ONE DIMENSION HIGHER OF THE WORD PROBLEM

A. SINAN ÇEVIK

Abstract

Just as a group presentation $\mathscr{P}$ can be regarded as a 2-complex with a single 0-cell, so we can consider a 3-complex with a single 0-cell, known as a 3-presentaton. In this paper, by using a geometric way, called spherical pictures, we show that there exist a finite 3-presentation which has unsolvable generalised identity problem which can be thought as one-dimension higher of the word problem.

1. Introduction

Let G be a group defined by a presentation

$$(1) \qquad \mathscr{P} = \langle \mathbf{x} \,;\, \mathbf{r} \rangle.$$

Therefore we have a free group $F(\mathbf{x})$ on $\mathbf{x}$, a normal closure set N of $\mathbf{r}$ in $F(\mathbf{x})$ and a quotient group, that is the group defined by $\mathscr{P}$, $G(\mathscr{P}) = F(\mathbf{x})/N$ such that $G = G(\mathscr{P})$ for finite presentation $\mathscr{P}$. We recall that a typical element of $G(\mathscr{P})$ is represented by $\overline{W} = [W]N$ where W is a word on $\mathbf{x}$ and $[W]$ is the free equivalence class of W.

The word problem is the question of asking for the existence of an algorithm to determine of an arbitrary word W on $\mathbf{x}$ whether or not $\overline{W} = 1$ in G.

Let H be a finitely generated subgroup of G. Then the *subgroup word problem* for H in G is the problem of deciding for an arbitrary word W on $\mathbf{x}$, whether or not W defines an element of H. It is clear that if H is trivial then the subgroup word problem is just simply the word problem.

If we think $\mathscr{P}$ as a finite connected 2-complex then the word problem for $\mathscr{P}$ is the problem of determining the existence of algorithm to decide whether for any arbitrary element of the first homotopy group $\pi_1(\langle \mathbf{x} \rangle)$, its image under the inclusion induced homomorphism

$$\psi_* : \pi_1(\langle \mathbf{x} \rangle) \longrightarrow \pi_1(\langle \mathbf{x} \,;\, \mathbf{r} \rangle)$$

is trivial. The main unsolvabilty result belong to Boone [4] and Novikov [12] which states that *there exists a finitely presented group with unsolvable word problem.*

As a natural extension of the above formulation of the word problem, for a finite connected 3-complex $\mathcal{K}$, one can ask whether for any arbitrary element of the second homotopy group $\pi_2(\mathscr{P})$, its image under the inclusion induced homomorphism

$$\theta_* : \pi_2(\mathscr{P}) \longrightarrow \pi_2(\mathcal{K})$$

is trivial.

Since a 2-complex with a single 0-cell can be regarded just as a presentation, we can consider a 3-complex with a single 0-cell, which is known as a 3-*presentation*. A 3-presentation $\mathcal{K}$ is a triple

$$\langle \mathbf{x} \, ; \, \mathbf{r} \, ; \, \mathbf{Y} \rangle$$

where $\mathbf{Y}$ is a set of spherical pictures over $\mathscr{P}$. We say that $\mathcal{K}$ is finite if $\mathbf{x}$, $\mathbf{r}$ and $\mathbf{Y}$ are all finite. In fact this definition comes from the definition of an extension group presentation, defined by Fenn [9].

By using [3, Theorem 1.6] and [14, Theorem 2.6], we can formulate the generalised identity problem as "*is there an algorithm to decide for any spherical picture* P *over* $\mathscr{P} = \langle \mathbf{x} \, ; \, \mathbf{r} \rangle$ *whether* P *is equivalent (relative to* $\mathbf{Y}$) *to the empty picture, for a given finite 3-presentation* $\mathcal{K} = \langle \mathbf{x} \, ; \, \mathbf{r} \, ; \, \mathbf{Y} \rangle$?"

In this paper we work on the generalised identity problem which is the analogue of the one dimension higher of the word problem, by using the above contruction on 3-complexes.

Thus the main result of this paper is the following.

THEOREM 1.1. *There is a finite 3-presentation* $\mathcal{K}$ *with unsolvable generalised identity problem. Furthermore, since* $\mathcal{K}$ *can be chosen, the word problem for the presentation* $\mathscr{P}$ *is solvable.*

REMARK 1.2. It is well known that the word problem plays a central role in decision problems. Thus one can generate the above theorem to the global properties such as the p-Cockcroft properties or the efficiency. The reader can find the details and some applications of the p-Cockcroft property and the efficiency, for instance, in [6], [7].

