\sigma_2}^{\sigma_1}(a)]\\
&\quad\le n[f_{d_2}^{d_1}\circ f_{\delta_2}^{\delta_1}(a)] \quad \text{(by Corollary 2.6 (2))}\\
&\quad\le n[f_{d_4}^{d_3}((1-p)f_{\delta_2}^{\delta_1}(a)(1-p))]+n[f_{d_4}^{d_3}(pf_{\delta_2}^{\delta_1}(a)p)] \quad \text{(by Corollary 2.7)}\\
&\quad\le (n+3)[f_{d_6}^{d_5}(pf_{\delta_2}^{\delta_1}(a)p)] \quad \text{(by (2))}\\
&\quad\le 3(n+1)[f_{d_8}^{d_7}\circ f_{\delta_2}^{\delta_1}(\hat{a})] \quad \text{(by Lemma 2.5 (5))}\\
&\quad\le [f_{d_{20}}^{d_{19}}\circ f_{\delta_4'}^{\delta_3'}(\hat{b})] \quad \text{(by (e-9))}\\
&\quad\le [f_{d_{22}}^{d_{21}}(pf_{\delta_4'}^{\delta_3'}(b)p)] \quad \text{(by Lemma 2.5 (5))}\\
&\quad\le [f_{d_{22}}^{d_{21}}(pf_{\delta_4'}^{\delta_3'}(b)p)]+[f_{d_{22}}^{d_{21}}((1-p)f_{\delta_4'}^{\delta_3'}(b)(1-p))]\\
&\quad\le [f_{d_{24}}^{d_{23}}\circ f_{\delta_4'}^{\delta_3'}(b)] \quad \text{(by Corollary 2.7)}\\
&\quad\le [f_{\sigma_4}^{\sigma_3}(b)] \quad \text{(by Corollary 2.6 (2)).}
\end{aligned}$$

THEOREM 4.6. *Let $\mathscr{A}$ be a unital C^*-algebra having* (P_0). *Then* $\mathrm{TR}(\mathscr{A}) = 0$.

PROOF. Let $\epsilon > 0$ and $n > 0$ be an integer and $\mathscr{F} \subset \mathscr{A}$ be a finite subset containing nonzero elements $a \in \mathscr{A}_+$ and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$. By Lemma 4.5, there are a projection $p_1 \in \mathscr{A}$ and a finite dimensional C^*-subalgebra $\mathscr{B}$ of $\mathscr{A}$ with $1_{\mathscr{B}} = p_1$ such that

(1) $\|p_1x - xp_1\| < \epsilon$, $p_1xp_1 \in_\epsilon \mathscr{B}$, for all $x \in \mathscr{F}$,

(1) $9(n+1)\big[f_{d_2}^{d_1}((1-p_1)a(1-p_1))\big] \le 3\big[f_{d_4}^{d_3}(p_1ap_1)\big]$.

Thus by Lemma 4.5, we have $n[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \le [f_{\sigma_4}^{\sigma_3}(pap)]$. It follows that $\mathrm{TR}(\mathscr{A}) = 0$.

5. The main result

THEOREM 5.1. *Let*

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow 0$$

be a short exact sequence of C^*-algebras, where $\mathscr{A}$ is a unital C^*-algebra. Suppose that $\mathrm{TR}(\mathscr{J}) = 0$ *and* $\mathrm{TR}(\mathscr{B}) = 0$. *Then* $\mathrm{TR}(\mathscr{A}) = 0$ *if* $(\mathscr{A}, \mathscr{J})$ *is tracially quasidiagonal.*

PROOF. Fix $\epsilon > 0$, an integer n, a finite subset $\mathscr{F} \subset \mathscr{A}$ containing a positive element a and $\sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$. Let $\sigma_3 < d_{22} < d_{21} < \cdots < d_1 < \sigma_2$.

Without loss of generality, we may assume that $\mathscr{F}$ is in the unit ball of $\mathscr{A}$. Choose a positive number η which depends only on ϵ, $\mathscr{F}$, $\sigma_i's$ and $d_i's$.

