

# FOXBY EQUIVALENCE AND COTORSION THEORIES RELATIVE TO SEMI-DUALIZING MODULES

EDGAR ENOCHS and SIAMAK YASSEMI\*

## Abstract

Foxby duality has proven to be an important tool in studying the category of modules over a local Cohen-Macaulay ring admitting a dualizing module. Recently the notion of a semi-dualizing module has been given [2]. Given a semi-dualizing module the relative Foxby classes can be defined and there is still an associated Foxby duality. We consider these classes (separately called the Auslander and Bass classes) and two naturally defined subclasses which are equivalent to the full subcategories of injective and flat modules. We consider the question of when these subclasses form part of one of the two classes of a cotorsion theory. We show that when this is the case, the associated cotorsion theory is not only complete but in fact is perfect. We show by examples that even when the semi-dualizing module is in fact dualizing over a local Cohen-Macaulay ring it both may or may not occur that we get this associated cotorsion theory.

## 1. The Foxby classes

Throughout this paper  $R$  will always be a commutative noetherian ring. For use throughout the paper we quote the following easy result.

LEMMA A. *Let  $R, S$  be rings, let  $F, G$  be covariant right-exact (resp., contravariant left-exact) additive functors from the category of  $R$ -modules to the category of  $S$ -modules and let  $\psi : F \rightarrow G$  be a natural transformation such that  $\psi(M)$  is an isomorphism for all finitely generated modules  $M$ . If furthermore  $\psi(M)$  is an isomorphism for all free modules  $M$ , then  $\psi$  is an isomorphism.*

DEFINITION 1.1. A finitely generated  $R$ -module  $C$  is said to be a *semi-dualizing module* for  $R$  if

- (i)  $\text{Ext}^i(C, C) = 0$  for  $i \geq 1$
- (ii) the canonical map  $R \rightarrow \text{Hom}(C, C)$  is an isomorphism.

If furthermore  $C$  has finite injective dimension then  $C$  is said to be a *dualizing module* for  $R$ .

---

\*This paper was prepared while the second author was on sabbatical from the University of Tehran. He would like to thank the University of Kentucky for its hospitality during his stay there. Received October 22, 2001; in revised form August 11, 2003.

Throughout the rest of this paper  $C$  will always denote a semidualizing module for  $R$ .

It is not obvious that a local ring admits semi-dualizing modules other than itself and, possibly, a dualizing module. The question concerning their existence was posed in 1985 by Golod (see [7]) and in 1987 Foxby gave examples of rings with three different semi-dualizing modules (see Christensen ([2], pg. 1874) for examples of local Cohen-Macaulay rings having at least  $n$  different semi-dualizing modules (for any  $n \geq 1$ )).

In our situation there are two classes of modules associated with  $C$ .

**DEFINITION 1.2.** The Auslander class of  $R$  (relative to  $C$ ) is denoted  $\mathcal{A}$  and consists of modules  $M$  such that:

- (i)  $\text{Tor}_i(C, M) = 0$  for  $i \geq 1$
- (ii)  $\text{Ext}^i(C, C \otimes M) = 0$  for  $i \geq 1$
- (iii) the canonical map  $\mu_M : M \rightarrow \text{Hom}(C, C \otimes M)$  is an isomorphism.

**DEFINITION 1.3.** The Bass class of  $R$  (relative to  $C$ ) is denoted  $\mathcal{B}$  and consists of modules  $N$  such that:

- (i)  $\text{Ext}^i(C, N) = 0$  for  $i \geq 1$
- (ii)  $\text{Tor}_i(C, \text{Hom}(C, N)) = 0$  for  $i \geq 1$
- (iii) the canonical  $\nu_N : C \otimes \text{Hom}(C, N) \rightarrow N$  is an isomorphism.

When  $R$  is local Cohen-Macaulay and  $C$  is dualizing  $\mathcal{A}$  and  $\mathcal{B}$  have a nice description in terms of Gorenstein projective and injective dimensions ([6], corollaries 2.4 and 2.6).

We note that for any  $C, R \in \mathcal{A}$  and  $C \in \mathcal{B}$ , both classes are closed under direct sums, direct summands and direct limits. and products. Since  $C$  is finitely presented,  $\text{Tor}_i(C, -)$  and  $\text{Ext}_i(C, -)$  commute with products for any  $i \geq 0$ . So both classes are closed under products. If  $F$  is flat, then by Lazard's thesis  $F$  is a direct limit of projective modules. So if  $M \in \mathcal{A}$  and  $N \in \mathcal{B}$ , then  $F \otimes M \in \mathcal{A}$  and  $F \otimes N \in \mathcal{B}$ . And so  $R \in \mathcal{A}$  gives  $F \in \mathcal{A}$ . The next result gives that  $E \in \mathcal{B}$  for all injective  $E$ .

**PROPOSITION 1.4.** *For any module  $M$ ,  $M \in \mathcal{A}$  if and only if  $\text{Hom}(M, E) \in \mathcal{B}$  for all injective  $E$ .*

**PROOF.** For any injective  $E$  and any  $M$  we have  $\text{Hom}(\text{Tor}_i(C, M), E) \cong \text{Ext}^i(C, \text{Hom}(M, E))$  for all  $i$ , so easily  $\text{Tor}_i(C, M) = 0$  for all  $i \geq 1$  if and only if  $\text{Ext}^i(C, \text{Hom}(M, E)) = 0$  for all  $i \geq 1$  and all injective  $E$ .