By the paper written by Otto ([13]) it has been stated that the word problem for groups is reducible to the conjugacy problem, and so the decidability of the conjugacy problem implies the decidability of the word problem but the situation is different for monoids. As a result of this, he showed that the word problem and the conjugacy problem are independent of one another for monoids. Therefore, for a future project, by using the same technique and similar

construction as in our this paper, the same result (Theorem 1.1 above) has been obtained seperately for monoids in the joint paper [8] written by Cevik and Ates.

2. Preliminaries

In this section we recover some basic concepts briefly which are needed in the next section. We refer to the reader [14] for the details.

2.1. Spherical pictures

Let G be a finitely presented group with the presentaton $\mathscr{P}$, as in (1). If we regard $\mathscr{P}$ as a 2-complex with single 0-cell whose 1-cells are in bijective correspondence with the elements of $\mathbf{x}$, and whose 2-cells are attached by the boundary path determined by the spelling of the correponding element of $\mathbf{r}$ in the standard way, then G is just the fundamental group of $\mathscr{P}$. Thus there is also, of course, the second homotopy group $\pi_2(\mathscr{P})$ of $\mathscr{P}$, which is a left $\mathsf{Z}G$-module. The elements of $\pi_2(\mathscr{P})$ can be represented by geometric configurations called *spherical pictures*. These are described in detail in [3] and [14]. In this paper we need only one basepoint on each disc of our pictures (so we will actually use $*$-pictures, as described in Section 2.4 of [14]). Also, as described in [14], there are certain operations on spherical pictures.

For a non-spherical picture Q over $\mathscr{P}$, we have the term *boundary label* which is the word read off by travelling around boundary of Q once in the clockwise direction starting from the basepoint of the Q. If a word W in $\mathbf{x} \cup \mathbf{x}^{-1}$ represents an element in the normal closure N of $\mathbf{r}$ in $F(\mathbf{x})$, then it is easy to construct a based picture over $\mathscr{P}$ with boundary label W. So there is the following pictorial version of the "van Kampen lemma" [5], [10].

PROPOSITION 2.1 ([3, Theorem 1.4]). *A word W in $\mathbf{x} \cup \mathbf{x}^{-1}$ represents an element of N, that is $\overline{W} = 1$ in $G(\mathscr{P})$, if and only if there is a based picture over $\mathscr{P}$ with boundary label W.*

Suppose $\mathbf{Y}$ is a collection of spherical pictures over P. Then, by [14], one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *equivalence (rel $\mathbf{Y}$) of spherical pictures*. We note that an example of using pictures can be found in [6].

In [14], Pride also proved the following important result about the equivalence of spherical pictures.

THEOREM 2.2. *The elements $\langle \mathsf{P} \rangle$ ($\mathsf{P} \in \mathbf{Y}$) generate $\pi_2(\mathscr{P})$ as a module if and only if every spherical picture is equivalent (rel $\mathbf{Y}$) to the empty picture.*

Therefore it is easy to see that if the elements $\langle \mathsf{P} \rangle$ ($\mathsf{P} \in \mathbf{Y}$) generate $\pi_2(\mathscr{P})$ then we say that $\mathbf{Y}$ *generates* $\pi_2(\mathscr{P})$.

2.2. *Projective resolutions*

Let

$$\longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathsf{Z} \longrightarrow 0$$

be any arbitrary projective resolution of the trivial G-module Z. This resolution is called *n-finite* if $P_0, P_1, \ldots, P_n$ are finitely generated. We say that G is *of type FP_n* $(0 \leq n < \infty)$ if G has an *n*-finite projective resolution. If the group G is given by a presentation $\mathscr{P}$, as in (1), then we have the Lyndon resolution

(2) $$0 \longrightarrow \pi_2(\mathscr{P}) \xrightarrow{\mu} P_2 \longrightarrow P_1 \longrightarrow \mathsf{Z}G \longrightarrow \mathsf{Z} \longrightarrow 0$$

where P_1 and P_2 are the free left $\mathsf{Z}G$-modules with basis $\{e_x : x \in \mathbf{x}\}$ and $\{e_R : R \in \mathbf{r}\}$, respectively. By [14], the embedding μ is given as follows. Let $\langle \mathsf{P} \rangle \in \pi_2(\mathscr{P})$ and suppose P has discs $\Delta_1, \Delta_2, \ldots, \Delta_n$, which each of them has single basepoint, with clockwise label $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \ldots, R_n^{\varepsilon_n}$ respectively ($R_i \in \mathbf{r}$, $\varepsilon_= \pm 1$ and $i = 1, 2, \ldots, n$). Let γ_i be a transverse path from the basepoint of P to the basepoint of each Δ_i. Let W_i be the label on γ_i. Then

$$\mu(\langle \mathsf{P} \rangle) = \sum_{i=1}^{n} \varepsilon_i g_i e_{R_i}$$

where g_i is the element of G represented by W_i.