Since $\mathrm{TR}(\mathscr{B}) = 0$, there is a finite dimensional C^*-subalgebra $\bar{F} \subset \mathscr{B}$ with $1_{\bar{F}} = \bar{q}$ such that

(1) $\|\bar{q}\pi(x) - \pi(x)\bar{q}\| < \eta$ for all $x \in \mathscr{F}$,

(2) $\bar{q}\pi(x)\bar{q} \in_\eta \bar{F}$ for all $x \in \mathscr{F}$ and

(3) $(n+1)\big[f_{d_{14}}^{d_{13}}((1-\bar{q})\pi(a)(1-\bar{q}))\big] \leq \big[f_{d_{16}}^{d_{15}}(\bar{q}\pi(a)\bar{q})\big]$.

Let $\{\bar{e}_j\}$ be a finite set of standard (partial isometries and projections) generators of $\bar{F}$. Let $\{a_j\} \subset \mathscr{A}$ such that $\|a_j\| \leq 1$ and $\pi(a_j) = \bar{e}_j$. Let x_i ($i = 1, 2, \ldots, n+1$) be in $\mathscr{B}$ such that $x_i^* x_i = f_{d_{14}}^{d_{13}}((1-\bar{q})\pi(a)(1-\bar{q}))$ and $x_i x_i^*$ ($i = 1, 2, \ldots, n+1$) are mutually orthogonal elements in $\mathrm{Her}(f_{d_{16}}^{d_{15}}(\bar{q}\pi(a)\bar{q}))$. Put $y_i \in \mathscr{A}$ with $\pi(y_i) = x_i$ and $\|y_i\| \leq 1$ ($i = 1, 2, \ldots, n+1$) and $0 \leq b_i \leq 1$ in $\mathscr{A}$ such that $\pi(b_i) = x_i x_i^*$ and $b_1, b_2, \ldots, b_{n+1}$ are mutually orthogonal. Let $0 \leq q' \leq 1$ be in $\mathscr{A}$ such that $\pi(q') = \bar{q}$. Set

$$\mathscr{G} = \mathscr{F} \cup \big\{a_j, y_i, y_i^*, q', f_{d_{14}}^{d_{13}}((1-q')a(1-q')), f_{d_{16}}^{d_{15}}(q'aq')\big\}.$$

Choose $1/64 > \delta > 0$ which depends only on what η depends on and on $\mathscr{G}$. Since $(\mathscr{A}, \mathscr{J})$ is tracially quasidiagonal, there exists a projection $p \in \mathscr{A}$ and a C^*-subalgebra $C_0 \subset \mathscr{A}$ such that $1_{C_0} = p$, $C_0 \cap \mathscr{J} = p\mathscr{J}p$ and $(C_0, p\mathscr{J}p)$ is quasidiagonal and such that

(4) $\|px - xp\| < \delta$ for all $x \in \mathscr{G}$,

(5) $pxp \in_\delta C_0$ for all $x \in \mathscr{G}$ and

(6) $\mathrm{n}\big[f_{d_2}^{d_1}((1-p)a(1-p))\big] \leq \big[f_{d_4}^{d_3}(pap)\big]$.

Since $\|\pi(p)\bar{q} - \bar{q}\pi(p)\| < \delta$, we obtain a projection $\bar{q}_1 \in \pi(C_0)$ such that

(7) $\|\pi(pq'p) - \bar{q}_1\| < 4\delta$.

With sufficiently small δ, there is a finite set of standard generators $\{e_j'\}$ of a finite dimensional C^*-subalgebra $\bar{F}_1$ in $\pi(C_0)$ with $1_{\bar{F}_1} = \bar{q}_1$ which is isomorphic to (possibly a summand of) $\bar{F}$ such that $\|\bar{e}_j' - \bar{q}_1 e_j \bar{q}_1\| < 5\delta$. With sufficiently small δ, by Lemma 2.5 (5), we have

$$\begin{aligned}
&\big[f_{d_8}^{d_7}((\pi(p) - \bar{q}_1)\pi(a)(\pi(p) - \bar{q}_1))\big] \\
&\qquad \leq \big[f_{d_{10}}^{d_9}((\pi(p) - \pi(p)\bar{q}\pi(p))\pi(a)(\pi(p) - \pi(p)\bar{q}\pi(p)))\big] \\
&\qquad \leq \big[f_{d_{12}}^{d_{11}}(\pi(p)(1-\bar{q})\pi(a)(1-\bar{q})\pi(p))\big].
\end{aligned}$$

