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SOME REMARKS ON THE C^*-ALGEBRAS ASSOCIATED WITH SUBSHIFTS

TOKE MEIER CARLSEN and KENGO MATSUMOTO

Abstract

We point out incorrect lemmas in some papers regarding the C^*-algebras associated with subshifts written by the second named author. To recover the incorrect lemmas and the affected main results, we will describe an alternative construction of C^*-algebras associated with subshifts. The resulting C^*-algebras are generally different from the originally constructed C^*-algebras associated with subshifts and they fit the mentioned papers including the incorrect results. The simplicity conditions and the K-theory formulae for the originally constructed C^*-algebras are described. We also introduce a condition called $(*)$ for subshifts such that under this condition the new C^*-algebras and the original C^*-algebras are canonically isomorphic to each other. We finally present a subshift for which the two kinds of algebras have different K-theory groups.

1. Introduction

Throughout this paper a finite set $\Sigma = \{1, 2, \ldots, n\}$, $n \geq 2$ is fixed. Let Σ^{Z}, Σ^{N} be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on Σ^{Z} given by $(\sigma(x)_i)_{i\in\mathsf{Z}} = (x_{i+1})_{i\in\mathsf{Z}}$ for $x = (x_i)_{i\in\mathsf{Z}} \in \Sigma^{\mathsf{Z}}$ is called the (full) shift. Let Λ be a shift invariant closed subset of Σ^{Z} i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system (Λ, σ) is called a subshift. Let $X_\Lambda = \{(x_1, x_2, \ldots) \in \Sigma^{\mathsf{N}} \mid (x_i)_{i\in\mathsf{Z}} \in \Lambda\}$: the set of all right-infinite sequences that appear in Λ.

In [17], the second named author has introduced a class of C^*-algebras associated with subshifts. The class of the C^*-algebras is wider than the class of the Cuntz-Krieger algebras that are associated with topological Markov shifts. The K-groups for the C^*-algebras have been computed in [18]. In the subsequent papers [19], [20], [23], [24], some results on the C^*-algebras associated with subshifts have been published. They are the results on dimension groups, ideal structure of the algebras ([19]), operator relations among the canonical generating partial isometries of the algebras ([20]), automorphisms of the algebras ([23]) and algebraic invariance of the stabilized C^*-algebras under topological conjugacy ([24]).

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However, there are some lemmas in these four papers which are not correct. They are: [19, Lemma 4.6, Corollary 4.7, Proposition 4.12, Lemma 5.3], [20, Lemma 3.1], [23, Lemma 4.1] and [24, Lemma 2.1 (i)].

All of them arise from the inaccurate statement ($\sharp$) below. For a subshift (Λ, σ) and $k \in \mathsf{N}$, let Λ^k be the set of all words with length k occurring in some $x \in \Lambda$. We put $\Lambda_l = \cup_{k=0}^{l} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$. Let $S_1, \dots, S_n$ be the canonical generating partial isometries of the C^*-algebra associated with Λ as in [17]. For an admissible word $\mu = \mu_1 \cdots \mu_l$ of Λ, we write $S_{\mu_1} \cdots S_{\mu_l}$ as S_μ. For $l \in \mathsf{N}$, let $\Omega_l = X_\Lambda / \sim_l$ be the l-past equivalence classes of the right one-sided subshift X_Λ (see [19, Introduction]).

($\sharp$): The C^*-algebra $\mathscr{A}_l$ generated by the projections $S_\mu^* S_\mu$, $\mu \in \Lambda_l$ is isomorphic to the the commutative C^*-algebra $C(\Omega_l)$ of all continuous functions on Ω_l.

There is a subshift for which the above statement ($\sharp$) does not hold (see Section 4). The arguments to deduce the main results [19, Theorem 4.1, Corollary 6.11], [20, Theorem 3.5], [23, Theorem 5.12], [24, Theorem 6.1] of the above mentioned four papers depend upon the lemmas in the above list. We shall consider the following two ways to establish the main results in the four papers without any inaccuracy:

(1) Describe an alternative construction of C^*-algebras associated with subshifts such that the statement ($\sharp$) for these C^*-algebras hold.
(2) Restrict the results to the class of subshifts for which the statement ($\sharp$) holds.

Consequently, we know that for the C^*-algebras given by the above mentioned alternative construction (1), all of the results in the previously mentioned four papers are valid to subshifts satisfying the condition (I) (cf. page 149, and [19, Section 5]). And also, if we consider the subshifts with a certain condition, that will be written as ($*$) in Section 3, the discussions in the four papers for the C^*-algebras originally defined in [17] are valid.

There are three purposes of this paper.

The first one is to recover the main results of the four papers by going through the above two ways. We will construct C^*-algebras associated with subshifts by a slightly different way from the original construction in [17] (Definition 2.1), and we will describe a condition, which will be called ($*$), such that for subshifts satisfying this condition, the statement ($\sharp$) holds. In [19, Corollary 6.11], a simplicity condition for the algebra defined originally in [17] is described in terms of the underlying symbolic dynamics. Since the simplicity condition is deduced by going through ($\sharp$), it only holds for subshifts which satisfy the condition $(*)$. On the other hand the simplicity condition fits the C^*-algebras given by the alternative construction.

The second purpose of this paper is to describe a precise simplicty condition for the originally constructed C^*-algebra associated with subshifts without going through the condition $(*)$. This criterion for the C^*-algebras to be simple is a new result.

The third purpose of this paper is to clarify the relationship between the C^*-algebras given by the alternative construction and the C^*-algebras given by the original construction, and to present an example of a subshift for which the two kinds of C^*-algebras are not isomorphic.

In Section 2, we present the alternative construction of C^*-algebras associated with subshifts. The resulting C^*-algebras have previously been seen in [25, Section 3]. In this paper, we write these algebras as $\mathcal{O}_\Lambda$ whereas the originally defined C^*-algebras associated with subshifts are written as $\mathcal{O}_{\Lambda^*}$. In the second half of Section 2, we will study the algebras $\mathcal{O}_{\Lambda^*}$ and describe their precise simplicity condition and their K-theory formula in terms of the symbolic dynamical system.