3. Proof of the main theorem

To construct this section we will pick a special group M (see below) and then we will use it to obtain a finite 3-presentation $\mathscr{K}$. By taking an arbitrary word W on the generating set of M, we will consider the set of P_W (where each P_W is a spherical picture over the presentation of M) and show that P_W is equivalent to the empty picture if and only if W defines an element of a special normal subgroup of M, say L (see below). Thus we can conclude that $\mathscr{K}$ has unsolvable generalised identity problem since, by the assumption on L, the subgroup word problem for L in M is unsolvable.

Let M be finitely presented group defined by $\mathscr{P}_M = \langle \mathbf{x} ; \mathbf{r} \rangle$ such that

1) the word problem for the group M is solvable,

2) $\pi_2(\mathscr{P}_M)$ is finitely generated,

3) there exists a finitely generated normal subgroup L of M such that the subgroup word problem for L in M is unsolvable.

We should note that the above analogue is a direct combination of the Novikov-Boone Theorem ([4], [12]) and the so-called *Rips construction* [15].

We could take the group of Miller [2, Corollary 1] for a specific example of a such group. One of the fact for this choise is the Miller group has an aspherical presentation and so $\pi_2(\mathscr{P}_M) = 0$. Therefore the second condition holds.

Let Z_2 be cyclic group of order 2 with a presentation $\mathsf{P}_{\mathsf{Z}_2} = \langle s \; ; \; s^2 \rangle$ and let T be a group $M \times \mathsf{Z}_2$ given by the presentation

(3) $\mathscr{P}_T = \langle \mathbf{x}, s \; ; \; \mathbf{r}, s^2, [x, s] \, (x \in \mathbf{x}) \rangle.$

We note that T has solvable word problem since M and Z_2 have both solvable word problem.

We need the following definition for the proof.

DEFINITION 3.1.   Let $x_1^{\varepsilon_1} \ldots x_j^{\varepsilon_j} x_{j+1}^{\varepsilon_{j+1}} \ldots x_n^{\varepsilon_n}$ be a word on $\mathbf{x}$. Then a *commutator picture* $\mathsf{D}_{x_1^{\varepsilon_1} \ldots x_j^{\varepsilon_j} x_{j+1}^{\varepsilon_{j+1}} \ldots x_n^{\varepsilon_n}}$ is a picture over $\langle \mathbf{x}, s \; ; \; [x, s] \, (x \in \mathbf{x}) \rangle$ of the form as depicted in Figure 1(a). Moreover, if $\mathsf{D}_{x_1^{\varepsilon_1} \ldots x_j^{\varepsilon_j} x_{j+1}^{\varepsilon_{j+1}} \ldots x_n^{\varepsilon_n}}$ is a picture over $\langle \mathbf{x}, s \; ; \; s^2, [x, s] \, (x \in \mathbf{x}) \rangle$, then it is equivalent to a picture as shown in Figure 1(b).
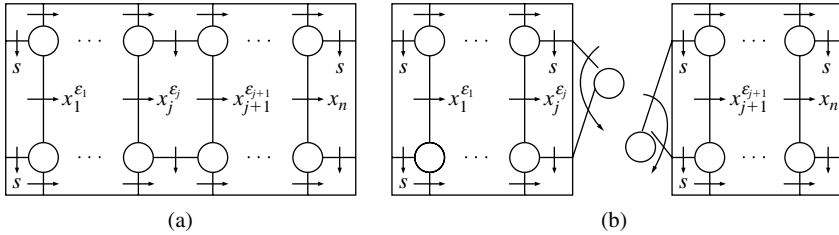

FIGURE 1

Suppose that $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ is a set of words on $\mathbf{x}$ which represents a finite set of generators of L. Let $\mathbf{Y_1}$ be a finite set of spherical pictures which generates $\pi_2(\mathscr{P}_M)$, and for each $R \in \mathbf{r}$, let $\mathbf{Y_2}$ be the finite set of spherical pictures (see Figure 2(a)) over the presentation $\mathscr{P}_T$, as given in (3).