Similarly the natural isomorphisms  $\text{Hom}(\text{Ext}^i(C, C \otimes M), E) \cong \text{Tor}_i(C, \text{Hom}(C \otimes M, E)) \cong \text{Tor}_i(C, \text{Hom}(C, \text{Hom}(M, E)))$  give that  $\text{Ext}^i(C, C \otimes M) = 0$  for all  $i \geq 1$  if and only if  $\text{Tor}_i(C, \text{Hom}(C, \text{Hom}(M, E))) = 0$  for all  $i \geq 1$  and all injective  $E$ .

$M) = 0$  for all  $i \geq 1$  if and only if  $\text{Tor}_i(C, \text{Hom}(C, \text{Hom}(M, E))) = 0$  for all  $i \geq 1$  and all injective  $E$ .

Finally consider the commutative diagram:

$$\begin{array}{ccc} \text{Hom}(\text{Hom}(C, C \otimes M), E) & \xrightarrow{\text{Hom}(\mu_M, E)} & \text{Hom}(M, E) \\ \cong \uparrow & & \uparrow v_{\text{Hom}(M, E)} \\ C \otimes \text{Hom}(C \otimes M, E) & \xrightarrow{\cong} & C \otimes \text{Hom}(C, \text{Hom}(M, E)). \end{array}$$

We see that  $\mu_M$  is an isomorphism if and only if  $v_{\text{Hom}(M, E)}$  is an isomorphism for all injective  $E$ .

**COROLLARY 1.5.** *For any semi-dualizing  $C$ ,  $E \in \mathcal{B}$  for all injective  $E$ .*

**PROOF.**  $R \in \mathcal{A}$  so  $\text{Hom}(R, E) = E \in \mathcal{B}$  for all injective  $E$ .

The preceding Proposition raises the question of whether we get the analogous result concerning  $N \in \mathcal{B}$ .

**PROPOSITION 1.6.** *For any module  $N$ ,  $N \in \mathcal{B}$  if and only if  $\text{Hom}(N, E) \in \mathcal{A}$  for all injective modules  $E$ .*

**PROOF.** Let  $E$  be an injective module. Then the canonical map  $M \otimes \text{Hom}(N, E) \rightarrow \text{Hom}(\text{Hom}(M, N), E)$  is an isomorphism for all finitely generated modules  $M$  by Lemma A. Deriving both sides and putting  $M = C$  we obtain the isomorphism

$$\text{Tor}_i(C, \text{Hom}(N, E)) \cong \text{Hom}(\text{Ext}_i(C, N)E).$$

Applying Lemma A again we obtain that, hence, the canonical map  $\text{Hom}(N, E) \rightarrow \text{Hom}(M \otimes \text{Hom}(C, N)E)$  is an isomorphism for all finitely generated modules  $M$ . Deriving both sides as functors in  $M$  and putting  $M = C$ , we obtain the isomorphism

$$\text{Ext}^i(C, C \otimes \text{Hom}(N, E)) \cong \text{Hom}(\text{Tor}_i(C, \text{Hom}(C, N))E).$$

These two isomorphisms immediately imply Proposition 1.6.

## 2. Foxby duality and cotorsion theories

In this section we again let  $C$  be a semi-dualizing module for  $R$  and again let  $\mathcal{A}$  and  $\mathcal{B}$  denote the associated Auslander and Bass classes.

**PROPOSITION 2.1** (Foxby equivalence). *The functors  $C \otimes - : \mathcal{A} \rightarrow \mathcal{B}$  and  $\text{Hom}(C, -) : \mathcal{B} \rightarrow \mathcal{A}$  give a well-defined equivalence between  $\mathcal{A}$  and  $\mathcal{B}$  (viewed as full subcategories of the category of  $R$ -modules).*

PROOF. If  $M \in \mathcal{A}$  then  $\text{Ext}^i(C, C \otimes M) = 0$  for  $i \geq 1$ . Also  $\text{Hom}(C, C \otimes M) \cong M$  so  $\text{Tor}_i(C, \text{Hom}(C, C \otimes M)) \cong \text{Tor}_i(C, M) = 0$  for  $i \geq 1$ .

If we consider the commutative

$$\begin{array}{ccc} C \otimes \text{Hom}(C, C \otimes M) & \xrightarrow{\nu_{C \otimes M}} & C \otimes M \\ \parallel & & \parallel \\ C \otimes \text{Hom}(C, C \otimes M) & \xleftarrow[\cong]{C \otimes \mu_M} & C \otimes M \end{array}$$

we see that  $\nu_{C \otimes M}$  is an isomorphism. So  $C \otimes M \in \mathcal{B}$ .

The argument that  $\text{Hom}(C, N) \in \mathcal{A}$  for  $N \in \mathcal{B}$  is similar. Now it is clear that the two functors given an equivalence of categories.

