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LOCAL DYNAMICS OF UNIFORMLY QUASIREGULAR MAPPINGS

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Abstract

We investigate local dynamics of uniformly quasiregular mappings, give new examples and show in particular that there is no quasiconformal analogue of the Leau-Fatou linearization of parabolic dynamics.

1. Introduction

Uniformly quasiregular (uqr) mappings $f : \overline{\mathsf{R}}^n \to \overline{\mathsf{R}}^n$ are quasiregular maps such that all the iterates f^k have a common distortion bound. These are natural higher (real) dimensional analogues of holomorphic functions and appeared for the first time in the paper [5] of Iwaniec and Martin.

In this paper we investigate local dynamics of these maps. The first problem is to give a classification of the different fixed point types. Recall that for holomorphic maps the multiplier, meaning the derivative of the map at the fixed point, is used for that. Quasiregular maps need not be differentiable, and even though they are locally Hölder continuous, they may be so with exponent less than 1. In the case of uqr maps, however, we are able to establish Lipschitz estimates near a fixed point x_0 which is not a branch point. Such estimates show that $\mathscr{F} = \{f_\lambda : \lambda > 1\}$ is a normal family, where $f_\lambda(z) = \lambda f(z/\lambda)$. A limit of a convergent subsequence of $\mathscr{F}$ is in fact a uniformly quasiconformal map and we call it a generalized derivative of f at x_0. Using the classification of uniformly quasiconformal maps (they are either loxodromic, elliptic or parabolic) we get an analytic classification of the different fixed points which generalizes in a natural way the usual one of holomorphic functions. As an application we get that uqr maps do have precisely the same type of stable components as rational functions.

Then we turn to the existence problem. Examples of uqr maps with attracting, repelling or super-attracting fixed points are known [5], [15]. We obtain

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new examples with parabolic dynamics and we also show that such a map can be constructed in a way that it does not admit a quasiconformal linearization in its attracting parabolic petal. We then complete the picture by showing that the natural candidates for a linearization are not affine maps but the generalized derivatives described above. In fact, we show that a K-uqr map can always be K-quasiconformally conjugated near an attracting or repelling fixed point to a generalized derivative.

2. Definitions and general facts

Let $D \subset \overline{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ be a domain and $f : D \to \overline{\mathsf{R}}^n$ a mapping of Sobolev class $W^{1,n}_{loc}(D)$. We consider only orientation preserving mappings, that means that the Jacobian determinant $J_f(x) \geq 0$ for a.e. $x \in D$. Such a mapping is said to be K-quasiregular, where $1 \leq K < \infty$, if

$$\max_{|h|=1} |f'(x)h| \leq K \min_{|h|=1} |f'(x)h| \qquad \text{for a.e.} \quad x \in D.$$

The smallest number K for which the above inequality holds is called the *linear dilatation* of f. A non-constant quasiregular mapping can be redefined on a set of measure zero so as to make it continuous, open and discrete, and we shall always assume that this has been done. If D is a domain in the compactification $\overline{\mathsf{R}}^n$ [equipped with the spherical metric; thus $\overline{\mathsf{R}}^n$ is isometric via stereographic projection to the n-sphere S^n], then we use the chart at infinity $x \mapsto x/|x|^2$ to extend in the obvious manner the notion of quasiregularity to mappings $f : D \to \overline{\mathsf{R}}^n$. Such mappings are also said to be *quasimeromorphic*. A mapping f of a domain D into itself is called *uniformly quasiregular (uqr)* if there is some K with $1 \leq K < \infty$ such that all the iterates f^k are K-quasiregular. We abbreviate this as $f \in UQR(D)$.

If f is a quasiregular mapping defined on the domain D with any range in $\overline{\mathsf{R}}^n$ then, at a given point $x \in D$, we can only consider the iterates $f^k(x)$ as long as the orbit $x \mapsto f(x) \mapsto \ldots \mapsto f^k(x)$ stays in D. We will call such an f a *local uqr mapping* if, for a fixed $1 \leq K < \infty$, the dilatation of f^k does not exceed K for every k and for a.e. $x \in D$ so that $f^k(x)$ is defined.

The branch set B_f is the set of points $x \in D$ for which f is not locally homeomorphic at x. In the whole paper we always assume that a uqr map is non-injective or has a non-empty branch set. Homeomorphic quasiregular maps are called *quasiconformal*. For further details on quasiregular maps we refer to [20].

In the setting of quasiregular mappings we have the following version of Picard's and Montel's Theorem which is due to Rickman [20].

PICARD'S THEOREM. *For every $K \geq 1$ there is an integer $q = q(n, K)$*

so that any K-quasiregular mapping $f : \mathsf{R}^n \to \overline{\mathsf{R}}^n$ that omits q values is constant.

MONTEL'S THEOREM. *For any $K \geq 1$ and any dimension $n \geq 2$ there is a positive integer $q = q(n, K)$ so that the following holds: If $\mathscr{F}$ is a family of K-quasiregular mappings defined on a domain $\Omega \subset \overline{\mathsf{R}}^n$ such that each element $f \in \mathscr{F}$ omits q values $a_1^f, \ldots, a_q^f$ that have spherical distance $\sigma(a_i^f, a_j^f) > \varepsilon$ $(i \neq j)$, where $\varepsilon > 0$ is independent of $f \in \mathscr{F}$, then $\mathscr{F}$ is a normal family.*

Another important normality criterion is the following quasiregular version of Zalcman's Lemma. It is due to Miniowitz [7]. We denote the unit ball in R^n by B and write $\mathsf{B}(x, r) = \{y \in \mathsf{R}^n : |y - x| < r\}$. Further, we write $\mathsf{S}^{n-1} = \partial\mathsf{B}$ and $\mathsf{S}^{n-1}(r) = \partial\mathsf{B}(0, r)$.

ZALCMAN'S LEMMA. *A family $\mathscr{F}$ of K-quasiregular mappings $f : \mathsf{B} \to \overline{\mathsf{R}}^n$ is not normal at $x_0 \in \mathsf{B}$ if, and only if, there are positive numbers $\rho_j \searrow 0$, points $x_j \to x_0$ and mappings $f_j \in \mathscr{F}$ such that*

$$f_j(x_j + \rho_j x) \to \Psi(x)$$

spherically uniformly on compact subsets of R^n, where $\Psi : \mathsf{R}^n \to \overline{\mathsf{R}}^n$ is a non-constant quasimeromorphic map.

The condition $x_j \to x_0$ does not appear in Miniowitz's paper but it can easily be achieved with slight modifications.

3. Basic dynamical properties

The dynamical behavior of a uqr mapping of $\overline{\mathsf{R}}^n$ splits the sphere into two parts: the *Fatou set* $\mathscr{F}_f$, which is the set of points x for which $\{f^k\}$ is a normal family in a neighborhood of x, and the *Julia set* $\mathscr{J}_f = \overline{\mathsf{R}}^n \setminus \mathscr{F}_f$. A connected component Ω of the Fatou set $\mathscr{F}_f$ is called *Fatou component* and it is called a *stable component* or a *stable domain* if it is forward invariant: $f(\Omega) \subset \Omega$. In that case, in fact, $f(\Omega) = \Omega$. The Fatou set is open, the Julia set is closed, and they are both completely invariant under f. Recall that a set E is said to be completely invariant under f if $f(E) \subset E$ and $f^{-1}(E) \subset E$. Moreover, the Julia set of any non-injective uqr map cannot be empty. See [5] for this and for a more detailed presentation.

