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A NOTE ON A THEOREM OF SPARR

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Abstract

We prove that, regardless of the choice of a positive, concave function ψ on R_+ and a "weight function" λ, the weighted ℓ_2-space $\ell_2(\psi(\lambda))$ is c-interpolation with respect to the couple $(\ell_2, \ell_2(\lambda))$, where $c \leq \sqrt{2}$. Our main result is that $c = \sqrt{2}$ is best possible here; a fact which is implicit in the work of G. Sparr.

1. A lemma on Pick functions

Of general interest in the theory of interpolation spaces is the class P' of functions representable in the form

$$(1) \qquad h(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda}\, d\rho(t), \qquad \lambda \in \mathsf{R}_+,$$

where ρ is some positive Radon measure on R_+. This class is known as the set of *positive Pick functions on* R_+ (cf. [2] or [4]). It is easy to see that P' constitutes a subcone of the convex cone of positive concave functions on R_+.

In the following, it will be convenient besides (1) to work with a modified representation for P'-functions (cf. [5], p. 266)

$$(2) \qquad h(\lambda) = \alpha + \beta\lambda + \int_0^\infty \frac{\lambda t}{\lambda + t}\, d\nu(t),$$

where $\alpha \geq 0, \beta \geq 0$ and ν is a positive measure on R_+ such that $\int_0^\infty d\nu(t)/(1+t^{-1}) < \infty$.

We have the following basic lemma.

LEMMA 1.1. *Let ψ be a positive concave function on R_+. Then there exists a function $h \in P'$ such that $h \leq \psi \leq 2h$.*

PROOF (Cf. Peetre [11], bottom of p. 168.). It is well-known that an arbitrary positive, concave function can be represented in the form (cf. [3], p. 117)

$$(3) \qquad \psi(\lambda) = \alpha + \beta\lambda + \int_0^\infty \min(\lambda, t)\, d\nu(t),$$

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where $\alpha \geq 0$, $\beta \geq 0$ and ν a positive measure on R_+ such that $\int_0^\infty d\nu(t)/(1+t^{-1}) < \infty$. Next observe that for $\lambda, t > 0$

$$\frac{\lambda t}{\lambda + t} \leq \min(\lambda, t) \leq 2\frac{\lambda t}{\lambda + t}.$$

The lemma now follows from (2) and (3) on integration with respect to ν.

2. The Foiaş -Ong-Rosenthal question

As we shall see presently, Lemma 1.1 is closely related to an interpolation theorem of Foiaş , Ong and Rosenthal [8], which goes back to the work of Jaak Peetre [10], [11]. Before we formulate this theorem, let us remind of some notions from the theory of interpolation spaces. (For more details on this theory, we refer to [3]).

Relative to a Hilbert couple $\overline{\mathscr{H}} = (\mathscr{H}_0, \mathscr{H}_1)$, we have the K_2-functional

$$K_2(t, f) = K_2(t, f; \overline{\mathscr{H}}) = \inf_{f=f_0+f_1} (\|f_0\|_0^2 + t\|f_1\|_1^2)^{1/2}.$$

Let $\mathscr{H}_0 \cap \mathscr{H}_1$ be dense in $\mathscr{H}_0$ and in $\mathscr{H}_1$. The basic fact for K_2 is the following (see e.g. [1]). Denote by A the unbounded, densely defined, positive, injective operator in $\mathscr{H}_0$ such that

$$\|f\|_1^2 = (Af, f)_0, \qquad f \in \mathscr{H}_0 \cap \mathscr{H}_1,$$

then

$$K_2(t, f)^2 = \left(\frac{tA}{1 + tA} f, f\right)_0. \tag{4}$$

With respect to $\mathscr{H}_0$ and $\mathscr{H}_1$ it will be advantageous to make use of several notations for the operator norms.

$$\begin{aligned} \|T\|^2 &= \|T\|^2_{\mathscr{L}(\mathscr{H}_0)} = \sup_{(f,f)_0 \leq 1} (T^*Tf, f)_0 \\ \|T\|_A^2 &= \|T\|^2_{\mathscr{L}(\mathscr{H}_1)} = \sup_{(Af,f)_0 \leq 1} (T^*ATf, f)_0. \end{aligned} \tag{5}$$

Let $\mathscr{L}(\overline{\mathscr{H}})$ be the set of linear operators on $\mathscr{H}_0 + \mathscr{H}_1$ such that the restriction of T to $\mathscr{H}_i$ belongs to $\mathscr{L}(\mathscr{H}_i)$, $i = 0, 1$. A Banach space norm on $\mathscr{L}(\overline{\mathscr{H}})$ is defined by

$$\|T\|_{\mathscr{L}(\overline{\mathscr{H}})} = \max(\|T\|_{\mathscr{L}(\mathscr{H}_0)}, \|T\|_{\mathscr{L}(\mathscr{H}_1)}) = \max(\|T\|, \|T\|_A).$$

We note that $K_2(t,\cdot)$ is an *exact interpolation norm with respect to* $\overline{\mathscr{H}}$, i.e.