There always exists a canonical unital surjective $*$-homomorphism from the algebra $\mathcal{O}_{\Lambda^*}$ onto the algebra $\mathcal{O}_\Lambda$. In Section 3 we will introduce the condition $(*)$ for subshifts. For the subshifts satisfying $(*)$, the two algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ become canonically isomorphic under the condition (I) so that the canonical generating partial isometries of $\mathcal{O}_{\Lambda^*}$ satisfy $(\sharp)$ and all of the statements in [19], [20], [23], [24] are valid for such subshifts. We also show that both of the algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are constructed as the C^*-algebras associated with λ-graph systems discussed in [26]. In Section 4 we finally present an example of an irreducible sofic shift Λ for which the algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are not isomorphic by computing their K-groups.

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2. The C^*-algebras associated with subshifts

Let (Λ, σ) be a subshift over Σ. We denote by Λ^* the set of all admissible words of Λ.

The first half of this section is devoted to presenting the alternative construction of C^*-algebras assosiated with subshifts, so that the resulting C^*-algebras with the canonical generating partial isometries $S_1, \dots, S_n$ satisfy $(\sharp)$. Hence we will know that the main results of the above mentioned four papers hold for these C^*-algebras under the assumption that the subshifts satisfy condition (I). This construction has first appeared in [23, Lemma 4.1] and also in [25, Section 3]. A generalization of this construction has been studied in [26]. We denote by H_Λ the Hilbert space with its complete orthonormal basis $\{e_x \mid x \in X_\Lambda\}$. Let $S_1, \dots, S_n$ be the operators on H_Λ defined by

$$S_j e_x = \begin{cases} e_{jx} & \text{if } jx \in X_\Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S_1, \dots, S_n$ are partial isometries satisfying the relation: $\sum_{j=1}^{n} S_j S_j^* = 1$.

DEFINITION 2.1 (cf. [23, Lemma 4.1], [25, Section 3], [26]). The C^*-algebra $\mathcal{O}_\Lambda$ associated with a subshift Λ is defined as the C^*-algebra generated by the partial isometries $S_1, \dots, S_n$.

In [23, Lemma 4.1], the C^*-algebra $\mathcal{O}_\Lambda = C^*(S_1, \dots, S_n)$ has first appeared. But Lemma 4.1 in [23] and also Lemma 4.6 in [19] do not hold in general unless the subshift Λ satisfies condition $(*)$ stated in Section 3 and condition (I) stated in [19, p. 691]. There is a subshift Λ such that the algebra $\mathcal{O}_\Lambda$ is not isomorphic to the C^*-algebra associated with the subshift defined in [17] (cf. Theorem 4.1).

As in the introduction, we denote by $\Omega_l = X_\Lambda/\sim_l$ the l-past equivalence classes of X_Λ. Let F_i^l, $i = 1, 2, \dots, m(l)$ be the set of the l-past equivalence classes of X_Λ. Hence X_Λ is a disjoint union of the set F_i^l, $i = 1, 2, \dots, m(l)$. The projections $S_\mu^* S_\mu$, $\mu \in \Lambda^*$ are mutually commutative so that the C^*-algebra $\mathcal{A}_l$ is commutative. It is direct to see that the set of all minimal projections of $\mathcal{A}_l$ exactly corresponds to the set F_i^l, $i = 1, 2, \dots, m(l)$ of Ω_l. Thus we have the following lemma (cf. [19, Section 4]).

LEMMA 2.2. *The C^*-algebra $\mathcal{A}_l$ generated by the projections $S_\mu^* S_\mu$, $\mu \in \Lambda_l$ is isomorphic to the C^*-algebra $C(\Omega_l)$ of all complex valued continuous functions on Ω_l. That is, for the generators $S_1, \dots, S_n$ of the algebra $\mathcal{O}_\Lambda$, the statement* ($\sharp$) *holds.*

We put $a_\mu = S_\mu^* S_\mu$ for $\mu \in \Lambda^*$. We define the algebra $\mathcal{F}_\Lambda$ as the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by elements of the form $S_\mu a_{\gamma_1} \dots a_{\gamma_m} S_\nu^*$ with $|\mu| = |\nu|$ for $\mu, \nu, \gamma_1, \dots, \gamma_m \in \Lambda^*$, where $|\mu|, |\nu|$ denote the lengths of μ, ν respectively. Then by the same argument as in [17] we see that the algebra $\mathcal{F}_\Lambda$ is an AF-algebra (this also follows from [26, Proposition 3.4]). Since the algebra $\mathcal{O}_\Lambda$ is not constructed by creation operators on sub Fock space as in [17], it is not clear if the correspondence: $S_i \to zS_i$, $i = 1, \dots, n$ for $z \in \mathbb{C}$, $|z| = 1$ gives rise to an automorphism on $\mathcal{O}_\Lambda$. Hence it is not clear that a projection of norm one from $\mathcal{O}_\Lambda$ to the AF-algebra $\mathcal{F}_\Lambda$, that would be realized as the fixed point algebra of $\mathcal{O}_\Lambda$ under the above action, exists. Existence of such a projection of norm one plays a key rôle in the simplicity argument discussed in

[17]. To guarantee existence of such a projection of norm one, we assume the condition (I) for subshift defined in [19, Section 5]. By [19, Lemma 5.3], the condition (I) is equivalent to the condition (I_Λ) in [17, Section 5] for the algebra $\mathcal{O}_\Lambda$. We will in Theorem 3.6 see that under the condition (I) the C^*-algebra $\mathcal{O}_\Lambda$ is canonically isomorphic to a C^*-algebra associated with a λ-graph system ([26]). This C^*-algebra associated with a λ-graph system always (even when the the subshift Λ does not satisfy the condition (I)) satisfies (♯) and has an action given by $S_i \to zS_i$ for $z \in \mathbb{C}$, $|z| = 1$. This C^*-algebra can also be constructed as a Cuntz-Pimsner algebra ([26, Proposition 6.1], cf. [5]) and as a groupoid C^*-algebra ([26, Section 3], cf. [4]).