Also let $\mathbf{Y_3}$ consists of the single picture as drawn in Figure 2(b). Finally, for each $w_i \in \mathbf{w}$, let $\mathbf{Y_4}$ be a finite set of spherical pictures over $\mathscr{P}_T$ of the form as shown in Figure 3(a).

We note that the subpicture D_i is a commutator picture and fixed over the presentation $\langle \mathbf{x}, s \; ; \; [x, s] \, (x \in \mathbf{x}) \rangle$.

Let $\mathbf{Y} = \mathbf{Y_1} \cup \mathbf{Y_2} \cup \mathbf{Y_3} \cup \mathbf{Y_4}$. Since each $\mathbf{Y}_j$ $(1 \leq j \leq 4)$ is finite, $\mathbf{Y}$ is finite. Therefore we have a finite 3-presentation

$$\mathscr{K} = \langle \mathbf{x}, s \; ; \; \mathbf{r}, s^2, [x, s] \, (x \in \mathbf{x}) \; ; \; \mathbf{Y} \rangle$$
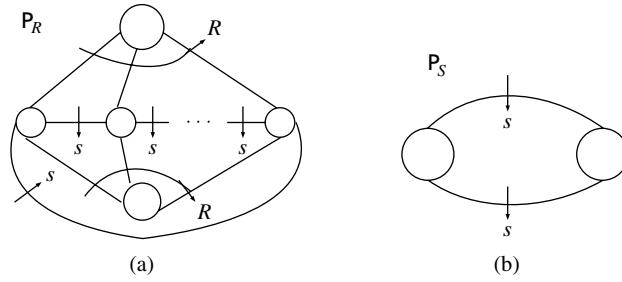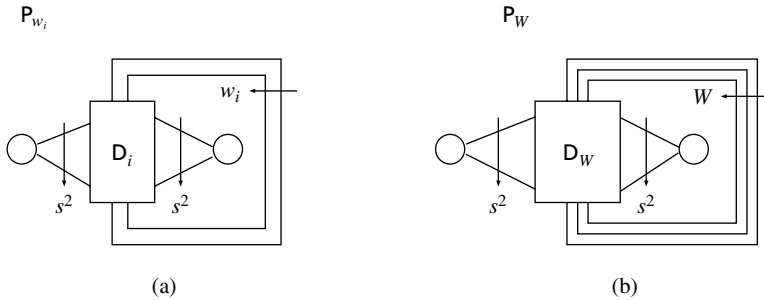
FIGURE 2

FIGURE 3

such that the underlying presentation $\mathscr{P}_T$ has solvable word problem.

Now suppose that M is finitely presented group defined by $\mathscr{P}_M$ as above. For any word W on $\mathbf{x}$, let P_W be a spherical picture of the form as shown in Figure 3(b). As in D_i, we again note that the commutator subpicture D_W is fixed over the presentation $\langle \mathbf{x}, s \,;\, [x, s] \ (x \in \mathbf{x}) \rangle$.