(6) $K_2(t,Tf) \le \|T\|_{\mathscr{L}(\overline{\mathscr{H}})} K_2(t,f), \quad T \in \mathscr{L}(\overline{\mathscr{H}}),\ f \in \mathscr{H}_0 + \mathscr{H}_1, \quad t > 0$

which property is immediate from the definition of K_2. Given a positive, concave function ψ on R_+, let an intermediate Hilbert space $\mathscr{H}_*$ be defined as the completion of $\mathscr{H}_0 \cap \mathscr{H}_1$ under the norm

$$\|f\|_*^2 = (\psi(A)f, f)_0.$$

In accordance with (5) we shall use different notations for the operator norms

$$\|T\|_{\psi(A)}^2 = \|T\|_{\mathscr{L}(\mathscr{H}_*)}^2 = \sup_{(\psi(A)f,f)_0 \le 1} (T^*\psi(A)Tf, f)_0. \tag{7}$$

By a theorem of Peetre [11], it is known that every positive, concave function ψ on R_+ is an *interpolation function of power* 2 meaning that (for any A, T)

$$\max(\|T\|, \|T\|_A) < \infty \qquad \textit{implies} \qquad \|T\|_{\psi(A)} < \infty. \tag{8}$$

From the proof of Peetre's theorem, it can also bee deduced that there exists a constant $c \ge 1$ such that $\mathscr{H}_*$ is a *c-interpolation space* with respect to $\overline{\mathscr{H}}$ in the sense that

$$\|T\|_{\psi(A)} \le c \max(\|T\|, \|T\|_A), \qquad T \in \mathscr{L}(\overline{\mathscr{H}}). \tag{9}$$

In 1972, Foiaş [6] noted that $c \le 2$ for the best c. In a later paper, Foiaş, Ong and Rosenthal proved that $c \le \sqrt{2}$, and also posed the question whether the constant $\sqrt{2}$ is best possible (cf. [8], question (i), p. 811). It is shown below that this is the case.

THEOREM 2.1. *The best c in* (9) *is* $c = \sqrt{2}$.

REMARK 2.2. This theorem is implicit in the work of Gunnar Sparr, cf. [12], Lemma 5.1. We shall here give a partially new proof, based on Lemma 1.1 and the following lemma.

LEMMA 2.3. *Every function h in the class P′ is* exact interpolation *in the sense that*

$$\|T\|_{h(A)} \le \max(\|T\|, \|T\|_A), \qquad T \in \mathscr{L}(\overline{\mathscr{H}}). \tag{10}$$

REMARK 2.4. The above lemma is the easy half of a theorem of Foiaş and Lions [7] (see also [9]) which states that, for a positive function defined on R_+, the condition $h \in P'$ is equivalent to that h fulfill (10) for every Hilbert couple $\overline{\mathscr{H}}$.

Proof of Lemma 2.3.. Denote by E the spectral measure of A and let ρ be the measure associated with h as in (2). Then by (4)

$$\|f\|_*^2 = (h(A)f, f)_0 = \int_0^\infty \left(\int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda}\, d\rho(t) \right) d(E_\lambda f, f)_0$$

$$= \int_{[0,\infty]} (1+t^{-1}) \left(\int_0^\infty \frac{t\lambda}{1+t\lambda}\, d(E_\lambda f, f)_0 \right) d\rho(t) \tag{11}$$

$$= \int_{[0,\infty]} (1+t^{-1}) K_2(t, f; \overline{\mathscr{H}})^2\, d\rho(t), \qquad f \in \mathscr{H}_0 \cap \mathscr{H}_1.$$

It is easy to see that the latter expression extends to an exact interpolation norm with respect to $\overline{\mathscr{H}}$, viz. (10) holds (use (6) and integrate with respect to $d\rho(t)$).

Proof of Theorem 2.1. Referring to the smallest constant in (9), we first show that $c \le \sqrt{2}$. Given an arbitrary concave, positive function ψ on R_+, let $h \in P'$ be such that $h \le \psi \le 2h$; then by Lemma 2.3,

(12)

$$\|T\|_{\psi(A)}^2 = \sup_{(\psi(A)f,f)_0 \le 1} (T^*\psi(A)Tf, f)_0 \le \sup_{(h(A)f,f)_0 \le 1} 2(T^*h(A)Tf, f)_0$$

$$= 2\|T\|_{h(A)}^2 \le 2\max(\|T\|^2, \|T\|_A^2), \qquad T \in \mathscr{L}(\overline{\mathscr{H}}),$$

and the estimate $c \le \sqrt{2}$ follows incidentally. Proving $c \ge \sqrt{2}$ is more subtle; we shall require a clever three-dimensional argument due to G. Sparr, cf. [12], Example 5.3. Let $\mathscr{H}_0 = \ell_2^3$ be the three-dimensional ℓ_2-space. For $n \in \mathsf{N}$ let us put