An immediate consequence of Montel's Theorem is that we can define the *exceptional set* $\mathscr{E}_f$ to be the largest discrete completely invariant set such that $\mathscr{E}_f$ has the following properties: for any open set U with $U \cap \mathscr{J}_f \neq \emptyset$ we have

$$\bigcup_{k \geq 0} f^k(U) \supset \overline{\mathsf{R}}^n \setminus \mathscr{E}_f \tag{1}$$

and for every point x that is not an exceptional point, $x \notin \mathscr{E}_f$, we have

$$\mathscr{J}_f \subset \overline{\bigcup_{k \geq 0} f^{-k}(x)} \quad . \tag{2}$$

Furthermore, $\mathscr{E}_f$ cannot contain more than $q = q(n, K)$ points. There are uqr mappings with zero, one or two exceptional points. It remains an exciting open question whether or not a uqr map may have more than two exceptional points.

We remark that an analysis similar to that for rational functions ([1], pp. 65–66) shows that there exists $n \geq 1$ such that each $x \in \mathscr{E}_f$ is a fixed point of f^n with $f^{-n}(x) = \{x\}$. Hence the local index of f^n at x is ≥ 2 (and is, in fact, equal to $(\deg f)^n$). It now easily follows from standard estimates ([20], Theorem III.4.7, p. 72) and the fact that the forward orbit of x under f lies in the finite set $\mathscr{E}_f$, first that x is a super-attracting fixed point of f^{nk} when k is large enough, and then that $x \in \mathscr{F}_f$. Hence $\mathscr{E}_f \subset \mathscr{F}_f$.

An example of the utility of the exceptional set is that we can describe the image of the limit functions of Zalcman's Lemma applied to the family $\{f^k\}$. Let $x_0 \in \mathscr{J}_f$. Then, by Zalcman's Lemma, there are $x_j \to x_0$ and $\rho_j \searrow 0$ such that

$$\Psi_j(x) = f^{k_j} \circ \alpha_j(x) \equiv f^{k_j}(x_j + \rho_j x) \to \Psi(x), \qquad x \in \mathsf{R}^n, \tag{3}$$

with uniform convergence on compact sets and such that the limit Ψ is a non-constant mapping.

LEMMA 3.1. *The image of such a limit function Ψ is $\Psi(\mathsf{R}^n) = \overline{\mathsf{R}}^n \setminus \mathscr{E}_f$.*

PROOF. Let $y \in \overline{\mathsf{R}}^n \setminus \mathscr{E}_f$ be any non-exceptional point. Then there is $k \in \mathsf{N}$ so that $f^{-k}(y)$ contains more than q points, with $q = q(n, K)$ the constant of Picard's Theorem. Now, since $\Psi = \lim_{j \to \infty} f^{n_j} \circ \alpha_j$ we also have

$$f^{n_j - k} \circ \alpha_j \to \Phi : \mathsf{R}^n \to \overline{\mathsf{R}}^n$$

where the non-constant quasimeromorphic map Ψ satisfies $f^k \circ \Phi = \Psi$. Because of Picard's Theorem, $\Phi(\mathsf{R}^n) \cap f^{-k}(y) \neq \emptyset$. Therefore, there is $z \in \mathsf{R}^n$ with $\Psi(z) = f^k \circ \Phi(z) = y$.

On the other hand, $\Psi(\mathsf{R}^n)$ cannot contain any exceptional point. Namely, since $x_0 \in \mathscr{J}_f$ and $\mathscr{E}_f$ is a finite set with $\mathscr{J}_f \cap \mathscr{E}_f = \emptyset$, it follows that for any compact set $E \subset \mathsf{R}^n$, we have $\alpha_j(E) \cap \mathscr{E}_f = \emptyset$ for all large j. Hence with $\Psi_j = f^{n_j} \circ \alpha_j$, we have $\Psi_j(E) \cap \mathscr{E}_f = \emptyset$ by the complete invariance of $\mathscr{E}_f$ under f. By the counterpart of Hurwitz's theorem for quasiregular maps, we have $\Psi(E) \cap \mathscr{E}_f = \emptyset$. Since this holds for every compact $E \subset \mathsf{R}^n$, we have $\Psi(\mathsf{R}^n) \cap \mathscr{E}_f = \emptyset$. This proves Lemma 3.1.

The counterpart of Hurwitz's theorem for quasiregular maps states that if f_n is a sequence of K-quasiregular maps in a domain D converging to a limit function f locally uniformly in D (so that then f is K-quasiregular by [20, Theorem VI.8.6, p. 159]) and if all f_n omit the value b in D then f omits b in D unless $f \equiv b$ in D. In the proof of Lemma 3.1 above we take D to be a relatively compact subdomain of R^n containing E. It follows, for example, from the counterpart of the argument principle for quasiregular maps proved by Rickman [19] in University of Helsinki lecture notes in 1973; these notes seem to have remained unpublished.

We can now give a precise version of the expanding property (1):

PROPOSITION 3.2. *Let $f \in UQR(\overline{\mathsf{R}}^n)$ and $x_0 \in \mathcal{J}_f$. Then there are arbitrarily small neighborhoods $\Omega \subset \overline{\mathsf{R}}^n \setminus \mathcal{E}_f$ of x_0 such that $\overline{\Omega} \subset f^N(\Omega)$, for some integer N, and such that $\Omega_k = f^{kN}(\Omega)$ is an increasing sequence exhausting $\overline{\mathsf{R}}^n \setminus \mathcal{E}_f$.*

PROOF. Let $\Psi = \lim_{j\to\infty} f^{k_j} \circ \alpha_j$ be a limit of a Zalcman sequence defined as in (3). Since $\Psi(\mathsf{R}^n) = \overline{\mathsf{R}}^n \setminus \mathcal{E}_f$ there is $a_0 \in \mathsf{R}^n$ such that $\Psi(a_0) = x_0$. Consider $\Omega = \mathsf{B}(x_0, \varepsilon) \subset \overline{\mathsf{R}}^n \setminus \mathcal{E}_f$ and let U_j be the component of $(f^{k_j} \circ \alpha_j)^{-1}(\Omega)$ that contains a_0. Then it is clear that

$$f^{k_j} : D_j = \alpha_j(U_j) \to \Omega$$

is a proper map and that $\overline{D_j} \subset \Omega$ provided j is big enough. From this one easily deduces the Proposition using Montel's Theorem.

COROLLARY 3.3. *Let $f \in UQR(\overline{\mathsf{R}}^n)$ and $x_0 \in \mathcal{J}_f$. Then no subsequence of (f^k) is normal in a neighborhood of x_0. In particular, we have $\mathcal{F}_{f^n} = \mathcal{F}_f$ and $\mathcal{J}_{f^n} = \mathcal{J}_f$ for all $n \geq 1$.*

PROOF. Suppose (f^{k_j}) is normal on $\overline{\Omega}$ where Ω is a neighborhood of x_0. Proposition 3.2 shows that we can choose Ω arbitrarily small and such that $\overline{\Omega} \subset f^N(\Omega)$ for some N. If we write $f^{k_j} = f^{r_j} \circ f^{l_j N}$ with $r_j \in \{0, 1, \dots, N-1\}$, then it follows from Proposition 3.2 that $(f^{l_j N})_j$ is not normal on Ω, contradicting the normality of the sequence (f^{k_j}).

