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LOWER BOUNDS FOR QUASIANALYTIC FUNCTIONS, II. THE BERNSTEIN QUASIANALYTIC FUNCTIONS

A. BORICHEV, F. NAZAROV and M. SODIN*

Abstract

Let $\mathscr{F}$ be a class of functions with the uniqueness property: if $f \in \mathscr{F}$ vanishes on a set E of positive measure, then f is the zero function. In many instances, we would like to have a quantitative version of this property, e.g. a lower bound for $|f|$ outside a small exceptional set. Such estimates are well-known and useful for polynomials, complex- and real-analytic functions, exponential polynomials. In this work we prove similar results for the Denjoy-Carleman and the Bernstein classes of quasianalytic functions.

In the first part, we considered quasianalytically smooth functions. Here, we deal with classes of functions characterized by exponentially fast approximation by polynomials whose degrees belong to a given very lacunar sequence. We also prove the polynomial spreading lemma and a comparison lemma which are of a certain interest on their own.

1. Introduction and the results

Let f be a continuous function on $[-1, 1]$, and let

$$E_n(f) = \min_{P \in \mathscr{P}_n} \|f - P\|_{[-1,1]}$$

be the *approximating sequence of the function* f. Here $\mathscr{P}_n$ is a space of all algebraic polynomials of degree $\le n$, and the norm $\|\,.\,\|_F$ denotes the uniform norm $\|\,.\,\|_{C(F)}$ on the set F. A classical result of S. Bernstein [1], [2] states that if for some $\beta > 0$

$$E_n(f) \le e^{-\beta n} \tag{1.1}$$

when n runs through a subsequence $\{n_j\} \subset \mathsf{N}$, and if the function f vanishes on a subset of $[-1, 1]$ of positive measure, then f is the zero function. If a sequence $\{n_j\}$ is not too lacunary:

$$\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} \le \Delta < \infty,$$

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then condition (1.1) describes a class of real-analytic functions on $[-1, 1]$ with analytic extension into a certain complex neighbourhood of $[-1, 1]$ whose size depends on the quotient β/Δ.

The functions satisfying (1.1) are called the *Bernstein quasianalytic functions*. Having the uniqueness property, generally speaking, they do not posses any smoothness. They were studied by Bernstein [1], [2], Beurling [3], Mergelyan [9, Chapter VII], Pleśniak [13], by no means is this list complete.

Here, we give an asymptotic upper bound for the size of the level sets

$$m_f(t) = |\{x \in [-1, 1] : \ |f(x)| \le t\}|$$

for $t = E^*_{n_j}(f)$ where $E^*_n(f) = \max\left(E_n(f), e^{-n}\right)$. The main result follows:

THEOREM A. *Suppose f is a Bernstein quasianalytic function satisfying condition* (1.1) *with sufficiently lacunar sequence $\{n_j\}$:*

$$\lim_{j\to\infty} \frac{n_{j+1}}{n_j} = +\infty. \tag{1.2}$$

Then

$$\lim_{j\to\infty} \frac{|\log m_f(E^*_{n_{j+1}}(f))|}{|\log m_f(E^*_{n_j}(f))|} = +\infty. \tag{1.3}$$

COROLLARY. *In the assumptions of Theorem A, we have*

$$\lim_{j\to\infty} \frac{\log|\log m_f(E^*_{n_j}(f))|}{j} = +\infty. \tag{1.4}$$

Mention that relations (1.3) and (1.4) are fulfilled with $E_{n_j}(f)$ instead of $E^*_{n_j}(f)$ if we assume additionally that

$$\lim_{j\to\infty} \frac{|\log E_{n_{j+1}}(f)|}{|\log E_{n_j}(f)|} = +\infty.$$

This follows from the proof of the Theorem A given below.