Now given our equivalence  $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$  we note that  $\mathcal{F} \subset \mathcal{A}$  and  $\mathcal{E} \subset \mathcal{B}$  where  $\mathcal{F}$  and  $\mathcal{E}$  are respectively the class of flat and of injective modules. The image of the class  $\mathcal{F}$  under  $\mathcal{A} \rightarrow \mathcal{B}$  is denoted  $\mathcal{W}(R)$  or  $\mathcal{W}$  and the image of  $\mathcal{E}$  under  $\mathcal{B} \rightarrow \mathcal{A}$  is denoted  $\mathcal{U}(\mathcal{B})$  or  $\mathcal{U}$ . So  $\mathcal{W}$  consists of all the modules  $C \otimes F$  with  $F$  flat and  $\mathcal{U}$  of the modules  $\text{Hom}(C, E)$  with  $E$  injective.

PROPOSITION 2.2. *We have for any  $C$*

- a)  $M \in \mathcal{U}$  implies  $\text{Hom}(M, E) \in \mathcal{W}$  for all injective  $E$  and
- b)  $N \in \mathcal{W}$  implies  $\text{Hom}(N, E) \in \mathcal{U}$  for all injective  $E$ .

PROOF. a) If  $M \in \mathcal{U}$  then  $M \cong \text{Hom}(C, E')$  for an injective  $E'$ . So  $\text{Hom}(M, E) \cong \text{Hom}(\text{Hom}(C, E'), E) \cong C \otimes \text{Hom}(E', E)$ . But  $\text{Hom}(E', E)$  is flat, so  $\text{Hom}(M, E) \in \mathcal{W}$ .

b) If  $N \in \mathcal{W}$  then  $N \cong C \otimes F$  for a flat module  $F$ . So if  $E$  is injective then  $\text{Hom}(N, E) \cong \text{Hom}(C \otimes F, E) \cong \text{Hom}(C, \text{Hom}(F, E)) \in \mathcal{U}$  since  $\text{Hom}(F, E)$  is injective.

It is well known that any module (over any ring) has an injective envelope. Recently it has also been shown that every module has a flat cover [1]. We will consider the analogous questions using the classes of modules  $\mathcal{W}$  and  $\mathcal{U}$ . We first note that both  $\mathcal{W}$  and  $\mathcal{U}$  are closed under direct sums, summands and direct limits and direct products.

DEFINITION 2.3. Given a class  $\mathcal{G}$  of modules, a linear  $\phi : G \rightarrow M$  with  $G \in \mathcal{G}$  is said to be a  $\mathcal{G}$ -precover of  $M$  if  $\text{Hom}(H, G) \rightarrow \text{Hom}(H, M)$  is surjective for all  $H \in \mathcal{G}$ . If moreover any  $f : G \rightarrow G$  such that  $\phi \circ f = \phi$  is an automorphism of  $G$ , then  $\phi$  is said to be a  $\mathcal{G}$ -cover.  $\mathcal{G}$ -preenvelopes and  $\mathcal{G}$ -envelopes are defined dually.

If, for example,  $\mathcal{G}$  is the class of flat modules, then a  $\mathcal{G}$ -cover is just called a flat cover.

**THEOREM 2.4.** *Every module has a  $\mathcal{W}$ -cover and a  $\mathcal{U}$ -envelope.*

**PROOF.** For any module  $M$  let  $C \otimes M \subset E$  be an injective envelope of  $C \otimes M$ . We will argue that  $M \rightarrow \text{Hom}(C, C \otimes M) \rightarrow \text{Hom}(C, E)$  is a  $\mathcal{U}$ -envelope. Given  $M \rightarrow \text{Hom}(C, \bar{E})$  with  $\bar{E}$  injective, we get a map  $C \otimes M \rightarrow C \otimes \text{Hom}(C, \bar{E}) \cong \bar{E}$  (recall that  $\bar{E} \in \mathcal{B}$ ). This map can be extended to a map  $E \rightarrow \bar{E}$  which in turn gives a map  $\text{Hom}(C, E) \rightarrow \text{Hom}(C, \bar{E})$ . But then the composite  $M \rightarrow \text{Hom}(C, E) \rightarrow \text{Hom}(C, \bar{E})$  is the original map  $M \rightarrow \text{Hom}(C, \bar{E})$ . So  $M \rightarrow \text{Hom}(C, E)$  is a  $\mathcal{U}$ -preenvelope.

Now if  $f : \text{Hom}(C, E) \rightarrow \text{Hom}(C, E)$  is such that  $f \circ \phi = f$  where  $\phi : M \rightarrow \text{Hom}(C, E)$  is our given map, then applying  $C \otimes -$  and using the fact that  $c \otimes \text{Hom}(C, E = E$  is an injective envelope, we see that  $C \otimes f$  is an automorphism of  $E$ .

But then  $f = \text{Hom}(C, C \otimes f) : \text{Hom}(C, E) \rightarrow \text{Hom}(C, E)$  is an automorphism of  $\text{Hom}(C, E)$ .

The argument that every module  $N$  has a  $\mathcal{W}$ -cover is similar. We just start with a flat cover  $F \rightarrow \text{Hom}(C, N)$  and argue that  $C \otimes F \rightarrow C \otimes \text{Hom}(C, N) \rightarrow N$  is the desired  $\mathcal{W}$ -cover.