Each element X of the $*$-algebra $\mathcal{P}_\Lambda$ of $\mathcal{O}_\Lambda$ algebraically generated by $S_\mu, S_\nu^*, \mu, \nu \in \Lambda^*$ can be written as a finite sum

$$X = \sum_{|\nu|\geq 1} X_{-\nu}S_\nu^* + X_0 + \sum_{|\mu|\geq 1} S_\mu X_\mu \qquad \text{for some} \quad X_{-\nu}, X_0, X_\mu \in \mathcal{F}_\Lambda.$$

By the same manner as the proof of [17, Theorem 5.2] and [17, Corollary 5.7] (also [26], cf. [5]), we have

LEMMA 2.3. *If a subshift Λ satisfies condition (I), the map $X \in \mathcal{P}_\Lambda \to X_0 \in \mathcal{F}_\Lambda$ can be extended to a projection of norm one from $\mathcal{O}_\Lambda$ to the AF-algebra $\mathcal{F}_\Lambda$.*

Hence we have

PROPOSITION 2.4. *If a subshift Λ satisfies condition (I), then the universal property for the algebra $\mathcal{O}_\Lambda$ as stated in [23, Lemma 2.3] holds.*

We then conclude with Lemma 2.2

PROPOSITION 2.5. *If a subshift Λ satisfies condition (I), then the results:* [19, Theorem 4.13, Corollary 6.11], [20, Theorem 3.5], [23, Theorem 5.2] *and* [24, Theorem 6.1] *hold for the algebra $\mathcal{O}_\Lambda$.*

In particular we can prove the following proposition in a similar manner to how [17, Theorem 6.3 and 7.5] and [18, Theorem 5.8] are deduced.

PROPOSITION 2.6. *If a subshift Λ satisfies condition (I) and is irreducible in past equivalence, then $\mathcal{O}_\Lambda$ is simple. In addition, if Λ is aperiodic in past equivalence, $\mathcal{O}_\Lambda$ is purely infinite.*

The second half of this secion is devoted to studying the originally constructed C^*-algebras associated with subshifts. The following is the original construction of C^*-algebras associated with subshifts ([17]). Let $\{e_1, \dots, e_n\}$

be an orthonormal basis of n-dimensional Hilbert space C^n. We put

$$F_\Lambda^0 = \mathsf{C}e_0 \quad (e_0\text{: the vacuum vector}),$$

$$F_\Lambda^k = \text{ the Hilbert space spanned by the vectors}$$
$$e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k},\ \mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k,$$

$$F_\Lambda = \oplus_{k=0}^\infty F_\Lambda^k \quad \text{(the direct sum of the Hilbert spaces)}.$$

We denote by T_i for $i \in \Sigma$ the creation operator on F_Λ of e_i defined by

$$T_i e_0 = e_i \qquad \text{and} \qquad T_i e_\mu = \begin{cases} e_i \otimes e_\mu, & \text{if } i\mu \in \Lambda^* \\ 0 & \text{otherwise} \end{cases}$$

which is a partial isometry. We denote by P_0 the rank one projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^n T_i T_i^* + \mathsf{P}_0 = 1$. We denote by T_μ for $\mu = \mu_1 \ldots \mu_k$ the operator $T_{\mu_1} \ldots T_{\mu_k}$. For $\mu, \nu \in \Lambda^*$, the operator $T_\mu \mathsf{P}_0 T_\nu^*$ is the rank one partial isometry from e_ν to e_μ. Hence, the C^*-algebra generated by elements of the form $T_\mu \mathsf{P}_0 T_\nu^*$, $\mu, \nu \in \Lambda^*$ is nothing but the C^*-algebra $\mathscr{K}(F_\Lambda)$ of all compact operators on F_Λ. Let $\mathscr{T}_\Lambda$ be the C^*-algebra on F_Λ generated by the elements T_ν, $\nu \in \Lambda^*$.

Definition 2.7 ([17]). The C^*-algebra $\mathscr{O}_{\Lambda^*}$ associated with the subshift Λ is defined as the quotient C^*-algebra $\mathscr{T}_\Lambda / \mathscr{K}(F_\Lambda)$ of $\mathscr{T}_\Lambda$ by $\mathscr{K}(F_\Lambda)$.

In this paper, we write the quotient algebra $\mathscr{T}_\Lambda / \mathscr{K}(F_\Lambda)$ as $\mathscr{O}_{\Lambda^*}$, although in [17], [19], [20], [23] and [24], it has been written as $\mathscr{O}_\Lambda$. We denote by S_i, S_μ the quotient image of the operator T_i, $i \in \Sigma$, T_μ, $\mu \in \Lambda^*$. Remark that $S_\emptyset = 1$. Hence $\mathscr{O}_{\Lambda^*}$ is generated by n partial isometries $S_1, \ldots, S_n$ which satisfy the relation $\sum_{i=1}^n S_i S_i^* = 1$. In this paper, We will use the following notation for the C^*-algebra $\mathscr{O}_{\Lambda^*}$. For a natural number l, we put

$$\mathscr{A}_{l*} = \text{ The } C^*\text{-subalgebra of } \mathscr{O}_{\Lambda^*} \text{ generated by } S_\mu^* S_\mu,\ \mu \in \Lambda_l.$$
$$\mathscr{A}_{\Lambda^*} = \text{ The } C^*\text{-subalgebra of } \mathscr{O}_{\Lambda^*} \text{ generated by } S_\mu^* S_\mu,\ \mu \in \Lambda^*.$$

As stated in the introduction, the algebra $\mathscr{A}_{l*}$ is not necessarily isomorphic to the algebra $C(\Omega_l)$. Hence the criterion for the algebra $\mathscr{O}_{\Lambda^*}$ to be simple is different from [19, Corollary 6.11] unless the subshift Λ satisfy the condition $(*)$ that will be stated in the next section. Also the K-theory formulas for $\mathscr{O}_{\Lambda^*}$ in terms of the underlying symbolic dynamics are different from [24, Corollary 6.4] because they are based on the property ($\sharp$) in the introduction. The rest of this section is devoted to finding a compact space Ω_l^* which can replace Ω_l such that $\mathscr{A}_{l*}$ is isomorphic to $C(\Omega_l^*)$. This will be done by using the underlying symbolic dynamics. As a result, simplicity condition and K-theory formulas for $\mathcal{O}_{\Lambda^*}$ will be described in terms of the underlying symbolic dynamics.