The more lacunary is the sequence $\{n_j\}$ in Theorem A, the worse is our bound (1.4). This is natural since as our second result shows, the Bernstein quasianalytic functions may have deep zeros of prescribed flatness:

THEOREM B. *Given decreasing functions $\varphi, \psi : [1, +\infty) \to (0, +\infty)$,*

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \psi(t) = 0,$$

there exist a function $f \in C[-1,1]$ *and a subsequence* $\{n_j\} \subset \mathsf{N}$ *such that for* $n \in \{n_j\}$

$$E_n(f) \le \psi(n), \tag{1.5}$$

and

$$|f(x)| \le e^{-n}, \qquad |x| \le \varphi(n). \tag{1.6}$$

In particular, if $\psi(s) = e^{-s}$, we get

$$m_f(E^*_{n_j}(f)) \ge 2\varphi(n_j), \qquad j = 1, 2, \ldots .$$

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2. The polynomial spreading lemma

The key ingredient in the proof of Theorem A is the following

POLYNOMIAL SPREADING LEMMA. *Let* $P \in \mathcal{P}_n$, $\|P\|_{[-1,1]} \le 1$, *and let*

$$\delta \ge \delta_0 > 0, \qquad 0 < c_0 \le c < 1, \qquad 0 < \varepsilon < \frac{1-c}{2-c},$$

be some parameters. Suppose $E \subset [-1,1]$ *is a measurable subset of sufficiently small measure*

$$|E| < \kappa(\delta_0, \varepsilon, c_0) \tag{2.1}$$

such that

$$\|P\|_E \le e^{-\delta n},$$

and suppose that I *is an interval,* $E \subset I \subset [-1,1]$. *Then*

$$\|P\|_I \le e^{-c\delta n}$$

provided

$$|I| \le |E|^{\frac{1}{2-c}+\varepsilon}. \tag{2.2}$$

If the set E is itself an interval, then (2.2) can be significantly improved to $|I| \le |E|^{c+\varepsilon}$ (cf. the end of the proof of comparison lemma below). This can be regarded as a polynomial version of the Hadamard three circle theorem. An 'ideal statement' would be

$$\|P\|_I \leq \|P\|_E^c \|P\|_{[-1,1]}^{1-c}$$

provided that $E \subset I \subset [-1, 1]$ are intervals such that

$$|I| \leq |E|^c 2^{1-c}.$$

This is too good to be true. Our result gives a reasonable approximation to such a logarithmic convexity.

The exponent $\frac{1}{2-c}+\epsilon$ in (2.2) is larger than the exponent $c+\epsilon$ we need. However, they are close to each other when c is close to one. To obtain Theorem A we first use a dyadic decomposition with a simple stopping-time rule, and then apply iteratively the spreading lemma for $c = 1 - \rho$ with small $\rho > 0$ to get the comparison lemma (Section 3) which claims that under natural conditions

$$\begin{aligned} &\left|\{x \in [-1, 1] : |P(x)| \leq e^{-t\delta n}\|P\|_{[-1,1]}\}\right| \\ &\qquad\qquad \geq \left|\{x \in [-1, 1] : |P(x)| \leq e^{-\delta n}\|P\|_{[-1,1]}\}\right|^{t+\gamma} \end{aligned}$$

with $0 < t_0 < t < 1$ and $0 < \gamma < 1 - t$. The proof of Theorem A is then completed in Section 4.

Proof of the spreading lemma. We use an argument adopted from Nadirashvili's work [10], [11]. Let $\eta = |I|/|E|$, and let x_0 be the centre of the interval I. Fix $k \geq 0$, and consider the Taylor polynomial

$$P_k(x) = \sum_{j=0}^{k} \frac{P^{(j)}(x_0)}{j!}(x - x_0)^j,$$

and the remainder $R_k = P - P_k$. Applying to P_k the classical Remez inequality [14] (cf. [4], [5]), we get

$$\begin{aligned} \|P\|_I &\leq \|P_k\|_I + \|R_k\|_I \\ &\leq (4\eta)^k \|P_k\|_E + \|R_k\|_{[-1,1]} \\ &\leq (4\eta)^k \left(\|P\|_E + \|R_k\|_{[-1,1]}\right) + \|R_k\|_{[-1,1]} \\ &\leq (4\eta)^k \left(e^{-\delta n} + 2\|R_k\|_{[-1,1]}\right) \end{aligned}$$