**DEFINITION 2.5** (see Salce [11]). For any class  $\mathcal{F}$  of  $R$ -modules, let  $\mathcal{F}^\perp$  consist of all modules  $C$  such that  $\text{Ext}^1(F, C) = 0$  for all  $F \in \mathcal{F}$ . Similarly, for a class  $\mathcal{C}$  with let  ${}^\perp\mathcal{C}$  consist of all  $F$  such that  $\text{Ext}^1(F, C) = 0$  for all  $C \in \mathcal{C}$ . The pair  $(\mathcal{F}, \mathcal{C})$  is said to be a *cotorsion theory* if  $\mathcal{F}^\perp = \mathcal{C}$  and  $\mathcal{F} = {}^\perp\mathcal{C}$ . A cotorsion theory  $(\mathcal{F}, \mathcal{C})$  is said to be *complete* if for every module there is an exact sequence  $0 \rightarrow N \rightarrow C \rightarrow F \rightarrow 0$  with  $C \in \mathcal{C}$  and  $F \in \mathcal{F}$  and if for every module  $M$  there is an exact sequence  $0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$  with  $C \in \mathcal{C}$  and  $F \in \mathcal{F}$ .

If  $(\mathcal{F}, \mathcal{C})$  is a complete cotorsion theory and  $0 \rightarrow N \rightarrow C \rightarrow F \rightarrow 0$  is exact as in the definition, and if  $D \in \mathcal{C}$ , then  $\text{Hom}(C, D) \rightarrow \text{Hom}(N, D) \rightarrow \text{Ext}^1(F, D) = 0$  is exact and so  $N \rightarrow C$  is a  $\mathcal{C}$ -preenvelope. Similarly if  $0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$  is as in the definition then  $F \rightarrow M$  is an  $\mathcal{F}$ -preenvelope.

**DEFINITION 2.6.** A complete cotorsion theory  $(\mathcal{F}, \mathcal{C})$  is said to be *perfect* if every module has an  $\mathcal{F}$ -cover and a  $\mathcal{C}$ -envelope.

For any ring  $R$ , if  $\mathcal{E}$  is the class of injective modules, then  ${}^\perp\mathcal{E}$  is the class of all modules. Clearly  $({}^\perp\mathcal{E}, \mathcal{E})$  is a perfect cotorsion theory. If  $\mathcal{F}$  is the class of flat modules then  $(\mathcal{F}, \mathcal{F}^\perp)$  is also a perfect cotorsion theory (by [12], Lemma 3.4.1 and [5] Theorem 7.4.4 and Theorem 7.26).

It is natural to ask when  $({}^\perp\mathcal{U}, \mathcal{U})$  and  $(\mathcal{W}, \mathcal{W}^\perp)$  are cotorsion theories. This question will be answered in Theorems 2.10 and 2.11 below. We begin with the following auxiliary result.

PROPOSITION 2.7. *The classes  $\mathcal{U}$  and  $\mathcal{W}$  are closed under extensions.*

PROOF. Let  $E', E''$  be injective and let  $0 \rightarrow \text{Hom}(C, E') \rightarrow G \rightarrow \text{Hom}(C, E'') \rightarrow 0$  be exact. Since  $\text{Tor}_1(C, \text{Hom}(C, E'')) = 0$  we have

$$0 \rightarrow C \otimes \text{Hom}(C, E') \rightarrow C \otimes G \rightarrow C \otimes \text{Hom}(C, E'') \rightarrow 0$$

is exact. But  $C \otimes \text{Hom}(C, E') \cong E'$  and  $C \otimes \text{Hom}(C, E'') \cong E''$  so  $0 \rightarrow E' \rightarrow C \otimes G \rightarrow E'' \rightarrow 0$  is exact and in fact split. Therefore  $C \otimes G$  is injective. Now applying  $\text{Hom}(C, -)$  we get as  $0 \rightarrow \text{Hom}(C, E) \rightarrow \text{Hom}(C, C \otimes G) \rightarrow \text{Hom}(C, E'') \rightarrow 0$  is exact. So  $G \cong \text{Hom}(C, C \otimes G)$  and so  $G \in \mathcal{U}$ .

Now let  $0 \rightarrow C \otimes F' \rightarrow G \rightarrow C \otimes F'' \rightarrow 0$  be exact where  $F'$  and  $F''$  are flat. Since  $\text{Ext}^1(C, C \otimes F') = 0$  we have  $0 \rightarrow \text{Hom}(C, C \otimes F') \rightarrow \text{Hom}(C, G) \rightarrow \text{Hom}(C, C \otimes F'') \rightarrow 0$  is exact. So  $0 \rightarrow F' \rightarrow \text{Hom}(C, G) \rightarrow F'' \rightarrow 0$  is exact. So  $F' \subset \text{Hom}(0, G)$  is pure and  $\text{Hom}(C, G)$  is flat. So applying  $C \otimes -$  we get

$$0 \rightarrow C \otimes F' \rightarrow C \otimes \text{Hom}(C, G) \rightarrow C \otimes F'' \rightarrow 0$$

exact. So  $G \cong C \otimes \text{Hom}(C, G) \in \mathcal{W}$ .

COROLLARY 2.8. *For any  $\mathcal{W}$ -cover  $\phi : \mathcal{W} \rightarrow M$ ,  $\ker(\phi) \in \mathcal{W}^\perp$  and for any  $\mathcal{U}$ -envelope  $\psi : N \rightarrow U$ ,  $\text{coker}(\psi) \in {}^\perp\mathcal{U}$ .*

PROOF. Apply Wakamatsu's lemma (see [5] Corollary 7.2.3 and Proposition 7.2.4).