Since $(n+1)[f_{d_{14}}^{d_{13}}((1-\bar{q})\pi(a)(1-\bar{q}))] \le [f_{d_{16}}^{d_{15}}(\bar{q}\pi(a)\bar{q})]$, by applying Lemma 2.12, we have

$$\begin{aligned}(n+1)\big[f_{d_{12}}^{d_{11}}(\pi(p)(1-\bar{q})\pi(a)(1-\bar{q})\pi(p))\big] &\le \big[f_{d_{18}}^{d_{17}}(\pi(p)\bar{q}\pi(a)\bar{q}\pi(p))\big]\\ &\le \big[f_{d_{20}}^{d_{19}}(\pi(p)\bar{q}\pi(a)\bar{q}\pi(p))\big]\\ &\le \big[f_{d_{22}}^{d_{21}}(\bar{q}_1\pi(a)\bar{q}_1)\big].\end{aligned}$$

Therefore

(8) $(n+1)\big[f_{d_8}^{d_7}((\pi(p)-\bar{q}_1)\pi(a)(\pi(p)-\bar{q}_1))\big] \le \big[f_{d_{22}}^{d_{21}}(\bar{q}_1\pi(a)\bar{q}_1)\big]$.

Moreover, with sufficiently small δ, by using (1), (2), (4), (5) and (7), we have

(9) $\|\bar{q}_1 y - y\bar{q}_1\| < 5(\delta+\eta)$ for $y \in \pi(p\mathscr{G}p)$, and

(10) $\bar{q}_1 y \bar{q}_1 \in_{5(\delta+\eta)} \bar{F}_1$ for $y \in \pi(p\mathscr{G}p)$.

Since $(C_0, p\mathscr{J}p)$ is quasidiagonal, it follows from Theorem 3.4, with sufficiently small δ and η and using (8), (9) and (10), that there is a finite dimensional C^*-subalgebra $F_1 \subset C_0$ with $1_{F_1} = q_1$, $\pi(q_1) = \bar{q}_1$ and $\pi(F_1) = \bar{F}_1$ and a projection $r_1 \in (p-q_1)\mathscr{J}(p-q_1)$ such that

(11) $\|(q_1+r_1)x - x(q_1+r_1)\| < \eta$ and $\|rx - xr\| < \eta$ for $x \in \mathscr{F}$,

(12) $(q_1+r_1)x(q_1+r_1) \in_\eta F_1 + r_1\mathscr{J}r_1$ for $x \in \mathscr{F}$ and

(13) $(n+1)\big[f_{d_6}^{d_5}((p-q_1-r_1)a(p-q_1-r_1))\big] \le \big[f_{\sigma_4}^{\sigma_3}((q_1+r_1)a(q_1+r_1))\big]$.

Note $q_1 \in C_0$. Let $C_1 = F_1 + r_1\mathscr{J}r_1$. Since $\mathrm{TR}(\mathscr{J}) = 0$ and $r_1 \in \mathscr{J}$, then $\mathrm{TR}(r_1\mathscr{J}r_1) = 0$. Note $q_1 \in C_0$ and q_1 and r_1 are orthogonal. So $\mathrm{TR}(C_0) = 0$. It follows that $\mathrm{TR}(C_1) = 0$ and $1_{C_1} = q_1 + r_1$. Now,