(due to conditions (2.1) and (2.2), we assume without loss of generality that $\eta > 1/4$). Using the Lagrange formula for the remainder, we have

$$\|R_k\|_{[-1,1]} \leq \left(\frac{|I|}{2}\right)^{k+1} \frac{\|P^{(k+1)}\|_{[-1,1]}}{(k+1)!} < \left(\frac{e}{2}\,\frac{|I|}{k+1}\right)^{k+1} \|P^{(k+1)}\|_{[-1,1]}.$$

Recalling the classical V. Markov inequality [8] for the $(k+1)$-st derivative of the polynomial P of degree n[1]

$$\|P^{(k+1)}\|_{[-1,1]} \le \frac{1}{2}\left(\frac{2}{k+1}\right)^{k+1} n^{2k+2}\|P\|_{[-1,1]},$$

we get

$$\|R_k\|_{[-1,1]} \le \frac{1}{2}\left(\frac{e|I|n^2}{(k+1)^2}\right)^{k+1},$$

and then

$$\|P\|_I \le (4\eta)^k\left\{e^{-\delta n} + \left(\frac{e|I|n^2}{(k+1)^2}\right)^{k+1}\right\}. \tag{2.3}$$

Now, our requirements to the choice of k are the following:

$$(4\eta)^k \le e^{\delta(1-c)n} \tag{2.4}$$

and

$$\left(\frac{e|I|n^2}{(k+1)^2}\right)^{k+1} \le e^{-\delta n}. \tag{2.5}$$

Naturally, relations (2.3)–(2.5) yield that

$$\|P\|_I \le 2e^{-c\delta n}. \tag{2.6}$$

Applying (2.6) to P^M with $M \in \mathsf{N}$, we obtain

$$\|P^M\|_I \le 2e^{-c\delta n M}$$

or

$$\|P\|_I = \|P^M\|_I^{1/M} \le 2^{1/M}e^{-c\delta n} = (1+o(1))e^{-c\delta n}, \qquad M \to \infty,$$

completing the proof of the lemma.

It remains to verify that under conditions (2.2) there exists k satisfying (2.4)–(2.5). Suppose that for some positive λ,

$$\lambda \le \delta\frac{1-c}{\log 4\eta}, \tag{2.7}$$

[1] There are several relatively simple proofs of this inequality, see e.g. [6] for one of them. In fact, we could use a slightly cruder version of Markov's estimate given in [7, Chapter VI, Lemma 4.III] with a proof found by Th. Bang.

$$\lambda \log \frac{\lambda^2}{e|I|} \geq \delta, \tag{2.8}$$

$$\lambda^2 \geq \frac{|I|}{e}, \tag{2.9}$$

and choose k such that

$$k \leq \lambda n < k + 1.$$

Then (2.4) follows from (2.7):

$$(4\eta)^k \leq (4\eta)^{\lambda n} \leq e^{\delta(1-c)n},$$

and (2.5) follows from (2.8) because the left-hand side of (2.8) increases as a function of λ satisfying (2.9):

$$\begin{aligned}\left(\frac{e|I|n^2}{(k+1)^2}\right)^{k+1} &= \exp\left[-(k+1)\log\frac{(k+1)^2}{e|I|n^2}\right] \\ &\leq \exp\left[-\lambda n \log \frac{\lambda^2}{e|I|}\right] \leq \exp[-\delta n].\end{aligned}$$

First of all, without loss of generality, we assume that $\delta \leq 1$ (otherwise, we set $n_1 := [\delta n] \geq n$, and apply the lemma with the parameters n_1 and 1 instead of n and δ). Now, we denote $A = \delta^2/(e|I|)$, $B = A/\log^2 A$, and set $\lambda = \delta/\log B$. We have to check that inequalities (2.7)–(2.9) hold for this choice of λ under the condition that the length of E (and therefore that of I) is sufficiently small, that is the value A is sufficiently large.