Fix $n \geq 1$. Clearly by definition, $\mathcal{F}_f \subset \mathcal{F}_{f^n}$ and so $\mathcal{J}_{f^n} \subset \mathcal{J}_f$. If there exists $x_0 \in \mathcal{J}_f \setminus \mathcal{J}_{f^n}$, then $x_0 \in \mathcal{F}_{f^n}$ so that the sequence $\{f^{mn}\}_{m=1}^{\infty}$ is normal in a neighborhood of x_0. Since $x_0 \in \mathcal{J}_f$, this contradicts what we have proved above already. It follows that $\mathcal{F}_{f^n} = \mathcal{F}_f$ and hence $\mathcal{J}_{f^n} = \mathcal{J}_f$.

Here is another fact which shows that uqr mappings do behave in many respects like holomorphic functions.

LEMMA 3.4. *Suppose* $f \in UQR(\overline{\mathsf{R}}^n)$ *and let* Ω *be a domain of* $\overline{\mathsf{R}}^n$. *If* (f^{k_j}) *is a subsequence of the iterates of* f *such that for every* j *we can define* $F_j : \Omega \to \overline{\mathsf{R}}^n$ *a branch of the inverse of* f^{k_j}, *then*

1) $\{F_j\}$ *is a normal family and,*
2) *if* $\Omega \cap \mathscr{J}_f \neq \emptyset$, *then any convergent subsequence of* $\{F_j\}$ *has a constant limit function.*

PROOF. We know that $B_f \neq \emptyset$. It follows then from [13] that B_f contains at least 2 points. Note that each F_j is one-to-one in Ω. The normality criterion for quasiconformal mappings ([28], pp. 69, 66) applies now since $B_f \subset \overline{\mathsf{R}}^n \setminus F_j(\Omega)$ and it gives the normality of the family $\{F_j\}$.

Suppose now that $\Omega \cap \mathscr{J}_f \neq \emptyset$ and that F_{j_l} converges on Ω to a non-constant map, which is therefore a quasiconformal map $\varphi : \Omega \to \Omega'$. Choose $D = \mathsf{B}(y_0, \varepsilon)$ relatively compact in Ω with $y_0 \in \mathscr{J}_f$. Let $D' = \varphi(D)$ and $x_0 = \varphi(y_0) \in \mathscr{J}_f$. Thus there is a neighborhood $D'' \subset D'$ of x_0 such that for all large l, we have $f^{k_{j_l}}(D'') \subset D$. Then it follows from Montel's Theorem that $(f^{k_{j_l}})_l$ is normal near $x_0 \in \mathscr{J}_f$ and this contradicts Corollary 3.3.

4. Fixpoint classification

We propose here a classification of the fixed points and therefore also of the cycles, i.e., the sets $\{x_1, \ldots, x_p\}$ with $f(x_i) = x_{i+1}$ for $i = 1, \ldots, p-1$ and $f(x_p) = x_1$. In what follows we will consider a local uqr mapping f of a domain U of $\overline{\mathsf{R}}^n$, that fixes some point $x_0 = f(x_0) \in U$.

The different fixed point types in the case of holomorphic mappings are determinated by the derivative of the function at the fixed point. For uqr mappings such a derivative need not exist. But we will see that instead there is a family of uniformly quasiconformal mappings, which reduces to the linear mapping $Df(x_0)$ when this derivative exists, and we show that these maps, which we call generalized derivatives, do determine the different fixed point types.

4.1. Lipschitz estimates near fixed points

The study of the distortion behavior that will follow is essential for the introduction of the generalized derivatives. A priori, a quasiregular mapping has a Hölder behavior near any point. Here we verify that in the case of uqr mappings we have in fact Lipschitz behavior. We write $\mathsf{B}(r) = \mathsf{B}(0, r)$. We further write $rE = \{rz : z \in E\}$ whenever $r > 0$ and $E \subset \mathsf{R}^n$.

LEMMA 4.1. *Suppose that* f *is a* K*-uqr map, that* $f(0) = 0$, *and that* f *is locally injective near the origin, i.e.,* $0 \notin B_f$. *Then there exist* $L \geq 1$ *and a* *neighborhood V of the origin such that*

$$\frac{1}{L}|x| \le |f(x)| \le L|x| \qquad \textit{for every } x \in V. \tag{4}$$

Proof. Suppose that f is injective on the unit ball B and that there exists a sequence $x_k \in \mathsf{B}$ with

$$\lim_{k\to\infty} \frac{|f(x_k)|}{|x_k|} = 0. \tag{5}$$

Necessarily $x_k \to 0$. This together with the usual distortion estimates ([29, Corollary 11.31, p. 147], whose proof can be adapted to cover the present situation also) implies that there is $2r_0 \in]0, 1[$ such that $f(\mathsf{B}(2r_0)) \subset \mathsf{B}(r_0)$. Consequently $f^k(\mathsf{B}(2r_0)) \subset \mathsf{B}(r_0)$ and hence $f^k|\mathsf{B}(2r_0)$ is K-quasiconformal for every k. This leads to uniform distortion estimates: there is $K^* = K^*(n, K) \ge 1$ such that

$$f^k(r\mathsf{S}^{n-1}) \subset A\left(\frac{\rho}{K^*}, K^*\rho\right) = \mathsf{B}(K^*\rho) \setminus \overline{\mathsf{B}\left(\frac{\rho}{K^*}\right)} \tag{6}$$

for every $k \in \mathsf{N}$ and $r \in]0, r_0]$, where we may take $\rho = |f^k(y)|$ for any y with $|y| = r$.

Denote $r_k = |x_k|$ and $\varepsilon_k = |f(x_k)|/|x_k|$. We may assume that $r_k \le r_0$ and $\varepsilon_k < 1/K^*$ for every $k \in \mathsf{N}$. It then follows from (6) with $y = x_k$ and hence $\rho = \varepsilon_k r_k$ that

$$A_k = \mathsf{B}(r_k) \setminus \overline{f(\mathsf{B}(r_k))} \supset \mathsf{B}(r_k) \setminus \overline{\mathsf{B}(K^*\varepsilon_k r_k)} = A(K^*\varepsilon_k r_k, r_k)$$

and this shows that

$$\operatorname{mod} A_k \le \operatorname{mod} A(K^*\varepsilon_k r_k, r_k) = \omega_{n-1}\left(\log\frac{1}{K^*\varepsilon_k}\right)^{1-n}.$$

Here ω_{n-1} is the measure of the unit sphere S^{n-1} of R^n and in [20], [28] one can find all the details needed concerning the modulus.

Consider $\Omega = \mathsf{B}(r_0) \setminus \overline{f^2(\mathsf{B}(r_0))}$ which is, so to say, a "double fundamental domain" of the action of f. Then there is an annulus $\mathscr{A}_k \subset \Omega$ such that $f^\nu(\mathscr{A}_k) = A_k$ for some ν. The quasi-invariance of the modulus for quasiconformal mappings implies

$$0 < \operatorname{mod}\Omega \le \operatorname{mod}\mathscr{A}_k \le K \operatorname{mod}(f^\nu(\mathscr{A}_k)) \le K\omega_{n-1}\left(\log\frac{1}{K^*\varepsilon_k}\right)^{1-n}$$

which is impossible for k sufficiently large.

We showed that the left hand side of the inequality (4) is true. The other part follows in the same way, one just has to consider $(f|\mathsf{B})^{-1}$ instead of f.