PROPOSITION 2.9. *Let  $0 \rightarrow W' \rightarrow W \rightarrow W'' \rightarrow 0$  be a short exact sequence where  $W, W'' \in \mathcal{W}$ . Then  $W' \in \mathcal{W}$  if and only if  $\text{Ext}^1(C, W') = 0$ .*

PROOF. The condition is necessary since  $\mathcal{W} \subset \mathcal{B}$ . Now assume  $\text{Ext}^1(C, W') = 0$ . Then

$$0 \rightarrow \text{Hom}(C, W') \rightarrow \text{Hom}(C, W) \rightarrow \text{Hom}(C, W'') \rightarrow 0$$

is exact. But  $\text{Hom}(C, W)$  and  $\text{Hom}(C, W'')$  are flat and hence so is  $\text{Hom}(C, W')$ . Since  $\text{Hom}(C, W'')$  is flat so is  $0 \rightarrow C \otimes \text{Hom}(C, W') \rightarrow C \otimes \text{Hom}(C, W) \rightarrow C \otimes \text{Hom}(C, W'') \rightarrow 0$ . But  $C \otimes \text{Hom}(C, W) \cong W$  and  $C \otimes \text{Hom}(C, W'') \cong W''$  so  $C \otimes \text{Hom}(C, W') \cong W'$ . Since  $\text{Hom}(C, W')$  is flat this gives that  $W' \in \mathcal{W}$ .

THEOREM 2.10. *The following are equivalent for  $\mathcal{U}$ :*

- a)  $({}^\perp\mathcal{U}, \mathcal{U})$  is a cotorsion theory
- b)  $\mathcal{E} \subset \mathcal{U}$  where  $\mathcal{E}$  is the class of injective modules
- c) every  $\mathcal{U}$ -envelope  $M \rightarrow U$  is injective

- d)  $\mu_M : M \rightarrow \text{Hom}(C, C \otimes M)$  is an injection for all  $M$   
 e)  $E \rightarrow \text{Hom}(C, C \otimes E)$  is an injection for all injective modules  $E$   
 f)  $E \rightarrow \text{Hom}(C, C \otimes E)$  is injective for an injective cogenerator  $E$

PROOF. a)  $\Rightarrow$  b). Clearly  $\mathcal{E} \subset ({}^\perp\mathcal{U})^\perp$ .

b)  $\Rightarrow$  c). Let  $M \rightarrow U$  be a  $\mathcal{U}$ -envelope. Let  $M \subset E$  with  $E$  injective. Since  $E \in \mathcal{U}$ ,  $M \rightarrow E$  can be factored  $M \rightarrow U \rightarrow E$ . So since  $M \rightarrow E$  is injective, so is  $M \rightarrow U$ .

d)  $\Leftrightarrow$  c) From the proof of Theorem 2.6 we know that a  $\mathcal{U}$ -envelope of  $M$  is of the form  $M \rightarrow \text{Hom}(C, C \otimes M) \rightarrow \text{Hom}(C, E)$  where  $C \otimes M \subset E$  is an injective envelope. So  $\text{Hom}(C, C \otimes M) \rightarrow \text{Hom}(C, E)$  is injective. So  $M \rightarrow \text{Hom}(C, E)$  is injective if and only if  $M \rightarrow \text{Hom}(C, C \otimes M)$  is injective.

d)  $\rightarrow$  e) trivially. To get e)  $\rightarrow$  d), let  $M \subset E$  with  $E$  injective. Then we have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & \text{Hom}(C, C \otimes M) \\ \downarrow & & \downarrow \\ E & \longrightarrow & \text{Hom}(C, C \otimes E) \end{array}$$

by e),  $E \rightarrow \text{Hom}(C, C \otimes E)$  is an injection. Since  $M \rightarrow E$  is an injection,  $M \rightarrow \text{Hom}(C, C \otimes M)$  is also an injection.

e)  $\rightarrow$  f) is trivial. To get f)  $\Rightarrow$  e), let  $\bar{E}$  be an injective cogenerator. Then if  $E$  is injective we have  $E \subset \bar{E}^{(I)}$  for some set  $I$ . Since  $\bar{E} \rightarrow \text{Hom}(C, C \otimes \bar{E})$  is injective, so is  $\bar{E}^{(I)} \rightarrow \text{Hom}(C, C \otimes \bar{E}^{(I)}) \cong \text{Hom}(C, C \otimes \bar{E}^{(I)})$  (since  $C$  is finite generated). But we have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & \text{Hom}(C, C \otimes E) \\ \downarrow & & \downarrow \\ \bar{E} & \longrightarrow & \text{Hom}(C, C \otimes \bar{E}^{(I)}) \end{array}$$

and we quickly see that  $E \rightarrow \text{Hom}(C, C \otimes E)$ .

d)  $\rightarrow$  a). Let  $M \in ({}^\perp\mathcal{U})^\perp$ . We want to show that  $M \in \mathcal{U}$ . Let  $M \rightarrow U$  be a  $\mathcal{U}$ -envelope. By d),  $M \rightarrow U$  is an injection. So we have the exact  $0 \rightarrow M \rightarrow U \rightarrow \frac{U}{M} \rightarrow 0$ . By Corollary 2.8,  $\frac{U}{M} \in ({}^\perp\mathcal{U})^\perp$ . But then since  $M \in ({}^\perp\mathcal{U})^\perp$  the sequence  $0 \rightarrow M \rightarrow U \rightarrow \frac{U}{M} \rightarrow 0$  splits. So  $M$  is a direct summand of  $U$ . Since  $\mathcal{U}$  is closed under direct summands we see that  $M \in \mathcal{U}$ .