Estimate (2.7) says that

$$(4\eta)^{1/(1-c)} \leq e^{\delta/\lambda} = B = \frac{\delta^2}{e|I|\log^2 \frac{\delta^2}{e|I|}},$$

or, equivalently,

$$\left(4\frac{|I|}{|E|}\right)^{1/(1-c)} \leq \frac{\delta^2}{e|I|\log^2 \frac{\delta^2}{e|I|}},$$

$$|I|^{\frac{2-c}{1-c}} \log^2 \frac{\delta^2}{e|I|} \leq \frac{\delta^2}{4^{1/(1-c)}e}|E|^{1/(1-c)}$$

that follows from (2.1) and (2.2).

Inequality (2.8) becomes

$$\frac{\delta}{\log B}\log\frac{A}{\log^2 B} \geq \delta,$$

that is
$$\frac{A}{\log^2 B} \geq \frac{A}{\log^2 A},$$
which is evidently true.

At last, inequality (2.9) becomes
$$\frac{1}{\log^2 B} \geq \frac{1}{e^2 A}$$
which is true when A is sufficiently large.

3. Comparison lemma

Here, we give a corollary to the spreading lemma which will be needed for the proof of Theorem A. For $P \in \mathscr{P}_n$, we set
$$\mathscr{E}_P(\delta) = \left\{x \in [-1,1] : |P(x)| \leq e^{-\delta n} \|P\|_{[-1,1]}\right\}.$$

COMPARISON LEMMA. *Let $P \in \mathscr{P}_n$ and let*
$$\delta \geq \delta_0 > 0, \qquad 0 < t_0 \leq t < 1, \qquad 0 < \gamma < 1 - t$$
be some parameters. Then

$$|\mathscr{E}_P(t\delta)| \geq |\mathscr{E}_P(\delta)|^{t+\gamma} \tag{3.1}$$

provided that the length of the set $\mathscr{E}_P(\delta)$ is (δ_0, γ, t_0)-sufficiently small.

PROOF. Without loss of generality we assume that $\|P\|_{[-1,1]} = 1$. First of all, we prove a weaker result:

CLAIM. *If*
$$0 < c_0 \leq c < 1, \qquad 0 < \varepsilon < \frac{1-c}{2-c},$$
and if
$$|\mathscr{E}_P(\delta)| \leq \kappa(\delta_0, \varepsilon, c_0),$$
where κ has the same value as in the spreading lemma, then
$$|\mathscr{E}_P(c\delta)| \geq |\mathscr{E}_P(\delta)|^{\frac{1}{2-c}+\varepsilon}.$$

Set $E = \mathscr{E}_P(\delta)$ and choose an integer
$$N \geq \kappa^{-1}(\delta_0, \varepsilon, c_0).$$

Let $\mathscr{J}$ be the collection of all maximal N-adic subintervals I of $[-1, 1]$ such that

$$|E \cap I|^{\frac{1}{2-c}+\varepsilon} \geq |I|.$$

Then the 'remainder set'

$$F = E \setminus \bigcup_{I \in \mathscr{J}} (E \cap I)$$

has zero length. Indeed, for any $\xi > 0$ we can cover F by disjoint N-adic intervals J_α of length $|J_\alpha| < \xi$, and

$$|F \cap J_\alpha| \leq |J_\alpha| \cdot \xi^{(\frac{1}{2-c}+\varepsilon)^{-1}-1}.$$

Summing up by α, we see that

$$|F| \leq \xi^{\left(\frac{1}{2-c}+\varepsilon\right)^{-1}-1}, \qquad \xi > 0,$$

and hence, $|F| = 0$.