In the case when the fixed point $x_0 = 0$ is also a branch point $x_0 \in B_f$ with local index $i = i(0, f) > 1$ the local distortion estimates of quasiregular mappings show that for every $k \in \mathsf{N}$ there are a neighborhood V of the origin and a constant $C > 0$ so that

$$(7) \qquad |f^k(x)| \leq C|x|^{\mu} \qquad \text{for every } x \in V \text{ with } \mu = \left(\frac{i^k}{K}\right)^{\frac{1}{n-1}}$$

(cf. [20, Theorem III.4.7, p. 72]).

4.2. Generalized derivatives

Let again the origin $x_0 = 0$ be a fixed point of a uqr map f at which f is locally injective, and let B_0 be a neighborhood of 0 in which f is injective so that the results of the previous subsection 4.1 are valid. So in particular we have $f(B_0) \subset LB_0$, where L is the Lipschitz constant from (4).

We consider now f_λ defined by $f_\lambda(z) = \lambda f(z/\lambda)$, where $\lambda \geq 1$, as a mapping defined on λB_0. Then, for a sequence of λ tending to infinity the associated sequence of mappings f_λ is normal on every ball of fixed radius centered at the origin. A limit function $\varphi = \lim_{j\to\infty} f_{\lambda_j}$ can be considered as a generalized derivative of f at x_0 and one knows that it is a constant or a quasiregular mapping of R^n.

DEFINITION 4.2. The set of limit mappings

$$\mathscr{D}f(x_0) = \left\{\varphi = \lim_{j\to\infty} f_{\lambda_j} \text{ where } \lambda_j \to \infty\right\}$$

is called the *infinitesimal space* of the uqr map f at the fixed point x_0.

REMARK 4.3. In case that f has a derivative at x_0, then $\mathscr{D}f(x_0)$ contains only the linear mapping $x \mapsto Df(x_0)x$.

Since f is locally injective near x_0, the Lipschitz estimates (4) imply that a limit function $\varphi \in \mathscr{D}f(x_0)$ is a quasiconformal homeomorphism of R^n. Moreover, φ is a uniformly quasiconformal mapping since $f^k_{\lambda_j}(z) = \lambda_j f^k(z/\lambda_j)$.

For uniformly quasiconformal mappings other than the identity map, we use the classification, as for Möbius transformations, of such maps into parabolic, loxodromic and elliptic mappings (see [3], [24]). In our case φ cannot be parabolic since it fixes 0 (and ∞). So either φ is *loxodromic* which means that φ^k or φ^{-k} converges uniformly on compact sets of R^n to the origin, or φ is *elliptic* and in this case the group $< \varphi >$ generated by φ is precompact.

LEMMA 4.4. *If one element $\varphi \in \mathscr{D} f(x_0)$ is loxodromic, then all the elements of the generalized derivative are loxodromic. Consequently, the same is true in the elliptic case.*

PROOF. Suppose that $\mathscr{D} f(x_0)$ contains a loxodromic element φ. We may assume that 0 is an attracting fixed point of φ. Then there is k with

$$\varphi^k(4K^2\mathsf{B}) \subset \frac{1}{4K^2}\mathsf{B}.$$

This means that the action of φ^k has a fundamental domain, which is the topological annulus A with boundary components $\partial\mathsf{B}(0, 4K^2)$ and $\varphi^k(\partial\mathsf{B}(0, 4K^2))$, and that this domain contains the annulus $A\big(\frac{1}{4K^2}, 4K^2\big) \subset A$.

Since $f_{\lambda_j}(z) = \lambda_j f(z/\lambda_j) \to \varphi(z)$ uniformly on compact sets, we see that f^k has, in any neighborhood V of the origin, a fundamental domain $\mathscr{A}_0$ that contains some annulus $A\big(\frac{1}{3K^2}r, 3K^2 r\big)$. Since φ is loxodromic, and so in particular not constant, the fixed point x_0 is not a branch point of f. If V has been chosen small enough so that the distortion estimate analogous to (6) is valid in V for f^k, then all the $\mathscr{A}_\nu = f^{\nu k}(\mathscr{A}_0)$ contain an annulus of the form $A\big(\frac{1}{K^*}r_\nu, K^* r_\nu\big)$ for a suitable number K^* independent of ν. Here we may take, for example, $r_\nu = |f^{\nu k}(x_1)|$ for any preassigned $x_1 \in \mathscr{A}_0$, say with $|x_1| = r$.

This uniform control of all the fundamental domains implies that every limit mapping $\psi = \lim f_{\mu_i} \in \mathscr{D} f(x_0)$ is loxodromic: use again uniform convergence to see that ψ^k has a fundamental domain containing an annulus of the form $A\big(\frac{1}{K^*}\rho, K^*\rho\big)$.

4.3. Classification of the fixed points and Fatou components

We showed that the elements of the infinitesimal space $\mathscr{D} f(x_0)$ share the common property of being either constant, elliptic or loxodromic. This allows us to give the following fixed point classification.

DEFINITION 4.5. Let x_0 be a fixed point of the uqr map f at which f is locally injective, and let $\mathscr{D} f(x_0)$ be the infinitesimal space of f at this point. Then we call x_0

1) *attracting* or *repelling* if one, and therefore every, element $\varphi \in \mathscr{D} f(x_0)$ is loxodromic and if the origin is a attracting or repelling fixed point of φ respectively (i.e., (φ^k) converges uniformly on compact subsets of R^n to the origin in the attracting case and to infinity on compact subsets of $\mathsf{R}^n \setminus \{0\}$ in the repelling case);
2) *neutral* if the elements of $\mathscr{D} f(x_0)$ are elliptic.

If x_0 is a fixed point of the uqr map f at which f is not locally injective, we call x_0 a *super-attracting* fixed point of f.

Note that in the attracting and super-attracting case the iterates of f converge uniformly to x_0 near this point. In the super-attracting case, choose k so large that $\left(\frac{i^k}{K}\right)^{\frac{1}{n-1}} > 1$ in the terminology of (7), and then use (7) to conclude that the iterates of f^k tend to x_0 uniformly in a sufficiently small neighborhood of x_0. This then implies that the whole sequence $f^m \to x_0$ uniformly on some such neighborhood as $m \to \infty$.

In particular, (super-)attracting fixed points are in the Fatou set. Similarly, when x_0 is repelling then the iterates of f are not equicontinuous near x_0 and $x_0 \in \mathscr{J}_f$.

Attracting and super-attracting fixed points can be characterized as follows (the first two assertions of this Proposition are also equivalent in the case of repelling fixed points):

PROPOSITION 4.6. *For a map $f \in UQR(\overline{\mathsf{R}}^n)$ and a fixed point x_0 of f, the following assertions are equivalent:*

1) *x_0 is an attracting or super-attracting fixed point of f.*
2) *x_0 is an attracting or super-attracting fixed point of some iterate f^k.*
3) *There is a stable component $\Omega \subset \mathscr{F}_f$ and a subsequence f^{k_j} converging locally uniformly to a point $x_0 \in \Omega$.*

Note that in 1) and 2), it is clear that x_0 is a fixed point of f and of any f^k. It is the type of the fixed point that is important.

PROOF. The equivalence between 1) and 2) follows directly from the definition since a uniformly quasiconformal map is loxodromic when some iterate of it is loxodromic. It is also clear that 1) implies 3).