For an admissible word $w \in \Lambda^*$ and $l \in \mathbf{N}$, we put $\Lambda_l(w) = \{\mu \in \Lambda_l \mid \mu w \in \Lambda^*\}$. Two admissible words $\mu, \nu \in \Lambda^*$ are said to be *l-past equivalent* if $\Lambda_l(\mu) = \Lambda_l(\nu)$and written as $\mu \sim_l \nu$. We consider the following subsets of the admissible words

$$\Lambda_l^* = \{w \in \Lambda^* \mid \text{The cardinality of the set } \{\mu \in \Lambda^* \mid \mu \sim_l w\} \text{ is infinite}\}.$$

LEMMA 2.8.
(i) *For $\mu, \nu \in \Lambda_l^*$ if $\mu \sim_l \nu$, then $\mu \sim_m \nu$ for $m < l$.*
(ii) *For $\mu, \nu \in \Lambda_l^*$ and $w \in \Lambda^k$ with $l > k$, if $\mu \sim_l \nu$ and $w\mu \in \Lambda_{l-k}^*$, then $w\nu \in \Lambda_{l-k}^*$ and $w\mu \sim_{l-k} w\nu$.*
(iii) *For $\mu \in \Lambda_l^*$, there exists a word $\nu \in \Lambda_{l+1}^*$ such that $\mu \sim_l \nu$.*
(iv) *For $\mu \in \Lambda_l^*$, there exist a word $\nu \in \Lambda_{l+1}^*$ and a symbol $j \in \Sigma$ such that $\mu \sim_l j\nu$.*

PROOF. The assertions (i) and (ii) are clear.

(iii) Let $\{\mu_i \mid i \in \mathbf{N}\}$ be the infinite set of all words in Λ^* for which $\mu_i \sim_l \mu$. Since the $l+1$-past equivalence classes $\Lambda^*/\sim_{l+1}$ of Λ^* is a finite set, there exists an infinite subset $\{\mu_{i_m} \mid m \in \mathbf{N}\}$ of $\{\mu_i \mid i \in \mathbf{N}\}$ such that the $\mu_{i_m}, m \in \mathbf{N}$ are $l+1$-past equivalent to each other. Hence we have $\mu_{i_m} \in \Lambda_{l+1}^*$ for $m \in \mathbf{N}$ so that we can take one of the words μ_{i_m} as ν.

(iv) Let $\{\mu_{i_m} \mid m \in \mathbf{N}\}$ be the set as above. Since the set $\{\mu_{i_m} \mid m \in \mathbf{N}\}$ is infinite, there exists a subset $\{\mu_{i_{m_k}} \mid k \in \mathbf{N}\}$ of $\{\mu_{i_m} \mid m \in \mathbf{N}\}$ such that the first letters of $\mu_{i_{m_k}}$ are the same. We denote by j the first letter. Hence there exists an infinite sequence of admissible words $\nu_{i_k}, k \in \mathbf{N}$ satisfying $\mu_{i_{m_k}} = j\nu_{i_k}$. Since the $l+1$-past equivalence classes $\Lambda^*/\sim_{l+1}$ of Λ^* is a finite set, there exists an infinite subset $\{\nu_{i_{k_p}} \mid p \in \mathbf{N}\}$ of $\{\nu_{i_k} \mid k \in \mathbf{N}\}$ such that $\nu_{i_{k_p}}, p \in \mathbf{N}$ are $l+1$-past equivalent to each other. Hence we have $\nu_{i_{k_p}} \in \Lambda_{l+1}^*, p \in \mathbf{N}$ so that we can take one of the words $\nu_{i_{k_p}} \in \Lambda_{l+1}^*, p \in \mathbf{N}$ as ν.

We denote by $\Omega_l^* = \Lambda_l^*/\sim_l$ the l-past equivalence classes of Λ_l^*. There is a natural surjection from Ω_{l+1}^* to Ω_l^*. It is easy to see that a subshift Λ is a sofic shift if and only if $\Omega_l^* = \Omega_{l+1}^*$ for some $l \in \mathbf{N}$. For a fixed $l \in \mathbf{N}$, let F_i^{*l}, $i = 1, 2, \ldots, m^*(l)$ be the set of the l-past equivalence classes of Λ_l^*. Hence Λ_l^* is a disjoint union of the set F_i^{*l}, $i = 1, 2, \ldots, m^*(l)$. The projections $S_\mu^* S_\mu$, $\mu \in \Lambda^*$ are mutually commutative so that the C^*-algebras $\mathcal{A}_{l*}$, $l \in \mathbf{N}$ are commutative. It is direct to see that the set of all minimal projections of $\mathcal{A}_{l*}$ exactly corresponds to the set F_i^{*l}, $i = 1, 2, \ldots, m^*(l)$ of Ω_l^*. Thus we have the following lemma (cf. [17, Section 3], [19, Section 4]).

LEMMA 2.9. *$\mathscr{A}_{l*}$ is isomorphic to the commutative C^*-algebra $C(\Omega_l^*)$ of all continuous functions on Ω_l^*.*

This lemma is the right one instead of ($\sharp$). We notice that in [19, Lemma 4.6] there is a corresponding result. However [19, Lemma 4.6] does not hold in general unless the subshift Λ satisfies the condition $(*)$ stated in the next section.

We introduce the following condition called (I^*) for subshifts:

(I^*): For any $l \in \mathsf{N}$ and $\mu \in \Lambda_l^*$, there exist distinct words $\xi_1, \xi_2 \in \Lambda^*$ with $|\xi_1| = |\xi_2|$ such that

$$\mu \sim_l \xi_1\gamma_1 \qquad \text{and} \qquad \mu \sim_l \xi_2\gamma_2$$

for some $\gamma_1, \gamma_2 \in \Lambda_{l+|\xi_i|}^*$.

In [17], the condition (I_Λ) for the C^*-algebras associated with subshifts has been introduced. In this paper, we denote it by (I_{Λ^*}). We can prove the following lemma by an argument similar to the proof of [19, Lemma 5.3].