Now, we apply the spreading lemma to the sets $E \cap I$, $I \in \mathscr{J}$. Since $I_0 = [-1, 1] \notin \mathscr{J}$, we have

$$|E \cap I| \leq |I| \leq \frac{1}{N} \leq \kappa(\delta_0, \varepsilon_0, c_0),$$

and the conditions of the lemma are satisfied. Hence,

$$\|P\|_I \leq e^{-c\delta n}, \qquad I \in \mathscr{J},$$

that is

$$\bigcup_{I \in \mathscr{J}} I \subset \mathscr{E}_p(c\delta).$$

Since I is the maximal interval, its N-adic 'supinterval' I^* does not belong to $\mathscr{J}$, that is

$$|E \cap I|^{\frac{1}{2-c}+\varepsilon} \leq |E \cap I^*|^{\frac{1}{2-c}+\varepsilon} \leq |I^*| = N|I|,$$

or $|I| \geq N^{-1}|E \cap I|^{\frac{1}{2-c}+\varepsilon}$. Therefore,

$$\begin{aligned} |\mathscr{E}_P(c\delta)| &\geq \sum_{I \in \mathscr{J}} |I| \geq \frac{1}{N} \sum_{I \in \mathscr{J}} |E \cap I|^{\frac{1}{2-c}+\varepsilon} \\ &\geq \frac{1}{N} \left(\sum_{I \in \mathscr{J}} |E \cap I| \right)^{\frac{1}{2-c}+\varepsilon} = \frac{|E|^{\frac{1}{2-c}+\varepsilon}}{N} = \frac{|\mathscr{E}_P(\delta)|^{\frac{1}{2-c}+\varepsilon}}{N}. \end{aligned}$$

Increasing slightly ε, we get the claim.

Now we choose an integer M and $\varepsilon > 0$ in such a way that

$$\left(\frac{1}{2-t^{1/M}}+\varepsilon\right)^M \le t+\gamma,$$

set $c_0 = t_0$, $c = t^{1/M}$ and apply the claim M times. If on each step the sets $\mathscr{E}_P(c^j\delta)$, $1 \le j \le M$ have small length, then this is legitimate, and we get

$$|\mathscr{E}_P(t\delta)| \ge |\mathscr{E}_P(\delta)|^{\left(\frac{1}{2-t^{1/M}}+\varepsilon\right)^M} \ge |\mathscr{E}_P(\delta)|^{t+\gamma}.$$

Otherwise,

$$|\mathscr{E}_P(t\delta)| \ge \kappa_1(\delta_0, \varepsilon, c_0).$$

In both cases we get (3.1), and thus the lemma is proved.

4. Proof of Theorem A

Theorem A is a simple corollary to the comparison lemma.

Let P_n be the polynomials of the best approximation to f, that is

$$\|f - P_n\|_{[-1,1]} = E_n(f).$$

Without loss of generality, we assume that $\|f\|_{[-1,1]} = 1$, and

$$1/2 \le \|P_{n_j}\|_{[-1,1]} \le 2.$$

Now, according to the definition of $E_n^*(f)$ and (1.1),

$$E_{n_j}^*(f) = e^{-\delta_j n_j}, \qquad \min(\beta, 1) \le \delta_j \le 1.$$

Due to the lacunarity condition (1.2), for any $\varepsilon > 0$,

$$\frac{1}{4}E_{n_{j-1}}^*(f) \ge (4E_{n_j}^*(f))^{\varepsilon}, \qquad j \ge j(\varepsilon), \tag{4.1}$$

and we consider only these sufficiently large j's. Then

$$\left\{x \in [-1,1] : |f(x)| \le E_{n_j}^*(f)\right\} \subset \left\{x \in [-1,1] : |P_{n_j}(x)| \le 4E_{n_j}^*(f)\|P_{n_j}\|_{[-1,1]}\right\}$$

and

$$\left\{x \in [-1,1]\colon |f(x)| \le E_{n_{j-1}}^*(f)\right\} \supset \left\{x \in [-1,1]\colon |P_{n_j}(x)| \le \frac{1}{4}E_{n_{j-1}}^*(f)\|P_{n_j}\|_{[-1,1]}\right\}.$$