Suppose then that 3) is true, so that there exists a constant limit function $x_0 = \lim f^{k_j}$ such that $x_0 \in \Omega$. The convergence is locally uniform in Ω. Thus there is a ball B centered at x_0 whose closure is contained in Ω such that for a certain integer p, which is among the k_j, we have $\overline{f^p(B)} \subset B$. Now the Brouwer fixed point theorem shows that f^p has a fixed point c in B, and, as $m \to \infty$, we have $f^{mp} \to c$, locally uniformly in B, but then also locally uniformly in Ω, in view of the definition of Ω as a component of the Fatou set of f. Each of the points $c_i = f^i(c) \in \Omega$ for $0 \le i < p$ is a fixed point of f^p. Since $c_i = f^{mp}(c_i) \to c$, we have (taking $i = 1$) $f(c) = c$. Since $f^{mp} \to c$, we have $f^{i+mp} \to f^i(c) = c$ as $m \to \infty$, locally uniformly in Ω, for each fixed i with $0 \le i < p$. Putting together the finitely many sequences f^{i+mp} for $0 \le i < p$, we find that the full sequence $f^m \to c$ as $m \to \infty$, locally uniformly in Ω. Hence it must be the case that $x_0 = c$, and 1) holds.

Attractors and repellors already have been studied by Hinkkanen and Martin in [4]. They used there the following

TOPOLOGICAL DEFINITION. A fixed point x_0 is *attracting* (*repelling*, respectively) if there is a neighborhood U of x_0 such that f is injective on U and such that $\overline{f(U)} \subset U$ ($f(U) \supset \overline{U}$, respectively).

This condition is equivalent to the present definition based on the generalized derivatives (which easily follows from [8, p. 420]). Let us mention, as an example of the utility of our new definition, that the implication "2) implies 1)" in the proof of Proposition 4.6 is a rather non-trivial fact if one uses only the topological definition.

In the same way, a fixed point is super-attracting if, and only if, it is a branch point, i.e., the definition used in [15]. Concerning the neutral fixed points, they can be classified into three different types:

DEFINITION 4.7. A neutral fixed point x_0 of a uqr mapping f is

1) a *Siegel point* if it is in the Fatou set $\mathscr{F}_f$,
2) a *parabolic fixed point* provided that there is a stable component $\Omega \subset \mathscr{F}_f$ with $x_0 \in \partial\Omega$ and a sequence $\{f^{n_j}\}$ such that $f^{n_j} \to x_0$ locally uniformly on Ω, and
3) a *Cremer point* if x_0 is in the Julia set and is not parabolic.

By Definition 4.7, a parabolic point and a Siegel point generates a stable component as do (super-)attracting fixed points. Recall that a component Ω of $\mathscr{F}_f$ is said to be stable if $f(\Omega) \subset \Omega$ (and, in fact, $f(\Omega) = \Omega$, as one can see in the same way as for rational functions).

DEFINITION 4.8. A stable component Ω of $\mathscr{F}_f$ is called an (immediate)

1) *(super-)attracting basin* if it contains a (super-)attracting fixed point,
2) *parabolic basin* if there is a fixed point $x_0 \in \partial\Omega$ and a sequence $\{f^{k_j}\}$ that converges locally uniformly on Ω to x_0, and
3) *rotation domain* provided $\overline{\langle f|\Omega\rangle}$ is a compact group. If such a domain contains a fixed point then it is also called a Siegel domain.

Note that in case 3), the definition makes sense only if $f|\Omega$ is a homeomorphism of Ω onto itself. Also, we have not proved that a given Ω could be a parabolic basin on account of at most one $x_0 \in \partial\Omega$.

As for rational functions, these are the only possible stable components:

PROPOSITION 4.9. *A stable component Ω of $\mathscr{F}_f$ is a (super-)attracting or parabolic basin or it is a rotation domain.*

PROOF. Let Ω be a stable component of $\mathscr{F}_f$. Suppose first that there are $x_0 \in \overline{\Omega}$ and a sequence of iterates f^{k_j} such that

$$f^{k_j} \to x_0 \in \overline{\Omega}$$

locally uniformly in Ω. Again, since $f^{k_j} \circ f = f \circ f^{k_j}$ converges to x_0 and to $f(x_0)$, the limit point x_0 is a fixed point. If $x_0 \in \Omega$ then x_0 must be an attracting or super-attracting fixed point and Ω an attracting or super-attracting basin (Proposition 4.6). Otherwise x_0 is parabolic and Ω a parabolic basin. So from now on we may assume that the limit of every convergent sequence f^{k_j} is non-constant.

Suppose then that there exists a non-constant limit function $\phi = \lim f^{k_j}$, for some sequence of integers $k_j \to \infty$ as $j \to \infty$, the convergence being locally uniform on Ω. Clearly $\phi(\Omega) \subset \overline{\Omega}$, and since ϕ is a non-constant quasiregular map and hence an open map, it is easily seen that $\phi(\Omega) \subset \Omega$. Hence for any compact subset E of Ω, the set $\phi(E)$ must be a compact subset of Ω.

Write $m_j = k_{j+1} - k_j \geq 1$. By replacing k_j by a subsequence, without changing notation, we may assume that $m_j \to \infty$ as $j \to \infty$. After that, we find a subsequence m_{j_p} such that $f^{m_{j_p}} \to \psi$ locally uniformly on Ω as $p \to \infty$. Since for any compact subset E of Ω, the set $\phi(E)$ is a compact subset of Ω, and since $f^{n_{j_p+1}} = f^{m_{j_p}} \circ f^{n_{j_p}}$, we find on the basis of locally uniform convergence that $\phi = \psi \circ \phi$, first on each compact subset of Ω, and hence on all of Ω. Therefore ψ is non-constant, so that both ϕ and ψ are non-constant K-quasiregular maps of Ω into itself.

Let B_ϕ denote the branch set of ϕ. Suppose that $x_0 \in \Omega \setminus B_\phi$. Then there is a branch h of ϕ^{-1} defined in a neighborhood of $\phi(x_0)$ taking the point $\phi(x_0)$ onto the point x_0. We have $\phi \circ h = \psi \circ \phi \circ h$ in a neighborhood of $\phi(x_0)$, which gives $\psi(z) = z$ for all z in in a neighborhood of $\phi(x_0)$. It follows that $\psi = Id$, the identity map, in $\phi(\Omega \setminus B_\phi)$. Since ($\Omega \setminus B_\phi$ is dense in Ω and) $\phi(\Omega \setminus B_\phi)$ is dense in $\phi(\Omega)$ (if dim refers to topological dimension, then $\dim B_\phi = \dim \phi(B_\phi) = \dim \phi^{-1}(\phi(B_\phi)) \leq n - 2$ by [27]), it follows by continuity that $\psi = Id$ in the subdomain $\phi(\Omega)$ of Ω and hence also in $\Omega \cap \overline{\phi(\Omega)}$. (For a rational function f in dimension 2, it now follows by analytic continuation that $\psi = Id$ in all of Ω since ψ is analytic in Ω, but this argument is not available in the general quasiregular case.)

Next, it is seen that f is one-to-one in $\Omega \cap \overline{\phi(\Omega)}$. For if $x, y \in \phi(\Omega)$ and $x \neq y$ while $f(x) = f(y)$, then $f^{m_{j_p}}(x) = f^{m_{j_p}}(y)$ for all p, so that $\psi(x) = \psi(y)$, which is a contradiction since $\psi(x) = x$ and $\psi(y) = y$. Thus indeed f is one-to-one in $\Omega \cap \overline{\phi(\Omega)}$ and in particular in $\phi(\Omega)$.