$$A_n = \begin{pmatrix} \frac{1}{4n^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4n^2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad f^n = \begin{pmatrix} n \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad T_n = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2n} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then $T_n f^n = g$ and a direct calculation yields that

$$\|T_n\| = \|T_n\|_{A_n} = \sqrt{1 + 1/4n^2}, \qquad n \in \mathsf{N}.$$

On the other hand, letting $\psi(\lambda) = \min(1, \lambda)$, we have

$$\|T_n f^n\|_{\psi(A_n)}^2 = \|g\|_{\psi(A_n)}^2 = (\psi(A_n)g, g)_0 = 1, \qquad n \in \mathsf{N},$$

whereas

$$\|f^n\|_{\psi(A_n)}^2 = (\psi(A_n)f^n, f^n)_0$$
$$= n^2 \min(1, 1/(4n^2)) + (1/4)\min(1, 4n^2) = 1/2, \qquad n \in \mathsf{N},$$

and it follows that

$$c \geq \frac{\|T\|_{\psi(A_n)}}{\sqrt{1+1/(4n^2)}} \geq \sqrt{\frac{2}{1+1/(4n^2)}} \nearrow \sqrt{2}, \qquad n \to \infty.$$

REMARK 2.5 (On Sparr's result). Let us introduce the modified K_2-functional

$$L_2(t, f)^2 = (\min(t, A)f, f)_0.$$

By Sparr's work ([12], Lemma 5.1) it is known that

$$K_2(t, g) \leq K_2(t, f) \quad \textit{implies} \quad L_2(t, g) \leq L_2(t, \sqrt{2}f), \tag{13}$$

where the constant $\sqrt{2}$ cannot be improved. Observe that, for an operator T, the condition $K_2(t, Tf) \leq K_2(t, f), t > 0$ is equivalent to that $\|T\|_{\mathscr{L}(\overline{\mathscr{H}})} \leq 1$. Moreover, by the representation (3) for a positive, concave function ψ, it is clear that

$$\|f\|_*^2 = (\psi(A)f, f)_0 = \alpha\|f\|_0^2 + \beta\|f\|_1^2 + \int_0^\infty L_2(t, f)^2\, d\nu(t)$$

with suitable α, β and ν. Hence the condition $L_2(t, Tf)^2 \leq 2L_2(t, f)^2, t > 0$ implies that $\|T\|_{\psi(A)} \leq \sqrt{2}$. Thus (13) yields that (for all A, T)

$$\|T\|_{\mathscr{L}(\overline{\mathscr{H}})} \leq 1 \qquad \textit{implies} \qquad \|T\|_{\psi(A)} \leq \sqrt{2},$$

where the constant $\sqrt{2}$ is best possible. Note that this yields an alternative proof of Theorem 2.1.

We note the following, sharp version of Lemma 1.1.

THEOREM 2.6. *The constant $c = 2$ is smallest possible with respect to the property thet for any positive concave function ψ on R_+, there exists $h \in P'$ such that $h \leq \psi \leq ch$.*

PROOF. Referring to the least constant, we have $c \leq 2$ by Lemma 1.1, and as in (12), one shows that for any positive concave ψ, any A, T

$$\|T\|_{\psi(A)}^2 \leq c \max(\|T\|^2, \|T\|_A^2).$$

By Theorem 2.1, the smallest possible c in the latter inequality is $c = 2$.

3. A note on K_2-functors

We consider an application of Lemma 1.1 to the more functorial aspects of the theory.

Given a positive Radon measure ρ on $[0, \infty]$, let an interpolation functor $K_2(\rho)$ be defined on the category of Banach couples by

$$\|f\|_{K_2(\rho)(\overline{\mathscr{A}})} = \left(\int_{[0,\infty]} (1 + t^{-1}) K_2(t, f; \overline{\mathscr{A}})^2 \, d\rho(t)\right)^{1/2}.$$

(Here the function $k : t \mapsto (1 + t^{-1}) K_2(t, f)^2$ is defined by continuity at the points 0 and ∞, $k(0) = \|f\|_1^2$ and $k(\infty) = \|f\|_0^2$ where we have used the convention: $\|f\|_i = \infty$ if $f \notin \mathscr{H}_i$, $i = 0, 1$.)

Corollary 3.1. *Let $\overline{\mathscr{H}}$ be a regular Hilbert couple with associated operator A. Then, given any positive, concave funtion ψ, there exists a positive Radon measure ρ on $[0, \infty]$ such that*

$$(1/\sqrt{2}) \|f\|_{K_2(\rho)(\overline{\mathscr{H}})} \leq \|f\|_{\psi(A)} \leq \sqrt{2} \, \|f\|_{K_2(\rho)(\overline{\mathscr{H}})}, \qquad f \in K_2(\rho)(\overline{\mathscr{H}}),$$

where the constant $\sqrt{2}$ cannot be improved.

Proof. This follows easily from Theorem 2.6 and (11).

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