Now, we will show that P_W is equivalent to the empty picture (relative to $\mathbf{Y}$) *if and only if* W defines an element of L, and hence $\mathscr{K}$ has unsolvable generalised identity problem since the subgroup word problem for L in M is unsolvable. Let us start to show this with the sufficiency part. So let us assume that W defines an element of L. Then $\overline{W} = \overline{w_1^{\varepsilon_1} w_2^{\varepsilon_2} \dots w_n^{\varepsilon_n}}$ in M for some w_i's which belong to $\mathbf{w}$ and $\varepsilon_i = \pm 1$. Thus, by Proposition 2.1, there is a picture Q over $\mathscr{P}_M = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ with boundary label $w_1^{\varepsilon_1} w_2^{\varepsilon_2} \dots w_n^{\varepsilon_n} W^{-1}$. Now let us consider the picture P_Q^1 as shown in Figure 4(a). We note that, by *cancelling pair operation* (see [14]), P_Q^1 is equivalent to the picture P_Q^2, as depicted in Figure 4(b). By [1], the set $\mathbf{Y}_1 \cup \mathbf{Y}_2$ generates $\pi_2(\langle \mathbf{x}, s \,;\, \mathbf{r}, [x, s] \ (x \in \mathbf{x}) \rangle)$. Then, by Theorem 14, P_Q^2 is equivalent (relative to $\mathbf{Y}_1 \cup \mathbf{Y}_2$) to the empty picture and so is P_Q^1.
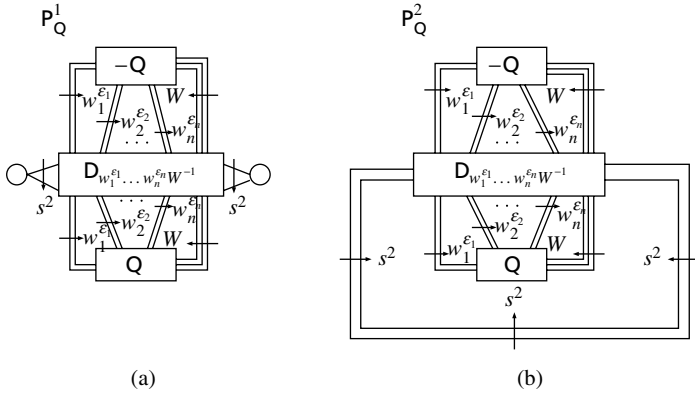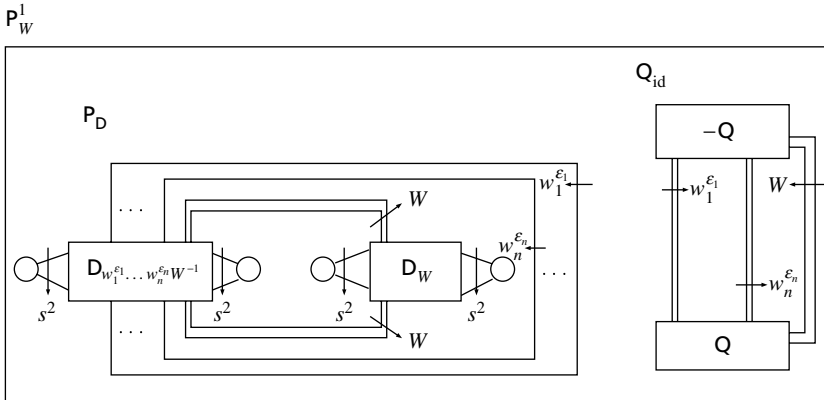
FIGURE 4

FIGURE 5

Now, by inserting P_Q^1 to the left side of P_W and performing some *bridge move operations* (see [14]), we have the picture P_W^1, as in Figure 5, which contains two subpictures P_D and Q_{id}.

Since the subpicture Q_{id} is equivalent to the empty picture (we can think it as a cancelling pair), we delete it. Moreover, by Definition 3.1, the subpicture P_D becomes the picture P_D' as in Figure 6. In this picture we can delete the subpicture S_W, depicted in Figure 7, since it is equivalent to the empty picture (by applying some bridge move and cancelling pair operations on it).

Furthermore, by applying a sequence of Definition 3.1, we obtain the picture P_W^2 (see Figure 8) which is equivalent (relative to Y_3) to the empty picture.

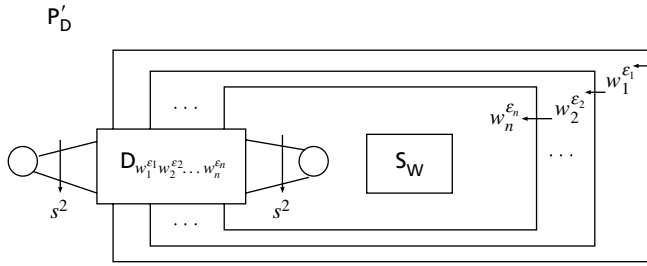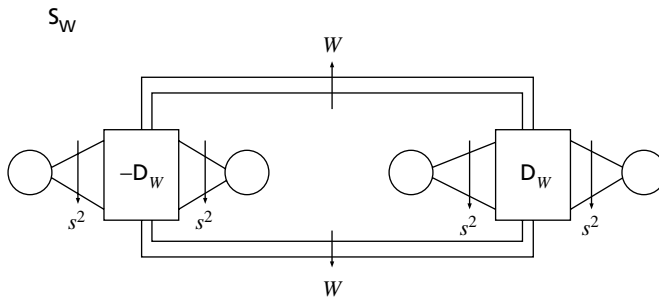These all above processes give us that the picture P_W is equivalent (relative