LEMMA 2.10 (cf. [19], [26]). *A subshift Λ satisfies condition (I^*) if and only if the C^*-algebra $\mathscr{O}_{\Lambda^*}$ satisfies condition (I_{Λ^*}).*

Put $\lambda_{\Lambda^*}(X) = \sum_{j=1}^n S_j^* X S_j$, $X \in \mathscr{A}_{\Lambda^*}$. The operator λ_{Λ^*} is said to be irreducible if there is no non-trivial λ_{Λ^*}-invariant ideal in $\mathscr{A}_{\Lambda^*}$. In addition, it is said to be aperiodic if for any number $l \in \mathsf{N}$, there exists $N \in \mathsf{N}$ such that $\lambda_{\Lambda^*}^N(p) \geq 1$ for any minimal projection p in $\mathscr{A}_{l*}$.

(i) A subshift Λ is said to be *irreducible in past equivalence of words* if for any $l \in \mathsf{N}$, $\mu \in \Lambda_l^*$ and a sequence $\nu^k \in \Lambda_k^*$, $k \in \mathsf{N}$ with $\nu^k \sim_k \nu^{k+1}$, $k \in \mathsf{N}$ there exist a number N and an admissible word ξ of length N such that $\mu \sim_l \xi\nu^{l+N}$.

(ii) A subshift Λ is *aperiodic in past equivalence of words* if for any $l \in \mathsf{N}$, and $\mu \in \Lambda_l^*$ there exists a number $N \in \mathsf{N}$ such that for any word $\nu \in \Lambda_{l+N}^*$ there exists an admissible word ξ of length N such that $\mu \sim_l \xi\nu$.

We know that if a subshift Λ is aperiodic in past equivalence of words or irreducible in past equivalence of words with an aperiodic point, then it satisfies the condition (I^*) (cf. [19, Proposition 5.2]).

By the same argument used in the proof of [19, Proposition 4.12], we see that λ_{Λ^*} is irreducible (resp. aperiodic) if and only if the subshift Λ is irreducible (resp. aperiodic) in past equivalence of words. Hence by the discussion of Section 6 in [17], we have

PROPOSITION 2.11 (cf. [17, Theorem 6.3 and 7.5], [18, Theorem 5.8], [26]). *If a subshift Λ satisfies condition (I^*) and is irreducible in past equivalence of words, then $\mathcal{O}_{\Lambda^*}$ is simple. In addition, if Λ is aperiodic in past equivalence of words, $\mathcal{O}_{\Lambda^*}$ is purely infinite.*

We next describe the K-theory of the algebra $\mathcal{O}_{\Lambda^*}$. Let $A^*_{l,l+1}(i,j)$ be the cardinality of the set

$$\{a \in \Sigma \mid a\mu \in F_i^{*l} \text{ for some } \mu \in F_j^{*l+1}\}.$$

We write $I^*_{l,l+1}(i,j) = 1$ if $F_j^{*l+1} \subset F_i^{*l}$ otherwise $I^*_{l,l+1}(i,j) = 0$. Hence we have two $m^*(l) \times m^*(l+1)$ matrices $I^*_{l,l+1}$ and $A^*_{l,l+1}$ with entries in $\{0,1\}$ and with entries in nonnegative integers respectively such that

$$I^*_{l,l+1} A^*_{l+1,l+2} = A^*_{l,l+1} I^*_{l+1,l+2}, \qquad l \in \mathsf{N}.$$

We denote by λ_l^* the restriction of the operator λ_{Λ^*} to $\mathscr{A}_{l*}$. It induces a homomorphism from $K_0(\mathscr{A}_{l*})$ to $K_0(\mathscr{A}_{l+1*})$. We denote by ι_l the natural embedding of $\mathscr{A}_{l*}$ into $\mathscr{A}_{l+1*}$. The homomorphisms λ_{l*}, ι_{l*} from $K_0(\mathscr{A}_{l*})$ to $K_0(\mathscr{A}_{l+1*})$ induced by λ_l, ι_l are regarded as the transposed matrices $A^{*t}_{l,l+1}$, $I^{*t}_{l,l+1}$ of $A^*_{l,l+1}, I^*_{l,l+1}$ respectively through isomorphisms between $K_0(\mathscr{A}_{j*})$ and $\mathsf{Z}^{m^*(j)}$, $j = l, l+1$. By [18], the K-groups for $\mathcal{O}_{\Lambda^*}$ are computed as

PROPOSITION 2.12 (cf. [18, Theorem 4.9]).
(i) $K_0(\mathcal{O}_{\Lambda^*}) \cong \varinjlim \mathsf{Z}^{m^*(l+1)}/(I^{*t}_{l,l+1} - A^{*t}_{l,l+1})\mathsf{Z}^{m^*(l)}$,
(ii) $K_1(\mathcal{O}_{\Lambda^*}) \cong \varinjlim \operatorname{Ker}(I^{*t}_{l,l+1} - A^{*t}_{l,l+1})$ *in* $\mathsf{Z}^{m^*(l)}$,
where the sequences of homomorphisms in $\varinjlim$ *are coming from the inclusions* $I^{*t}_{l,l+1} : \mathsf{Z}^{m^*(l)} \hookrightarrow \mathsf{Z}^{m^*(l+1)}$.

3. Relations between $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_{\Lambda}$

We consider the following condition for a subshift Λ:

$(*)$: There exists for each $l \in \mathsf{N}$ and each infinite sequence of admissible words $\mu_i, i \in \mathsf{N}$ satisfying $\Lambda_l(\mu_i) = \Lambda_l(\mu_j), i, j \in \mathsf{N}$ a right infinite sequence $x \in X_\Lambda$ such that

$$\Lambda_l(x) = \Lambda_l(\mu_i), \qquad i \in \mathsf{N}.$$

The topological Markov shifts, the β-shifts and a synchronizing counter shift Z refered as the context free shift in [16, Example 1.2.9] satisfy the above condition $(*)$ (cf. [11], [21]).