We wish to prove next that $\psi = Id$ in all of Ω. For this purpose, choose a subsequence of m_{j_p}, denoted briefly just by κ_p, and a subsequence of k_j, denoted by λ_p, such that $\kappa_p - \lambda_p \to \infty$ as $p \to \infty$, and such that $f^{\kappa_p - \lambda_p} \to \chi$ as $p \to \infty$, locally uniformly on Ω. Since $f^{\kappa_p} = f^{\kappa_p - \lambda_p} \circ f^{\lambda_p}$, and since $f^{\lambda_p} \to \phi$, so that for any compact subset E of Ω, the sets $f^{\lambda_p}(E)$ remain in a compact subset of Ω (this is why we have to consider first $f^{\kappa_p - \lambda_p} \circ f^{\lambda_p}$ rather than $f^{\lambda_p} \circ f^{\kappa_p - \lambda_p}$), we deduce that $\psi = \chi \circ \phi$. Thus χ is a non-constant quasiregular map (if χ were constant then this would force ψ to be constant, which is not the case) with $\chi(\Omega) \subset \Omega$. Next, from $f^{\kappa_p} = f^{\lambda_p} \circ f^{\kappa_p - \lambda_p}$ we now get, in the same way, $\psi = \phi \circ \chi$. Hence

$$\Omega \cap \overline{\phi(\Omega)} = \psi(\Omega \cap \overline{\phi(\Omega)}) \subset \psi(\Omega) = \phi(\chi(\Omega)) \subset \phi(\Omega).$$

If $\phi(\Omega) \neq \Omega$, this gives a contradiction, as there would then exist a point in $\Omega \cap \partial\phi(\Omega)$. Such a point is in $\Omega \cap \overline{\phi(\Omega)}$ but not in $\phi(\Omega)$, which is the contradiction.

We deduce that $\phi(\Omega) = \Omega$, and it follows that $\psi = Id$ on Ω. Since now $\phi \circ \chi = \chi \circ \phi = Id$, it follows that ϕ is a homeomorphism of Ω onto itself. Also f is one-to-one in Ω, so that f is also a homeomorphism of Ω onto itself.

Since ϕ was an arbitrary limit function (all of them assumed to be non-constant), it follows that all limit functions are homeomorphisms of Ω onto itself. All of them are also K-quasiregular maps on Ω. Above, we also saw that ϕ^{-1} is such a limit function. Further, if ϕ_1 and ϕ_2 are such limit functions, with, say $f^{k_j} \to \phi_1$ and $f^{l_j} \to \phi_2$, then

$$\phi_1 \circ \phi_2 = \lim_{j\to\infty} f^{k_j} \circ f^{l_j} = \lim_{j\to\infty} f^{l_j} \circ f^{k_j} = \phi_2 \circ \phi_1,$$

so that the limit functions form a group, which further is an abelian group. If ϕ_m is a sequence of such limit functions tending to a non-constant function ϕ, then there is clearly a sequence of iterates of f tending to ϕ, so that ϕ is also a homeomorphism in this abelian group. Thus the group of limit functions is closed, in this sense. This shows that Ω is a rotation domain.

We end this section by discussing the case when the derivative of f exists at a fixed point x_0. In this case it is easy to check what kind of a fixed point we have. It suffices to consider the matrix $A = Df(x_0)$. If we set

$$\|A\| = \max\{|Ah| : |h| = 1\}$$

then the fact that f is uniformly quasiregular implies that

$$\|A^k\|^n \leq K \det A^k \qquad \text{for every} \quad k \in \mathsf{N}.$$

Such a matrix is known to be an affine conjugate of an element of the similarity group $\mathsf{R}O(n)$ (see [9] for this and more details). Therefore, x_0 is an attracting, neutral or repelling fixed point if, and only if, all the eigenvalues of $Df(x_0)$ (note that they all coincide with each other) are strictly less than, equal to, or strictly greater than 1, respectively.

However, it is not clear whether the different types of neutral fixed points can be distinguished if one only looks at the derivative. One might hope that the fixed point is parabolic if some power of $Df(x_0)$ is the identity (and in general, when the generalized derivative $\mathscr{D}f(x_0)$ contains an element of finite order). But conversely there are parabolic fixed points for which $Df(x_0)$ generates a non-discrete subgroup of the group of orthogonal matrices. In fact, a parabolic Möbius transformation which has a non-periodic rotation part is such an example.

5. Examples of uniformly quasiregular mappings

Finding explicit examples of uqr maps is a particularly difficult thing. Also, as we showed in [12], there is a rigidity phenomenon which says that, on the Julia set, uqr maps are always of a very special kind (i.e., of Lattès-type) provided that the Julia set is large enough and that the maps have some kind of expansive property on the Julia set. So, there are not too many such maps. On the other hand, there is some freedom to modify uqr maps on their Fatou set. We illustrate this in Proposition 5.1.

Our main motivation here is to analyse whether or not there are uqr mappings that have fixed points of the different types we defined. Examples with super-attracting, attracting and repelling fixed points are known. We give new examples having parabolic fixed points and they will be used later to construct quasiconformally wild parabolic examples, meaning that the quasiconformal analogue of Leau and Fatou's petal linearization theorem is not true. We do not know of any (higher-dimensional) uqr mapping with a Cremer or Siegel point.

The first family of examples has been found by Iwaniec-Martin:

THEOREM ([5]). *There are uqr maps of* $\overline{\mathsf{R}}^n$ *with non-empty branch set, with attracting and repelling fixed points and with Julia set a Cantor set (on which the map does act like a Schottky group).*

5.1. Lattès-type and related examples

The Lattès-type mappings introduced in [15], [16] are uqr analogues of the rational functions that are called critically finite with parabolic orbifold. They are obtained by semi-conjugating an expanding similarity by an automorphic map. We call a quasimeromorphic map $h : \mathsf{R}^n \to \overline{\mathsf{R}}^n$ *automorphic*, or more precisely automorphic with respect to a group Γ of isometries of R^n, if $h \circ \gamma = h$ for all $\gamma \in \Gamma$ and if in addition Γ acts transitively on fibers of h: for every $x_1, x_2 \in \mathsf{R}^n$ with $h(x_1) = h(x_2)$ there is $\gamma \in \Gamma$ such that $\gamma(x_1) = x_2$. Now, a uqr map f is of *Lattès-type* if

$$f \circ h(x) = h \circ A(x) \qquad \text{for every} \quad x \in \mathsf{R}^n \tag{8}$$

where $h : \mathsf{R}^n \to \overline{\mathsf{R}}^n \setminus \mathscr{E}_f$ is a automorphic map and $A(x) = \lambda U(x)$ with $\lambda > 1$ and $U \in O(n)$ an orthogonal matrix.

For example, when h is automorphic with respect to a cocompact group Γ and f a corresponding solution of (8), then $\mathscr{J}_f = \overline{\mathsf{R}}^n$ and the repelling cycles of f are dense in $\overline{\mathsf{R}}^n$. We call such a map *chaotic Lattès-type* map. Other Lattès-type mappings are power mappings, i.e., uqr analogues of $z \mapsto z^d$. They are obtained by taking h to be an analogue of the exponential function and Γ to be a group so that R^n / Γ is a cylinder (see [15]). These are examples of uqr mappings with a super-attracting fixed point. Starting from such a power mapping we obtain new examples of a different nature.