We also have

**THEOREM 2.11.** *The following are equivalent:*

- a)  $(\mathcal{W}, \mathcal{W}^\perp)$  is a cotorsion theory
- b)  $\mathcal{F} \subset \mathcal{W}$  where  $\mathcal{F}$  is the class of flat module
- c) for all  $M$ ,  $C \otimes \text{Hom}(C, M) \rightarrow M$  is surjective
- d) every  $\mathcal{W}$ -cover  $\mathcal{W} \rightarrow M$  is surjective

PROOF. The proof is analogous to that of the preceding theorem.

PROPOSITION 2.12. *If  $({}^\perp\mathcal{U}, \mathcal{U})$  (or  $(\mathcal{W}, \mathcal{W}^\perp)$ ) is a cotorsion theory then it is a perfect cotorsion theory.*

PROOF. Suppose  $({}^\perp\mathcal{U}, \mathcal{U})$  is a cotorsion theory. Then by Theorem 2.6 every  $M$  has a  $\mathcal{U}$ -envelope  $M \rightarrow U$  which is injective by d) of Theorem 2.12. By Corollary 2.8 the cokernel is in  ${}^\perp\mathcal{U}$ .

By an argument of Salce ([10] or see [4] Proposition 7.1.7)  $({}^\perp\mathcal{U}, \mathcal{U})$  is complete. By ([4], Theorem 7.2.6)  $({}^\perp\mathcal{U}, \mathcal{U})$  is perfect.

A similar argument works for  $(\mathcal{W}, \mathcal{W}^\perp)$ .

### 3. Examples with dualizing modules

We recall that a module  $D$  is said to dualizing if it is semi-dualizing and if  $\text{inj. dim } D < \infty$ . In this section  $D$  will always be dualizing for  $R$  and  $\mathcal{U}$  will always be the class of modules  $\text{Hom}(D, E)$  with  $E$  injective. Our rings  $R$  will always be local and Cohen-Macaulay. We will show that if  $\dim R = 0$ , then  $({}^\perp\mathcal{U}, \mathcal{U})$  is a cotorsion theory if and only if  $R$  is Gorenstein. Of course, if  $R$  is Gorenstein of any dimension then  $({}^\perp\mathcal{U}, \mathcal{U})$  is a cotorsion theory ( $\mathcal{U}$  is just the class of injective modules). For any  $d \geq 1$  we will show there are examples of our rings  $R$  of dimension  $d$  which are not Gorenstein but for which  $({}^\perp\mathcal{U}, \mathcal{U})$  is a cotorsion theory and such examples where  $({}^\perp\mathcal{U}, \mathcal{U})$  is not a cotorsion theory.

PROPOSITION 3.1. *If  $R$  is local and artinian and  $R$  is not Gorenstein, then  $({}^\perp\mathcal{U}, \mathcal{U})$  is not a cotorsion theory.*

PROOF. By Theorem 2.12 e) it suffices to argue that  $D \rightarrow \text{Hom}(D, D \otimes D)$  since  $D = E(k)$  where  $k$  is the residue field of  $k$ . We let  $M^v = \text{Hom}(M, D)$  be the Matlis dual of  $M$  for any  $M$ . We have  $(D \otimes D)^v = \text{Hom}(D, R)$ . But if  $\sigma \in \text{Hom}(D, R)$  then  $\sigma(k) = 0$  since  $R$  is not injective (recall that  $\text{length } D = \text{length } R$ ). Now

$$D \otimes D \cong ((D \otimes D)^v)^v \cong \text{Hom}(D, R)^v \cong \text{Hom}(\text{Hom}(D, R), D)$$

So  $D \rightarrow \text{Hom}(D, D \otimes D)$  becomes

$$D \rightarrow \text{Hom}(D, \text{Hom}(\text{Hom}(D, R), D))$$



This map is  $x \mapsto (y \mapsto (\sigma(\mapsto \sigma(x)(y))))$ . By the remark above,  $k$  is in the kernel of this map.

REMARK 3.2. If  $R$  is local artinian, let  $S \subset R$  be the socle of  $R$  and let  $\dim_k S = p$  where  $k$  is the residue field. If  $p$  is a prime, then the only semi-dualizing modules  $C$  for  $R$  are  $E(k)$  (the dualizing module) and  $C = R$ . For if  $m$  is the maximal ideal of  $R$  and  $T \subset C$  is the socle of  $C$  then  $\text{Hom}(C/mC, T)$  is the socle of  $R = \text{Hom}(C, C)$ . But its dimension over  $k$  is  $\dim C/mC \cdot \dim T$ . So either  $\dim C/mC = 1$  or  $\dim T = 1$ . Then it is not hard to see that  $C = R$  in the first case and  $C = E(k)$  in the second.