P'_D

FIGURE 6

S_W

FIGURE 7

P^2_W

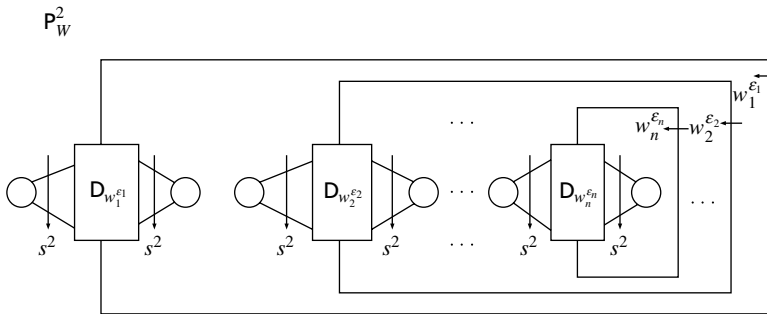FIGURE 8

to **Y**) to the empty picture, as required.

For the necessity part of the above process, we will show that if W does not define any element of L, then P_W is not equivalent to the empty picture (relative to **Y**) over $\mathscr{P}_T$.

Suppose that P_W can be obtained from **Y**. Let P_2 be the free $\mathbf{Z}T$-module with basis $\{e_R : R \in \mathbf{r}\} \cup \{e_{s^2}\} \cup \{e_{[x,s]} : x \in \mathbf{x}\}$. By considering the exact

sequence, as given in (2), we will determine the image of P_W in P_2. We recall that each γ (as used in below) defines a transverse path from the basepoint of the concerned picture to the basepoint of each discs of this picture.

Let us take $P_2 = \mathsf{Z}T e_{s^2} \oplus P_2'$, where P_2' is the free $\mathsf{Z}T$-module with the above basis excluding $\{e_{s^2}\}$. Then the image of P_W in P_2 is

$$\gamma_W = (\overline{W} - 1)e_{s^2} + \lambda_W,$$

for some $\lambda_W \in P_2'$, and the image of P_{w_i} $(\mathsf{P}_{w_i} \in \mathbf{Y}_4)$ is

$$\gamma_i = (\overline{w_i} - 1)e_{s^2} + \lambda_i,$$

for some $\lambda_i \in P_2'$. Also let the image of each P_R $(R \in \mathbf{r})$ be γ_R and the image of B $(\mathsf{B} \in \mathbf{Y}_1)$ be γ_{B}. We should note that γ_R and γ_{B} contained in P_2'.

By the assumption, since P_W is obtainable from $\mathbf{Y}$, we then have

$$\gamma_W = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_n\gamma_n + \alpha_0\gamma_{s^2} + \sum_{R \in \mathbf{r}} \alpha_R\gamma_R + \sum_{\mathsf{B} \in \mathbf{Y}_1} \alpha_{\mathsf{B}}\gamma_{\mathsf{B}},$$

where each of the α and β belongs to $\mathsf{Z}T$. Let us equate the coefficients of e_{s^2}. Then we get

$$\overline{W} - 1 = \beta_1(\overline{w_1} - 1) + \beta_2(\overline{w_2} - 1) + \cdots + \beta_n(\overline{w_n} - 1) + \alpha_0(\overline{s} - 1).$$

After all, if we consider the induced ring homomorphism

$$\mathsf{Z}T \longrightarrow \mathsf{Z}M \longrightarrow \mathsf{Z}(M/L), \quad \overline{x} \longmapsto \overline{x} \longmapsto \overline{x}L \; (x \in \mathbf{x}), \;\; \overline{s} \longmapsto 1 \longmapsto 1L$$

arising from the group homomorphism, then we have $\overline{W}L - 1L = 0$. In other words W defines the element $\overline{W}$ of L which makes contradiction to our assumption.

Hence the result.

FINAL REMARK 3.2.   The finite 3-presentation that we produce in our main result (Theorem 1.1) can be taken to have as two-skeleton a presentation $\mathscr{P}$ for a group G that not only has solvable word problem, but which is also of homological finiteness type F_∞: the group G admits a $K(G, 1)$ with finitely many cells in each dimension. To identify a finite three-skeleton, we need only replace the set $\mathbf{Y}_4$ of spherical pictures P_{w_i} $(w_i \in \mathbf{w})$ with the analogous pictures P_x $(x \in \mathbf{x})$.

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BALIKESIR UNIVERSITESI
FEN-EDEBIYAT FAKULTESI
MATEMATIK BOLUMU
10100, BALIKESIR
TURKEY
E-mail: scevik@balikesir.edu.tr