Wolfgang Krieger kindly informed to the authors that the condition $(*)$ is not an invariant property for topological conjugacy and is strictly weaker than the existence of an instantaneous presentation of subshifts (cf. [15]).

We first note the following proposition.

PROPOSITION 3.1. *Suppose that a subshift Λ satisfies condition $(*)$. Then we have*

(i) *Λ satisfies condition (I^*) if and only if Λ satisfies condition (I).*

(ii) *Λ is irreducible in past equivalence of words if and only if Λ is irreducible in past equivalence.*

(iii) *Λ is aperiodic in past equivalence of words if and only if Λ is aperiodic in past equivalence.*

For $x \in X_\Lambda$, we denote by $[x]_l \in X_\Lambda/\sim_l$ its l-past equivalence class. Put

$$\Lambda_l^c(x) = \{\mu \in \Lambda^l \mid \mu x \text{ does not belong to } X_\Lambda\}.$$

Since the cardinality of the set $\Lambda_l^c(x)$ is finite, there exists a number $N_{x,l}$ such that

$$\Lambda_l(x) = \Lambda_l(x_{[1,\dots,n]}) \qquad \text{for all} \quad n \geq N_{x,l}$$

where $x_{[1,\dots,n]} = x_1 \cdots x_n$ for $(x_i)_{i \in \mathsf{N}} \in X_\Lambda$. As $x_{[1,\dots,n]} \sim_l x_{[1,\dots,m]}$ for $n, m > N_{x,l}$, we know that $x_{[1,\dots,n]}$ belongs to Λ_l^* and the map

$$x \in X_\Lambda \longrightarrow x_{[1,\dots,n]} \in \Lambda_l^* \qquad \text{for} \quad n \geq N_{x,l}$$

induces a map

$$[x]_l \in X_\Lambda/\sim_l = \Omega_l \longrightarrow [x_{[1,\dots,n]}]_l \in \Lambda_l^*/\sim_l = \Omega_l^*.$$

We will denote this map by π_l.

LEMMA 3.2. *The map $\pi_l : \Omega_l \to \Omega_l^*$ is injective and compatible to ι and λ, that is,*

$$\pi_l \circ \iota_{l+1} = \iota_{l+1}^* \circ \pi_{l+1}, \qquad \pi_l([jx]_l) = [jx_{[1,\dots,N_{x,l+1}]}]_l$$

for $j \in \Sigma$ with $jx \in X_\Lambda$. Furthermore π_l is surjective for all $l \in \mathsf{N}$ if and only if Λ satisfies condition $(*)$.

PROOF. Injectivity for π_l is clear by its construction.
For $x \in X_\Lambda$, we have

$$\pi_l(\iota_{l+1}([x]_{l+1})) = \pi_l(([x]_l)) = [x_{[1,\dots,N_{x,l}]}]_l.$$

As $x_{[1,\dots,N_{x,l}]} \sim_l x_{[1,\dots,N_{x,l+1}]}$, it follows that

$$[x_{[1,\dots,N_{x,l}]}]_l = [x_{[1,\dots,N_{x,l+1}]}]_l = \iota_{l+1}^*(x_{[1,\dots,N_{x,l+1}]}) = \iota_{l+1}^*(\pi_{l+1}([x]_{l+1})).$$

Hence we see $\pi_l(\iota_{l+1}([x]_{l+1})) = \iota_{l+1}^*(\pi_{l+1}([x]_{l+1}))$.

For $j \in \Sigma$ and $x \in X_\Lambda$ with $jx \in X_\Lambda$, put $y = jx$. As $y_{[1,\dots,N_{y,l}]} \sim_l j \cdot x_{[1,\dots,N_{x,l+1}]}$, we see $\pi_l([jx]_l) = [jx_{[1,\dots,N_{x,l+1}]}]_l$.

It is immediate that π_l is surjective for all $l \in \mathsf{N}$ if and only if Λ satisfies condition $(*)$.

Since we may regard the algebras $\mathscr{A}_{l*}$ and $\mathscr{A}_l$ as the algebras of all continuous functions on the sets Ω_l^* and Ω_l respectively, we have an induced map $\pi_l^* : \mathscr{A}_{l*} \to \mathscr{A}_l$ for each $l \in \mathsf{N}$.

COROLLARY 3.3. *The sequence of the induced maps $\pi_l^* : \mathscr{A}_{l*} \to \mathscr{A}_l$, $l \in \mathsf{N}$ yields a unital surjective $*$-homomorphism π_Λ from $\mathscr{A}_{\Lambda*}$ to $\mathscr{A}_\Lambda$. The map π_Λ is injective if and only if the subshift Λ satisfies condition* $(*)$.

THEOREM 3.4. *The correspondence $\pi : S_i \in \mathcal{O}_{\Lambda^*} \to S_i \in \mathcal{O}_\Lambda, i \in \Sigma$ gives rise to a surjective $*$-homomorphism $\pi : \mathcal{O}_{\Lambda^*} \to \mathcal{O}_\Lambda$. If in particular the subshift Λ satisfies conditions both $(*)$ and (I), the map π is injective so that the algebra $\mathcal{O}_{\Lambda^*}$ is canonically isomorphic to the algebra $\mathcal{O}_\Lambda$.*

COROLLARY 3.5. *If a subshift is one of the followings:*

(i) *a topological Markov shift for which its adjacency matrix is irreducible and not permutation,*

(ii) *β-shift for each $1 < \beta \in \mathsf{R}$,*

(iii) *the context free shift Z,*

then the associated C^*-algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are canonically isomorphic.

Although the constructions of the C^*-algebras $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are different, we can unify them by the following observations.