EXAMPLE 3.3. Let  $k$  be a field and let  $R = k[[x^3, x^4, x^5]]$ . Then  $R$  is local, Cohen-Macaulay and of dimension 1. But  $R$  is not Gorenstein since the submonoid of  $N$  generated by 3, 4 and 5 is not symmetric.

The  $R$ -submodule  $D$  of  $k[[x]]$  generated by  $x$  and  $x^2$  is dualizing for  $R$ . We argue that  $E \rightarrow \text{Hom}(D, D \otimes E)$  is an injection for the injective cogenerator  $E = E(k)$ .

If  $P$  is the maximal ideal of  $R$  then  $E(R/P) = E(k) = k[x^{-3}, x^{-4}, x^{-5}]$  (by Park [9]). To argue that  $k[x^{-3}, x^{-4}, x^{-5}] \rightarrow \text{Hom}(D, D \otimes k[x^{-3}, x^{-4}, x^{-5}])$  is an injection means we have to argue that if  $z \in k[x^{-3}, x^{-4}, x^{-5}]$  and if  $y \otimes z = 0$  in  $D \otimes k[x^{-3}, x^{-4}, x^{-5}]$  for all  $y \in D$  then  $z = 0$ .

We argue that if  $z \neq 0$  then  $x \otimes z \neq 0$  in  $D \otimes k[x^{-3}, x^{-4}, x^{-5}]$ . If  $x \otimes z = 0$  then  $x \otimes z + x^2 \otimes 0 = 0$ . But  $x$  and  $x^2$  generate  $D$ . So in matrix notation

$$\begin{bmatrix} x & x^2 \end{bmatrix} \otimes \begin{bmatrix} z \\ 0 \end{bmatrix} = 0$$

So for this to happen there must be  $y, y_2, y_3 \in k[x^{-3}, x^{-4}, x^{-5}]$  such that

$$\begin{bmatrix} x^4 & x^5 & x^6 \\ -x^3 & -x^4 & -x^5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix}.$$

Since it is easy to check that the modules of relations between  $x$  and  $x^2$  (as a submodule of  $R^2$ ) is generated by  $(x^4, -x^3)$ ,  $(x^5, -x^4)$  and  $(x^6, -x^5)$ . But  $-x^3y_1 - x^4y_2 - x^5y_3 = 0$  implies  $x^4y_1 + x^5y_2 + x^6y_3 = 0$ , i.e. that  $z = 0$ .

REMARK. It would be of interest to characterize the submonoids  $S = \langle a_1, a_2, \dots, a_5 \rangle \subset N$  such that  $R = k[[x^{a_1}, x^{a_2}, \dots, x^{a_5}]]$  are such that  $({}^\perp \mathcal{U}, \mathcal{U})$  is a cotorsion theory.

We now consider passage from the ring  $R$  to the ring  $R[[x]]$ . Given an  $R$ -module  $M$  we have the  $R[[x]]$ -module  $M[[x]]$ . If  $N$  is another  $R$ -module then  $\text{Hom}_{R[[x]]}(M[[x]], N[[x]]) \cong \text{Hom}_R(M, N[[x]]) \cong \text{Hom}_R(M, N)[[x]]$ .

This and some of the isomorphisms below can be found in (Park [9]). For completeness, we give short arguments for them.

Also, for any  $R[[x]]$ -module  $U$ ,  $R[[x]] \otimes_{R[[x]]} U \cong R \otimes_R U$  and so if  $P$  is a projective  $R$ -module,  $P[[x]] \otimes_{R[[x]]} U \cong P \otimes_R U$ . Hence  $M$  is any  $R$ -module, and  $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  is exact with  $P_1, P_0$  projective, we get the commutative diagram

$$\begin{array}{ccccccc} P_1 \otimes_R U & \longrightarrow & P_0 \otimes_R U & \longrightarrow & M \otimes_R U & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ P_1[[x]] \otimes_{R[[x]]} U & \longrightarrow & P_0[[x]] \otimes_{R[[x]]} U & \longrightarrow & M[[x]] \otimes_{R[[x]]} U & \longrightarrow & 0 \end{array}$$

with exact rows. So  $M \otimes_R U \cong M[[x]] \otimes_{R[[x]]} U$ .

If we furthermore assume  $U = N[[x]]$  with  $N$  an  $R$ -module, we get  $M[[x]] \otimes_{R[[x]]} N[[x]] \cong M \otimes_R N[[x]] \cong M[[x]] \otimes_R N$ . But if  $M$  is furthermore finitely generated, we get  $M \otimes_R N[[x]] \cong (M \otimes_R N)[[x]]$ . So we get  $M[[x]] \otimes_{R[[x]]} N[[x]] \cong (M \otimes_R N)[[x]]$  if either  $M$  or  $N$  is finitely generated.

These isomorphisms then give  $\text{Ext}_{R[[x]]}^n(M[[x]], N[[x]]) \cong \text{Ext}_R^n(M, N)[[x]]$  and  $\text{Tor}_n^{R[[x]]}(M[[x]], N[[x]]) \cong \text{Tor}_n^R(M, N)[[x]]$  (here if  $M$  or  $N$  is finitely generated).