PROPOSITION 5.1. *There is a uqr map f of $\overline{\mathsf{R}}^n$ with a super-attracting and an attracting fixed point, such that the union of the associated immediate basins is equal to the Fatou set $\mathscr{F}_f$.*

PROOF. Start with $g \in UQR(\overline{\mathsf{R}}^n)$ a power map. So in particular 0 and ∞ are super-attracting fixed points and the Julia set is the sphere S^{n-1}. We modify g in the attracting basin B of 0. In order to do this, take a ball $B = \mathsf{B}(0, r) \subset \mathsf{B}$ that is moved inside itself. We may even choose this ball so that $B' = g(B) \subset \mathsf{B}(0, \frac{r}{2})$. Take $a \in B'$ such that g is locally injective at a and denote $b = g(a)$. Modify then, using Sullivan's quasiconformal version of the Annulus Theorem [25], g near a such that $g : \mathsf{B}(a, 2\varepsilon) \to \mathsf{B}(b, \varepsilon)$ is conformal and maps a onto b. Here $\varepsilon > 0$ is at least so small such that $\mathsf{B}(a, 4\varepsilon) \subset B'$. Clearly, this new map, which we still denote by g, is quasiregular but not necessarily uqr. The map f we look for is $f = \varphi \circ g$ where φ is a quasiconformal map which is the identity outside of $B(0, \frac{r}{2})$, such that $\varphi(B') \subset B(a, \varepsilon)$ and whose restriction to $\mathsf{B}(b, \varepsilon)$ is a translation that maps $\mathsf{B}(b, \varepsilon)$ onto $\mathsf{B}(a, \varepsilon)$. It is now easy to check that f has all the properties we looked for.

5.2. Construction of parabolic uqr-maps

Here we give an example of a global non-injective uqr map that has a parabolic fixed point.

THEOREM 5.2. *There exists a uqr-map $f : \overline{\mathsf{R}}^n \to \overline{\mathsf{R}}^n$ having a parabolic fixed point, such that the Fatou set of f consists of one completely invariant parabolic basin of attraction and the Julia set of f is a Cantor set.*

To be more precise, in what follows we will construct a uqr map that behaves around one of its fixed points like the translation $x \mapsto x + 1$ near infinity. Our construction is based on the conformal trap technique introduced by Iwaniec and Martin in [5] and developed further in [10], [11].

We start the construction with an arbitrary non-injective quasiregular map $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$. Let $x_0 \in \overline{\mathbb{R}}^n$ be a point such that the following two properties hold:

(1) The preimages $\{x_1, \ldots, x_d\} = f^{-1}(x_0)$ are all disjoint and are not branch points.

(2) There is a small ball $U_0 = B(x_0, r)$ such that $f^{-1}(U_0)$ has pairwise disjoint components $U_1, \ldots, U_d$ and such that $f : U_j \to U_0$ is injective for $1 \leq j \leq d$.

We may suppose that $x_1 = x_0$, that is, f fixes this point x_1. Otherwise, when x_0 is not one of its preimages, then it suffices to consider the map $x \mapsto f(x) + (x_1 - x_0)$ instead of f. In the first step we modify our map in order to make f behave like the rotation

$$R(y) = R(y_1, y_2, \ldots, y_n) = (-y_1, -y_2, y_3, \ldots, y_n)$$

near x_1. For convenience, we assume that $x_1 = 0$, to avoid the use of a further translation when applying R. Choose a ball $B(x_1, \tau) \subset U_1$ and a second one $\overline{B}(x_1, \varepsilon) \subset B(x_1, \tau) \cap f(B(x_1, \tau))$. Then, using Sullivan's quasiconformal version of the Annulus Theorem [25], we can modify f so that $f = Id$ in $B(x_1, \varepsilon)$ and so that f remains unchanged outside $B(x_1, \tau)$. In what follows we will work with the quasiregular mapping $f_1 = R \circ f$. It has the properties:

(1) f_1 is the rotation R in the ball $V_1 = B(x_1, \varepsilon)$, and

(2) $f_1^{-1}(V_1)$ consists of pairwise disjoint components $V_1, V_2, \ldots, V_d$ such that $f_1 : V_j \to V_1$ is injective for $1 \leq j \leq d$. Note that $V_j \subset U_j$ for all j with $1 \leq j \leq d$.

Take now a ball B such that $x_1 \in \partial B$ and $\overline{2B} \subset V_1$, such that $y_1 > 0$ throughout B. Here, if $B = B(\zeta, \rho)$, we write $2B = B(\zeta, 2\rho)$. Denote by Ω_j the preimage of $2B$ in V_j ($j = 2, \ldots, d$) and choose further balls $\overline{B}_j \subset \Omega_j$ with $x_j \in \partial B_j$. Appealing a second time to Sullivan's quasiconformal version of the Annulus Theorem [25], we modify f_1 to a new map g_1 so that

(1) $g_1 = f_1$ in the complement of $\Omega_2 \cup \ldots \cup \Omega_d$ and

(2) $g_1 : B_j \to B$ conformal with $g_1(x_j) = x_1$ for $j = 2, \ldots, d$.

In the final step we create the conformal trap: Let Φ be a conformal mapping which exchanges the ball B with its complement. Then the mapping we looked for is $g = \Phi \circ g_1$. The dynamical properties of this mapping are those of the mapping of [5] except that the basin of attraction is parabolic. We only verify the properties of g important for our purposes, namely:

PROPOSITION 5.3. *The mapping* $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ *is uqr and, if we conjugate* g *by a Möbius transformation sending the fixed point* x_1 *to infinity, then* g *becomes a translation near infinity.*

PROOF. The uniform control of the dilatation is a consequence of the conformal trap. In fact, it suffices to follow the orbit of an arbitrary given point $x \in \overline{\mathbb{R}}^n$. Observe first that $g(B) \subset B$ and that g restricted to this ball B is the composition of the rotation R with the conformal inversion Φ. So, the map $x \mapsto g^n(x)$ is conformal in B for every $n \in \mathbb{N}$.

Let $B_1 = R(B)$ and suppose that x is in the complement of $B_1 \cup \cdots \cup B_d = g_1^{-1}(B)$. Then $g_1(x) \notin B$ and therefore $g(x) \in B$. This means that for those points x, the first step $x \mapsto g(x)$ is not conformal, but it ends in the ball B and this yields that all the following applications of g are conformal.

Consider finally a point $x \in B_1 \cup \cdots \cup B_d$. Then $x \mapsto g(x)$ is also conformal. Now two cases may occur: either the image $g(x) \in B_1 \cup \cdots \cup B_d$ and then the next application of g will also be conformal. The other possibility is $g(x) \notin B_1 \cup \cdots \cup B_d$. Then we are in the above situation and we see that after (at most) one more application of g, where the orbit gains some dilatation, we land in B and then we never leave this ball and have a conformal orbit. This shows that g and all its iterates g^n have the same dilatation.

The verification that x_1 is parabolic can be done in the following way: The mapping g is conformal around x_1 and so, by Liouville's Theorem, is the restriction of a Möbius transformation. Since B is an attracting petal and B_1 is a repelling one, this Möbius transformation must be parabolic.

6. Linearizations and dynamics of the branch points

6.1. Non-linearizable parabolic dynamics

Recall that Leau and Fatou showed that a holomorphic mapping f can be conformally linearized in each parabolic petal U, meaning that there is φ defined and univalent on $U \cap f^{-1}(U)$ with

$$(\varphi \circ f)(z) = \varphi(z) + 1 \qquad \text{for all} \quad z \in U \cap f^{-1}(U)$$

(see, e.g., [18]). Our aim here is to establish that the quasiregular analogue is false in dimension $n = 3$.