Let $\mathfrak{L}_\Lambda$ be ths canonical λ-graph system for the subshift Λ that is constructed from l-past equivalence classes of $\Omega_l = X_\Lambda/\sim_l$ ([22, p. 297]). Let $\mathfrak{L}_{\Lambda^*}$ be the λ-graph system constructed from l-past equivalence classes $\Omega_l^* = \Lambda_l^*/\sim_l$ of words. The λ-graph system $\mathfrak{L}_{\Lambda^*}$ corresponds to the symbolic matrix system $(\mathscr{M}_{l,l+1}^*, I_{l,l+1}^*)_{l \in \mathsf{N}}$ where the symbolic matrices $\mathscr{M}_{l,l+1}^*$, $l \in \mathsf{N}$ are defined as follows: Let $a_1, \dots, a_p$ be the set of all symbols in Σ for which $a_k\mu \in F_i^{*l}$ for some $\mu \in F_j^{*l+1}$. We then define the (i, j)-component of the matrix $\mathscr{M}_{l,l+1}^*(i, j)$ as $\mathscr{M}_{l,l+1}^*(i, j) = a_1 + \cdots + a_p$, the formal sum of $a_1, \dots, a_p$. In [26], general construction of the C^*-algebras associated with λ-graph systems have been introduced. The constructed C^*-algebras have universal property subject to certain operator relations that come from concatenations of vertices and edges of the λ-graph systems. We note that

(i) Λ satisfies condition (I^*) if and only if the λ-graph system $\mathfrak{L}_{\Lambda^*}$ satisfies condition (I).

(ii) Λ satisfies condition (I) if and only if the λ-graph system $\mathfrak{L}_\Lambda$ satisfies condition (I).

Then we have

THEOREM 3.6.

(i) *Suppose that a subshift Λ satisfies condition (I). Then the C^*-algebra $\mathcal{O}_\Lambda$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}_\Lambda}$ associated with the λ-graph system $\mathfrak{L}_\Lambda$.*

(ii) *Suppose that a subshift Λ satisfies condition (I^*). Then the C^*-algebra $\mathcal{O}_{\Lambda^*}$ is canonically isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}_{\Lambda^*}}$ associated with the λ-graph system $\mathfrak{L}_{\Lambda^*}$.*

PROOF. The assertions come from the universality of the C^*-algebras $\mathcal{O}_\Lambda$, $\mathcal{O}_{\Lambda^*}$, $\mathcal{O}_{\mathfrak{L}_\Lambda}$ and $\mathcal{O}_{\mathfrak{L}_{\Lambda^*}}$.

4. An example

We finally present an example of subshift Λ for which the C^*-algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are not isomorphic. Let Λ be the subshift given by the set

$$\mathfrak{F} = \{12^k1, 32^k12, 32^k13, 42^k14 \mid k \in \mathsf{N}_0\}$$

of forbidden words, where 2^k denotes $\underbrace{2\cdots 2}_{k\ \text{times}}$ and N_0 denotes the set of all nonnegative integers.

For an admissible word $\omega \in \Lambda^*$ we put

$$\Lambda_*(\omega) = \bigcup_{l=1}^{\infty} \Lambda_l(\omega) = \{\mu \in \Lambda^* \mid \mu\omega \in \Lambda^*\}.$$

Then for $k \in \mathsf{N}$ and $\nu \in \Lambda^*$, we have

$$\begin{aligned}
\Lambda_*(1) &= \Lambda^* \setminus \{\mu 12^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\};\\
\Lambda_*(2^k) &= \Lambda^* \setminus \{\mu 32^j1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\};\\
\Lambda_*(3\nu) &= \Lambda^* \setminus \{\mu 32^j1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 3\nu \in \Lambda^*;\\
\Lambda_*(4\nu) &= \Lambda^* \setminus \{\mu 42^j1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 4\nu \in \Lambda^*;\\
\Lambda_*(2^k3\nu) &= \Lambda^* \setminus \{\mu 32^j1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k3\nu \in \Lambda^*;\\
\Lambda_*(2^k4\nu) &= \Lambda^* \setminus \{\mu 32^j1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k4\nu \in \Lambda^*;\\
\Lambda_*(2^k1) &= \Lambda^* \setminus \{\mu 12^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\};\\
\Lambda_*(2^k12\nu) &= \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\}\\
&\qquad \text{where } 2^k12\nu \in \Lambda^*;
\end{aligned}$$

$$\Lambda_*(2^k13\nu) = \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k13\nu \in \Lambda^*;$$
$$\Lambda_*(2^k14\nu) = \Lambda^* \setminus \{\mu 12^j, \mu 42^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k14\nu \in \Lambda^*;$$
$$\Lambda_*(12\nu) = \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \text{ where } 12\nu \in \Lambda^*;$$
$$\Lambda_*(13\nu) = \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \text{ where } 13\nu \in \Lambda^*;$$
$$\Lambda_*(14\nu) = \Lambda^* \setminus \{\mu 12^j, \mu 42^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \text{ where } 14\nu \in \Lambda^*.$$

Thus for $l \geq 4$, we have that $m^*(l) = 5$,

$$\begin{aligned}
F_1^{*l} &= \{2^k12\nu, 2^k13\nu \in \Lambda^* \mid k \in \mathsf{N}_0,\ \nu \in \Lambda^*\},\\
F_2^{*l} &= \{2^k14\nu \in \Lambda^* \mid k \in \mathsf{N}_0,\ \nu \in \Lambda^*\},\\
F_3^{*l} &= \{2^k, 3\nu, 2^k3\nu, 2^k4\nu \in \Lambda^* \mid k \in \mathsf{N},\ \nu \in \Lambda^*\},\\
F_4^{*l} &= \{4\nu \in \Lambda^* \mid \nu \in \Lambda^*\},\\
F_5^{*l} &= \{2^k1 \in \Lambda^* \mid k \in \mathsf{N}_0\},
\end{aligned}$$

and

$$\mathscr{M}^*_{l,l+1} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 1 & 0\\ 0 & 3 & 2+3 & 2+3 & 3\\ 4 & 0 & 4 & 4 & 4\\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

It follows that

$$I^{*t}_{l,l+1} - A^{*t}_{l,l+1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0\\ 0 & 0 & -1 & 0 & 0\\ -1 & 0 & -1 & -1 & 0\\ 0 & -1 & -2 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

so that

$$K_1(\mathscr{O}_{\Lambda^*}) \cong \varinjlim \left(\mathrm{Ker}(I^{*t}_{l,l+1} - A^{*t}_{l,l+1}) \text{ in } \mathsf{Z}^{m^*(l)}\right) \cong \mathsf{Z},$$
$$K_0(\mathscr{O}_{\Lambda^*}) \cong \varinjlim \left(\mathsf{Z}^{m^*(l+1)}/(I^{*t}_{l,l+1} - A^{*t}_{l,l+1})\mathsf{Z}^{m^*(l)}\right) \cong \mathsf{Z}.$$