$$\begin{aligned}
&n[f_{\sigma_2}^{\sigma_1}((1-q_1-r_1)a(1-q_1-r_1))]\\
&\le n[f_{d_2}^{d_1}((1-p)q(1-p))]\\
&\qquad + n[f_{d_2}^{d_1}((p-q_1-r_1)a(p-q_1-r_1))] && \text{(by Corollary 2.7)}\\
&\le [f_{d_4}^{d_3}(pap)] + n[f_{d_2}^{d_1}((p-q_1-r_1)a(p-q_1-r_1))] && \text{(by (6))}\\
&\le [f_{d_6}^{d_5}((p-q_1-r_1)a(p-q_1-r_1))] + [f_{d_2}^{d_1}((q_1+r_1)a(q_1+r_1))]\\
&+ n[f_{d_2}^{d_1}((p-q_1-r_1)a(p-q_1-r_1))] && \text{(by Corollary 2.7)}\\
&\le (n+1)[f_{d_6}^{d_5}((p-q_1-r_1)a(p-q_1-r_1))] + [f_{d_6}^{d_5}((q_1+r_1)a(q_1+r_1))]\\
&\le 2[f_{\sigma_4}^{\sigma_3}(q_1+r_1)a(q_1+r_1))] && \text{(by (13))}.
\end{aligned}$$

In other words, $\mathscr{A}$ satisfies the property (P_0). Thus, by Theorem 4.6, we have shown $\mathrm{TR}(\mathscr{A}) = 0$.

THEOREM 5.2. *Let* $0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow 0$ *be a short exact sequence of* C^**-algebras, where* $\mathscr{A}$ *is a unital* C^**-algebra. Suppose that* $\mathrm{TR}(e\mathscr{J}e) = 0$ *for any projection* $e \in \mathscr{A}$ *and* $\mathrm{TR}(\mathscr{B}) = 0$*. Then* $\mathrm{TR}(\mathscr{A}) = 0$ *if and only if* $(\mathscr{A}, \mathscr{J})$ *is tracially quasidiagonal.*

PROOF. By Theorem 5.1, it suffices to show the "only if" part. So we assume that $\mathrm{TR}(\mathscr{A}) = 0$. For any $\epsilon > 0$, and finite subset $\mathscr{F} \subset \mathscr{A}$ containing nonzero element $a \in \mathscr{A}_+$ and any $0 < \sigma_2 < \sigma_1 < 1$, there exists a finite dimensional C^*-subalgebra D of $\mathscr{A}$ with $1_D = p$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathscr{F}$,

(2) $pxp \in_\epsilon D$ for all $x \in \mathscr{F}$,

(3) $[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \le [f_{\sigma_4}^{\sigma_3}(pap)]$ (where $0 < \sigma_4 < \sigma_3 < \sigma_2$).

Write $D = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_l}$. Let e_i^k, $i = 1, 2, \ldots, n_k$ be a maximal set of mutually orthogonal minimal projections in M_{n_k} $(k = 1, 2, \ldots, l)$. By the assumption $\mathrm{TR}(e_i^k \mathscr{J} e_i^k) = 0$. It follows from Corollary 2.16 that $e_i^k \mathscr{J} e_i^k$ admits an approximate identity $\{u_n^{ik}\}$ consisting of projections. Put $d_n^k = \mathrm{diag}(u_n^{1k}, \ldots, u_n^{n_k k})$ and $E_n = d_n^1 \oplus d_n^2 \oplus \cdots \oplus d_n^l$. Then $\{E_n\}$ commutes with every element in D. Furthermore, $\{E_n\}$ forms an approximate identity for $p\mathscr{J}p$ consisting of projections. Therefore

$$\|E_n x - x E_n\| \to 0 \qquad \text{as} \qquad n \to \infty$$

for all $x \in D + p\mathscr{J}p$. Let $C = D + p\mathscr{J}p$. Then $C \cap \mathscr{J} = p\mathscr{J}p$, $(C, p\mathscr{J}p)$ is quasidiagonal and $C/p\mathscr{J}p$ is a quotient of D which is finite dimensional. This shows that $(A, \mathscr{J})$ is tracially quasidiagonal.

REMARK 5.3. There are examples of tracially quasidiagonal extensions which are not quasidiagonal. These examples and unexpected interesting consequences will appear elsewhere(see [17]).

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