THEOREM 6.1. *In dimension* $n = 3$*, there are uqr maps with parabolic fixed point that cannot be quasiconformally linearized in their attracting parabolic petal* U*, i.e., there is no quasiconformal mapping* φ *so that* $(\varphi \circ f)(x) = \varphi(x) + 1$ *holds in* $U \cap f^{-1}(U)$.

We prove this result by modifying the examples obtained in the previous section so that they will behave like the quasiconformally wild parabolic mappings of [14]. We use the following properties of these latter mappings.

PROPOSITION 6.2. *There is a uniformly quasiconformal map h of $\mathsf{H}^3 = \{x_3 > 0\}$ onto itself, extending to a self-homeomorphism of the closure of H^3, whose restriction to $\{x_3 = 0\}$ is the translation $x = (x_1, x_2, x_3) \mapsto T(x) = (x_1 + 1, x_2, x_3)$, which is topologically conjugate to this translation but which does not admit a quasiconformal linearization in $V = \{x_1 > 0\} \cap \mathsf{H}^3$.*

PROOF. The maps of [14] are uniformly quasiconformal mappings h of R^3 that are obtained by homeomorphic conjugation of the translation T. It is shown in [14] that these maps do not admit a quasiconformal conjugation to this translation and an inspection of the proof given there shows that this result is local: h does not admit a quasiconformal linearization in V.

The explicit construction of these maps (which goes back to Tukia) shows that h preserves the half-space H^3. The uniformly quasiconformal map of the plane $h|\partial\mathsf{H}^3$ admits a quasiconformal conjugation to a translation [21], [22]. Since moreover every plane quasiconformal map can be extended to a quasiconformal map of H^3 (see for example [26]) we can conjugate the whole map $h|\mathsf{H}^3$ to a new map that has all the properties mentioned in Proposition 6.2.

PROOF OF THEOREM 6.1. Let f be a uqr mapping of $\overline{\mathsf{R}}^3$ with parabolic fixed point ∞ such that f is the translation T outside some ball B (cf. Proposition 5.3). Since f is a translation near ∞ it preserves a half-space $\mathscr{H} \subset \mathsf{R}^3 \setminus B$ and is the translation T there. We may suppose that $\mathscr{H} = \mathsf{H}^3$. This allows to consider a new map:

$$g(x) := \begin{cases} h(x) & \text{for } x \in \mathsf{H}^3, \\ f(x) & \text{elsewhere,} \end{cases}$$

where h is the map of Proposition 6.2. It is clear now that g is uqr, that ∞ is a parabolic fixed point of this mapping and that it does not admit a quasiconformal linearization in the attracting petal.

6.2. Linearization at attractors and repellors

In contrast to the parabolic case, a uqr map can be quasiconformally linearized near attractors and repellors. More precisely, Hinkkanen and Martin [4] showed that if x_0 is say a repelling fixed point of $f \in UQR(\overline{\mathsf{R}}^n)$, then there is a global quasiconformal map Ψ such that $\Psi(0) = x_0$ and

$$f \circ \Psi(y) = \Psi(2y) \qquad \text{for } y \text{ close to } 0. \tag{9}$$

Their proof involves topological methods which do not allow one to get a quantitative result. In fact, it turns out that the dilatation of Ψ does depend on the map f and not only on its dilatation [17]. We show now that the natural candidates of the linear model are the elements of the infinitesimal space since we get a conjugacy between the map and its generalized derivative that has at most the dilatation of the map f. If f is holomorphic, then the next result is precisely the Koenigs Theorem.

THEOREM 6.3. *Let f be a K-uqr map and let $x_0 = 0$ be an attracting or repelling fixed point of f.*

(i) *If x_0 is an attracting fixed point of f, let Ω be the component of $\mathscr{F}_f$ containing x_0. Then there is a K-quasiregular map Ψ of Ω and a generalized derivative $\varphi \in \mathscr{D}f(x_0)$ such that*

$$\Psi \circ f = \varphi \circ \Psi$$

holds in Ω.

(ii) *If x_0 is a repelling fixed point of f, there is a K-quasiregular map Ψ of R^n and a generalized derivative $\varphi \in \mathscr{D}f(x_0)$ such that*

$$f \circ \Psi = \Psi \circ \varphi$$

holds in a neighborhood of the origin.

PROOF. Suppose that $x_0 = 0$ is an attracting fixed point of f, and let Ω be the component of $\mathscr{F}_f$ containing x_0. Then we may suppose that f is injective on $\mathsf{B} \subset \Omega$ and furthermore that the uniform distortion control (6) is valid for every $r \in]0, 1[$. It follows that for every $k \geq 1$ there is $\rho_k > 0$, with $\rho_k \to 0$ as $k \to \infty$, such that

$$\Psi_k(\mathsf{S}^{n-1}) = \frac{1}{\rho_k} f^k(\mathsf{S}^{n-1}) \subset A\left(\frac{1}{K^*}, K^*\right).$$

Therefore, the sequence $(\Psi_k)_k$ is normal in B and the limit $\Psi : \mathsf{B} \to \Omega$ of a convergent subsequence Ψ_{k_j} is a quasiconformal map in B. Observe that

$$\frac{1}{\rho_{k_j}} f^{k_j+1}(z) = \frac{1}{\rho_{k_j}} f(\rho_{k_j} \Psi_j(z)).$$

Extracting a subsequence if necessary, we find a generalized derivative $\varphi \in \mathscr{D}f(x_0)$ such that $\lim_{j\to\infty} \frac{1}{\rho_{k_j}} f(\rho_{k_j} z) = \varphi(z)$ uniformly for z in a neighborhood of the origin. Now, passing to the limit in the above identity, one obtains $\Psi \circ f = \varphi \circ \Psi$. Hence $\Psi \circ f^n = \varphi^n \circ \Psi$. Using this identity the map Ψ can be extended in the usual way to a quasiregular map in Ω.

Suppose then that $x_0 = 0$ is a repelling fixed point of f. Again we may assume that f is injective on B and furthermore that the uniform distortion control (6) is valid for every $r \in]0, 1[$. It follows that for every $k \geq 1$ there is $\rho_k > 0$, with $\rho_k \to \infty$ as $k \to \infty$, such that

$$\Psi_k(\mathsf{S}^{n-1}) = f^k \left(\frac{1}{\rho_k} \mathsf{S}^{n-1} \right) \subset A \left(\frac{1}{K^*}, K^* \right).$$

Therefore, the sequence $(\Psi_k)_k$ is normal in B and the limit $\Psi : \mathsf{B} \to \Omega$ of a convergent subsequence Ψ_{k_j} is a quasiconformal map in B. Observe that

$$f^{k_j+1}(z/\rho_{k_j}) = f^{k_j} \left(\frac{1}{\rho_{k_j}} \left(\rho_{k_j} f(z/\rho_{k_j}) \right) \right).$$

Extracting a subsequence if necessary, we find a generalized derivative $\varphi \in \mathscr{D} f(x_0)$ such that $\lim_{j\to\infty} \rho_{k_j} f(z/\rho_{k_j}) = \varphi$. Now, passing to the limit in the above identity, one obtains $f \circ \Psi = \Psi \circ \varphi$. Hence $f^n \circ \Psi = \Psi \circ \varphi^n$, where φ is loxodromic, fixing 0 and ∞, and having 0 as its repelling fixed point. Using this identity the map Ψ can be extended to a global quasiregular map in R^n.

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