From these isomorphisms it easily follows that if  $C$  is semi-dualizing for  $R$ , then  $C[[x]]$  is semi-dualizing for  $R[[x]]$ . But also if  $D$  is dualizing for  $R$  then  $D[[x]]$  is dualizing for  $R[[x]]$  (see [4], Proposition 2.7).

Letting  $R$  be a local artinian ring which is not Gorenstein and  $D = E(k)$ , then we know  $D \rightarrow \text{Hom}_R(D, D \otimes_R D)$  is not injective. So with  $\bar{D} = D[[x_1, \dots, x_s]]$  and  $\bar{R} = R[[x \cdots x_s]]$  we get

$$\begin{aligned} \bar{D} &= D[[x_1, \dots, x_s]] \\ &\rightarrow \text{Hom}_{\bar{R}}(\bar{D}, \bar{D} \otimes_R \bar{D}) = \text{Hom}_R(D, D \otimes_R D)[[x_1, \dots, x_s]] \end{aligned}$$

Now let  $R$  be local with a dualizing module  $D$  and suppose  $R$  with  $D$  (as our  $C$ ) satisfies the conditions of Theorem 2.10. By f) of that theorem this means  $E(k) \rightarrow \text{Hom}(D, D \otimes E(k))$  is an injection where  $k$  is the residue field of  $R$ . The injective envelope of the residue field of  $R[[x]]$  is  $E(k)[x^{-1}]$  and  $D[[x]]$  is a dualizing module for  $R[[x]]$ . We argue that

$$E(k)[x^{-1}] \rightarrow \text{Hom}_{R[[x]]}(D[[x]], D[[x]] \otimes_{R[[x]]} E(k)[x^{-1}])$$

is also an injection.

We have  $D[[x]] \otimes_{R[[x]]} E(k)[x^{-1}] \cong D \otimes_R (E(k)[x^{-1}]) \cong (D \otimes_R E(k))[x^{-1}]$ . But an  $R[[x]]$ -linear map  $D[[x]] \rightarrow (D \otimes_R E(k))[x^{-1}]$  is uniquely determined by its  $R$ -linear restriction  $D \rightarrow (D \otimes_R E(k))[x^{-1}]$  (since ever

element of  $(D \otimes_R E(k))[x^{-1}]$  is annihilated by  $x^n$  for some  $n \geq 1$ ) and this restriction can be any  $R$ -linear map. So  $\text{Hom}_{R[[x]]}(D[[x]], D[[x]] \otimes_{R[[x]]} E(k)[x^{-1}]) \cong \text{Hom}_R(D, (D \otimes_R E(k))[x^{-1}])$ . But this  $R[[x]]$ -module is isomorphic to  $\text{Hom}(D, D \otimes E(k))[x^{-1}]$ . So our map becomes

$$E(k)[x^{-1}] \rightarrow \text{Hom}(D, D \otimes E(k))[x^{-1}].$$

Since  $E(k) \rightarrow \text{Hom}(D, D \otimes E(k))$  is an injection, so is this map.

Noting that  $R[[x]]$  is Gorenstein if and only if  $R$  is, we see that by this procedure we can use our zero dimensional example to get a local Cohen-Macaulay ring of any dimension  $d \geq 1$  admitting a dualizing module which is not Gorenstein but which does satisfy the conditions of Theorem 2.10 (with  $C$  being a dualizing module). Similarly we can get examples for every  $d \geq 0$  which do not satisfy these conditions.

#### REFERENCES

1. Bican, L., El Bashir, R., and Enochs, E., *All modules have flat covers*, Bull. London Math. Soc. 33 (2001), 385–390.
2. Winther Christensen, L., *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. 353 (2001), 1839–1883.
3. Winther Christensen, L., *Gorenstein dimensions*, Lecture Notes in Math. 1747 (2000).
4. Enochs, E. E., Herzog, I., and Park, S., *Cyclic quiver rings and polycyclic-by-finite group rings*, Houston J. Math. 25 (1999), 1–13.
5. Enochs, E. E., and Jenda, O. M. G., *Relative Homological Algebra*, De Gruyter, Berlin-New York (2000).
6. Enochs, E. E., Jenda, O. M. G., and Xu, J., *Foxby dualizing and Gorenstein injective and projective modules*, Trans. Amer. Math. Soc. 348 (1996) no. 8, 3223–3234.
7. Golod, E. S., Uspekhi Mat. Nauk 40 (1985) no. 1 (241), 234, (Question, in Russian).
8. Northcott, D. G., *Injective envelopes and inverse polynomials*, J. London Math. Soc. 68 (1974), 290–296.
9. Park, S., *The Macaulay-Northcott functor*, Arch. Math. (Basel) 63 (1994), 225–230.
10. Park, S., *Gorenstein rings and inverse polynomials*, Comm. Algebra 28 (2000), 785–789.
11. Salce, L., *Cotorsion theories for abelian groups*, Sympos. Math. 23 (1979), 11–32.
12. Xu, J., *Flat covers of modules*, Lecture Notes in Math. 1634 (1996).

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF KENTUCKY  
LEXINGTON, KENTUCKY 40506-0027  
USA  
*E-mail:* enochs@ms.uky.edu

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF TEHRAN  
IRAN

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF KENTUCKY  
LEXINGTON, KENTUCKY 40506-0027  
USA  
*E-mail:* yassemi@ipm.ir