In the second inclusion, we used that if $|P_{n_j}(x)| \le 4^{-1} E^*_{n_{j-1}} ||P_{n_j}||_{[-1,1]}$, then

$$|f(x)| \le E_{n_j}(f) + |P_{n_j}(x)| \le E^*_{n_j}(f) + \frac{1}{4} E^*_{n_{j-1}} \cdot 2 < E^*_{n_{j-1}}(f).$$

Therefore, applying the comparison lemma to the polynomials P_{n_j} with $t = \gamma = \varepsilon$, we get for sufficiently large j:

$$\begin{aligned} m_f(E^*_{n_{j-1}}(f)) &\ge \left| \left\{ x \in [-1,1] \colon |P_{n_j}(x)| \le \frac{1}{4} E^*_{n_{j-1}}(f) \|P_{n_j}\|_{[-1,1]} \right\} \right| \\ &\ge \left| \left\{ x \in [-1,1] \colon |P_{n_j}(x)| \le 4 E^*_{n_j}(f) \|P_{n_j}\|_{[-1,1]} \right\} \right|^{2\varepsilon} \ge m_f^{2\varepsilon}(E^*_{n_j}(f)). \end{aligned}$$

This proves the theorem.

5. Proof of Theorem B

We start with

LEMMA. *Let Q be a polynomial, $Q(0) = 0$. Then for any odd positive integer n and any sufficiently large integer $l \ge l_0(n)$ there is a polynomial P of degree at most ln* deg *Q such that*

$$\|P\|_{[-1,1]} \le C_1 \cdot \frac{\|Q\|_*}{n},$$

and

$$|(Q+P)(t)| \le C_1 \cdot (2n|t|)^{l+1} \cdot \|Q\|_*, \qquad |t| \le \frac{1}{n}.$$

Here $\|Q\|_*$ means the sum of the absolute values of the coefficients of Q, and C_1 is a constant.

PROOF. First, we prove a special case of the lemma with $Q(t) = t$. Set

$$\Phi_n(w) = n \sin\left(\frac{1}{n} \arcsin w\right), \qquad w = u + iv.$$

The functions Φ_n are analytic in the unit disc, continuous up to its boundary and uniformly bounded. Furthermore,

$$|\Phi_n(w)| \lesssim |w|, \qquad |w| \le 1,$$

where the notation $A \lesssim B$ means that $A \le C \cdot B$ for a positive numerical constant C. Also,

$$|\Phi_n^{(k)}(w)| \lesssim 2^k k!, \qquad |w| \le \frac{1}{2}, \quad k \in \mathbb{Z}_+.$$

Set

$$\Phi_{n,l}(w) = \sum_{k=0}^{l} \frac{\Phi_n^{(k)}(0)}{k!} w^k.$$

The function $u \mapsto \arcsin u$ has an absolutely convergent Taylor series in the closed unit disc. Since postcomposition with any entire function (in our case, with $\lambda \mapsto n\sin(\lambda/n)$) preserves this property, the polynomials $\Phi_{n,l}(u)$ converge to $\Phi_n(u)$ at the points $u = \pm 1$, and therefore converge to $\Phi_n(u)$ uniformly in $u \in [-1, 1]$, so that

$$|\Phi_{n,l}(u)| \lesssim 1, \qquad u \in [-1, 1], \quad l \ge l_0(n), \tag{5.1}$$

and

$$|(\Phi_n - \Phi_{n,l})(u)| \le \frac{|u|^{l+1}}{(l+1)!} \|\Phi_n^{(l+1)}\|_{[-u,u]} \lesssim (2|u|)^{l+1}, \qquad |u| \le \frac{1}{2}. \tag{5.2}$$