For a right infinite sequence $x \in X_\Lambda$ we put

$$\Lambda_*(x) = \bigcup_{l=1}^{\infty} \Lambda_l(x) = \{\mu \in \Lambda^* \mid \mu x \in \Lambda^*\}.$$

Then for $k \in \mathsf{N}$ and $x \in X_\Lambda$, we have

$$
\begin{aligned}
\Lambda_*(2^\infty) &= \Lambda^* \setminus \{\mu 32^j 1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\};\\
\Lambda_*(3x) &= \Lambda^* \setminus \{\mu 32^j 1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 3x \in X_\Lambda;\\
\Lambda_*(4x) &= \Lambda^* \setminus \{\mu 42^j 1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 4x \in X_\Lambda;\\
\Lambda_*(2^k 3x) &= \Lambda^* \setminus \{\mu 32^j 1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k 3x \in X_\Lambda;\\
\Lambda_*(2^k 4x) &= \Lambda^* \setminus \{\mu 32^j 1 \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \quad \text{where } 2^k 4x \in X_\Lambda;\\
\Lambda_*(2^k 12x) &= \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\}\\
&\qquad\qquad \text{where } 2^k 12x \in X_\Lambda;\\
\Lambda_*(2^k 13x) &= \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\}\\
&\qquad\qquad \text{where } 2^k 13x \in X_\Lambda;\\
\Lambda_*(2^k 14x) &= \Lambda^* \setminus \{\mu 12^j, \mu 42^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\}\\
&\qquad\qquad \text{where } 2^k 14x \in X_\Lambda;\\
\Lambda_*(12x) &= \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \ \text{where } 12x \in X_\Lambda;\\
\Lambda_*(13x) &= \Lambda^* \setminus \{\mu 12^j, \mu 32^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \ \text{where } 13x \in X_\Lambda;\\
\Lambda_*(14x) &= \Lambda^* \setminus \{\mu 12^j, \mu 42^j \in \Lambda^* \mid \mu \in \Lambda^*,\ j \in \mathsf{N}_0\} \ \text{where } 14x \in X_\Lambda.
\end{aligned}
$$

Thus for $l \geq 4$, we have that $m(l) = 4$,

$$
\begin{aligned}
F_1^l &= \{2^k 12x, 2^k 13x \in X_\Lambda \mid k \in \mathsf{N}_0,\ x \in X_\Lambda\},\\
F_2^l &= \{2^k 14x \in X_\Lambda \mid k \in \mathsf{N}_0,\ x \in X_\Lambda\},\\
F_3^l &= \{2^\infty, 3x, 2^k 3x, 2^k 4x \in X_\Lambda \mid k \in \mathsf{N},\ x \in X_\Lambda\},\\
F_4^l &= \{4x \in X_\Lambda \mid x \in X_\Lambda\}.
\end{aligned}
$$

Let $a_1, \dots, a_q$ be the set of all symbols in $\{1, 2, 3, 4\}$ for which $a_k \mu \in F_i^l$ for some $\mu \in F_j^{l+1}$. We then define the (i, j)-component of the matrix $\mathcal{M}_{l,l+1}(i, j)$ as $a_1 + \cdots + a_q$, the formal sum of $a_1, \dots, a_q$ and $A_{l,l+1}(i, j)$ as q respectively. We also define $I_{l,l+1}(i, j)$ as 1 if $F_j^{l+1} \subset F_i^l$, otherwise 0. It is straightforward to see that

$$
\mathcal{M}_{l,l+1} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 3 & 2+3 & 2+3 \\ 4 & 0 & 4 & 4 \end{bmatrix}
$$

and

$$I^t_{l,l+1} - A^t_{l,l+1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

so that

$$K_1(\mathcal{O}_\Lambda) \cong \varinjlim \left(\mathrm{Ker}(I^t_{l,l+1} - A^t_{l,l+1}) \ \text{ in } \ \mathsf{Z}^{m(l)}\right) \cong 0,$$
$$K_0(\mathcal{O}_\Lambda) \cong \varinjlim \left(\mathsf{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathsf{Z}^{m(l)}\right) \cong 0.$$

We note the four sets F^l_i, $i = 1, 2, \dots, 4$ for $l \geq 4$ exactly correspond to the vertex set of the left Krieger cover graph for the subshift Λ. The vertex set of the graph is finite so that Λ is sofic and also irreducible because the adjacency matrix of the graph is given by the irreducible matrix $\mathcal{M}_{l,l+1}$ (cf. [13], [14]). Therefore we obtain

THEOREM 4.1. *Let Λ be the subshift over $\{1, 2, 3, 4\}$ defined by the forbidden words*

$$\mathfrak{F} = \{12^k1, 32^k12, 32^k13, 42^k14 \mid k = 0, 1, 2, \dots\}$$

Then Λ is an irreducible sofic shift such that the K-groups for the associated C^*-algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are

$$K_0(\mathcal{O}_{\Lambda^*}) \cong K_1(\mathcal{O}_{\Lambda^*}) \cong \mathsf{Z}, \qquad K_0(\mathcal{O}_\Lambda) \cong K_1(\mathcal{O}_\Lambda) \cong 0.$$

Hence the C^*-algebras $\mathcal{O}_{\Lambda^*}$ and $\mathcal{O}_\Lambda$ are not stably isomorphic.

We remark that $\mathcal{O}_\Lambda$ is simple and purely infinite whereas $\mathcal{O}_{\Lambda^*}$ is not simple.

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MATHEMATICS INSTITUTE
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5,
DK-2100 COPENHAGEN
DENMARK
E-mail: toke@math.ku.dk

DEPARTMENT OF MATHEMATICAL SCIENCES
YOKOHAMA CITY UNIVERSITY
SETO 22-2, KANAZAWA-KU
YOKOHAMA 236-0027
JAPAN
E-mail: kengo@yokohama-cu.ac.jp