Let $t = \sin\left(\frac{1}{n} \arcsin u\right)$, then $u = u_n(t) = \sin(n \arcsin t)$. Since n is odd, $u_n(t)$ is a polynomial of degree n, and

$$|u_n(t)| \le \min(1, n|t|), \qquad |t| \le 1. \tag{5.3}$$

Indeed, it is sufficient to verify that

$$n \sin t - \sin(nt) \ge 0, \qquad 0 \le \sin t \le \frac{1}{n}.$$

This inequality holds for $t = 0$, and taking the derivative of the left-hand side we get

$$n[\cos t - \cos(nt)]$$

which is non-negative on the interval $[0, \arcsin \frac{1}{n}]$.

Set

$$R_{n,l}(t) = -\frac{1}{n} \left(\Phi_{n,l} \circ u_n\right)(t).$$

This is a polynomial of degree at most ln. We have

$$\begin{aligned} \left|t + R_{n,l}(t)\right| &= \left|\frac{1}{n} \left(\Phi_n - \Phi_{n,l}\right)(u_n(t))\right| \\ &\overset{(5.2)}{\lesssim} (2|u_n(t)|)^{l+1} \overset{(5.3)}{\lesssim} (2n|t|)^{l+1}, \qquad |t| \le \frac{1}{n}, \end{aligned}$$

and

$$|R_{n,l}(t)| \overset{(5.1)}{\lesssim} \frac{1}{n}, \qquad |t| \le 1.$$

This proves the special case of the lemma.

The general case follows if we set

$$Q(t) = \sum_{j\geq 1} c_j t^j,$$

and

$$P(t) = \sum_{j\geq 1} c_j R_{n,l}(t^j).$$

COROLLARY. *Given a polynomial Q, $Q(0) = 0$, $\varepsilon > 0$, $N < \infty$, and given a function φ decreasing to zero at infinity, there exist $M > N$, and a polynomial P, $P(0) = 0$, such that* $\deg P \leq M$,

$$\|P\|_{[-1,1]} \leq \varepsilon,$$

and

$$|(Q + P)(t)| \leq e^{-2M}, \qquad |t| \leq \varphi(M).$$

PROOF. We apply the lemma with

$$n = \frac{C_1 \cdot \|Q\|_*}{\varepsilon},$$

and l such that

$$M = ln \deg Q > N,$$

$$\varphi(M) \leq \frac{1}{2n} e^{-2n \deg Q},$$

$$e^{2n(l+1) \deg Q} \geq C_1 \cdot e^{2M} \|Q\|_*.$$

We get a polynomial P of degree at most M such that

$$\|P\|_{[-1,1]} \leq \varepsilon,$$

and

$$|(Q + P)(t)| \leq \frac{C_1 \|Q\|_*}{e^{2n(l+1) \deg Q}} \leq e^{-2M}, \qquad |t| \leq \varphi(M).$$

This establishes the corollary.

PROOF OF THEOREM B. We use the corollary in an inductive procedure. We build the function f in the form

$$f = \sum_{j\geq 1} P_j,$$

where P_j are polynomials such that for an increasing sequence of integers $\{n_j\}$ we have $\deg P_j \le n_j$, $\|P_j\|_{[-1,1]} < \psi(n_{j-1})/2$, and

$$\left|\left(\sum_{1\le j\le m} P_j\right)(x)\right| \le e^{-2n_m}, \qquad |x| \le \varphi(n_m).$$

We start with $P_1(x) = x$. After $P_1, \dots, P_m$ have been chosen, set

$$Q = \sum_{1\le j\le m} P_j,$$
$$\varepsilon = \psi(n_m)/2,$$
$$N = n_m,$$

and get $n_{m+1} = M > n_m$ and P_{m+1} from the corollary.

We may always assume that

$$\psi(x) \le e^{-2x}, \qquad x \ge 1, \tag{5.4}$$

otherwise, from the very beginning, we replace $\psi(x)$ by $\min(e^{-2x}, \psi(x))$; and that

$$\sum_{j\ge m+1} \psi(n_{j-1})/2 \le \psi(n_m). \tag{5.5}$$

Now, we check that f and $\{n_j\}$ satisfy conditions (1.5) and (1.6) of Theorem B. For $n = n_m$ we have

$$\sum_{j\ge m+1} \|P_j\|_{[-1,1]} \le \sum_{j\ge m+1} \psi(n_{j-1})/2 \overset{(5.5)}{\le} \psi(n)$$

that proves (1.5). Finally, for $|x| \le \varphi(n)$

$$|f(x)| \le \left|\left(\sum_{1\le j\le m} P_j\right)(x)\right| + \sum_{j\ge m+1} \|P_j\|_{[-1,1]} \le e^{-2n} + \psi(n) \le e^{-n}$$

that proves (1.6).

6. Remarks and questions

6.1. Beurling's theorem.

Beurling [3, p. 396–403] gave a general quasianalyticity condition which contains those of Bernstein and Denjoy-Carleman. Here, we formulate a special case of his result. Given a sequence $1 \ge e_n \downarrow 0$, consider the Bernstein class

$$\mathscr{F}_{\{e_n\}} = \{f \in C[-1,1] : E_n(f) \le e_n\} \setminus \{0\}.$$

THEOREM (Beurling). *The class $\mathscr{F}_{\{e_n\}}$ contains no function vanishing on a subset of positive measure if and only if*

$$\sum_{n \ge 1} \frac{\log^- e_n}{n^2} = +\infty,$$

where $\log^- a = \max\bigl(\log \frac{1}{a}, 0\bigr)$.

Beurling's proof uses the Laplace transform combined with the harmonic estimation in the "if part" and the Paley-Wiener theorem in the "only if part". One can extract a quantitative estimate from his proof which however is essentially weaker than Theorem A above and Theorem B from the part I [12] of this work.

It seems to be interesting to obtain another proof of Beurling's theorem by means of the constructive function theory and to get its quantitative version in a sharp form.

6.2. Potential theory approach

A minute's reflection suggests that there could be a natural generalization of the spreading lemma and comparison lemma for the logarithmic potentials of probability measures.

Let u be a subharmonic function in the complex plane $\mathbf{C}$ with compactly supported Riesz measure μ, $\mu(\mathbf{C}) \le 1$, and let $u\big|_{[0,1]} \le 0$. Let $E \subset [0,1]$ be a subset of positive measure such that $u\big|_E \le -\delta$.

PROBLEM. *Given c, $0 < c < 1$, estimate from below the length of the set* $\{x \in [0,1] : u(x) \le -c\delta\}$.

An 'ideal' lower bound would be $|E|^c$, which is true in the trivial limiting cases $c = 0$ and $c = 1$. Probably, our polynomial comparison lemma (combined with a suitable atomization of the Riesz measure μ) yields an 'asymptotic' lower bound $|E|^{c+\epsilon}$. However, it seems more natural to treat this problem by means of potential theory.

One can also ask a similar question replacing the unit interval by the unit disk. In this case, probably, one should deal with capacity instead of linear measure.

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LABORATOIRE DE MATHÉMATIQUES PURES
UNIVERSITÉ BORDEAUX I
TALENCE, 33405
FRANCE
E-mail: borichev@math.u-bordeaux.fr

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
USA
E-mail: fedja@math.msu.edu

SCHOOL OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
RAMAT AVIV, 69978
ISRAEL
E-mail: sodin@post.tau.ac.il