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ON THE DIOPHANTINE SYSTEM $x^2 - Dy^2 = 1 - D$ AND $x = 2z^2 - 1$

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Abstract

Let D be a positive integer such that $D - 1$ is an odd prime power. In this paper we give an elementary method to find all positive integer solutions (x, y, z) of the system of equations $x^2 - Dy^2 = 1 - D$ and $x = 2z^2 - 1$. As a consequence, we determine all solutions of the equations for $D = 6$ and 8.

1. Introduction

Let Z, N be the sets of all integers and positive integers respectively. Let D be a positive integer with $D > 1$. The determination of all solutions (x, y, z) of the system of equations

$$(1) \quad x^2 - Dy^2 = 1 - D, \quad x = 2z^2 - 1, \qquad x, y, z \in \mathsf{N}, \ \gcd(x, y) = 1$$

is an interesting problem concerning the arithmetic properties of recurrence sequences and the solution of exponential-polynomial equations over real quadratic fields. In 1995 Mignotte and Pethö [9] determined all solution (x, y, z) for $D = 6$. Their proof relied upon deep tools related to linear form in logarithms and reduction technigues. In 1998, Cohn [4] gave an elementary proof of the above mentioned result.

In this paper we give an elementary method to find all solutions of (1) for the general case that $D - 1$ is an odd prime power. We now introduce some needful notations and known results given by Petr [10].

LEMMA 1. *Let D be a nonsquare positive integer, and let $u_1 + v_1\sqrt{D}$ be the fundamental solution of Pell equation*

$$(2) \qquad u^2 - Dv^2 = 1, \qquad u, v \in \mathsf{Z}.$$

Then we have

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(i) *All solutions* (u, v) *of* (2) *can be expressed as*

$$u + v\sqrt{D} = \left(u_1 + v_1\sqrt{D}\right)^t, \qquad t \in \mathbb{Z}. \tag{3}$$

(ii) *For any positive integer n, let*

$$u_n + v_n\sqrt{D} = \left(u_1 + v_1\sqrt{D}\right)^n. \tag{4}$$

Then $(u, v) = (u_n, v_n)$ $(n \in \mathbb{N})$ *are all positive integer solutions of* (2).

LEMMA 2. *Let D be an even nonsquare positive integer, and let*

$$D' = \begin{cases} D, \\ \dfrac{D}{4}, \end{cases} \qquad \varepsilon = \begin{cases} 1, & \text{if } v_1 \text{ is even,} \\ 2, & \text{if } v_1 \text{ is odd.} \end{cases} \tag{5}$$

For a fixed D, there exists a unique positive integers pair (D_1, D_2) *such that* $D_1 > 1$, $D_1 D_2 = D'$ *and the equation*

$$D_1U^2 - D_2V^2 = 1, \qquad U, V \in \mathbb{N} \tag{6}$$

has solution (U, V).

LEMMA 3. *Let* D_1, D_2 *be positive integers with* $D_1 > 1$. *If* (6) *has solutions* (U, V), *then it has a unique solution* (U_1, V_1) *satisfying* $V_1 \le V$, *where V runs through all solutions* (U, V) *of* (6). *The solution* (U_1, V_1) *is called the least solution of* (6). *Then we have:*

(i) $\left(U_1\sqrt{D_1} + V_1\sqrt{D_2}\right)^2 = u_1 + v_1\sqrt{D}$.

(ii) *For any odd positive integer m, let*

$$U_m\sqrt{D_1} + V_m\sqrt{D_2} = \left(U_1\sqrt{D_1} + V_1\sqrt{D_2}\right)^m. \tag{7}$$

Then $(U, V) = (U_m, V_m)$ *for* $m = 1, 3, \ldots$ *are all solutions of* (6).

Under the mentioned notations, using a result of [5], we prove a general result as follows.

THEOREM. *Let D be a positive integer such that* $D-1$ *is an odd prime power. If D is a square, then* $D = 4$ *and* (1) *has only the solution* $(x, y, z) = (1, 1, 1)$. *If D is not a square, then all solutions of* (1) *can be classified into the following five shapes.*

(i) $(x, y, z) = (1, 1, 1)$.

(ii) $(x, y, z) = \left(u_{2n} + Dv_{2n}, u_{2n} + v_{2n}, \sqrt{u_n(u_n + Dv_n)}\right)$.

(iii) $(x, y, z) = \left(-u_{2n} + Dv_{2n}, u_{2n} - v_{2n}, \sqrt{Dv_n(u_n - v_n)}\right)$.

(iv) $(x, y, z) = \left(u_m + Dv_m, u_m + v_m, \sqrt{D_1 U_m (U_m + \varepsilon D_2 V_m)}\right)$.

(v) $(x, y, z) = \left(-u_m + Dv_m, u_m - v_m, \sqrt{D_2 V_m (\varepsilon D_1 U_m - V_m)}\right)$.

Under the assumption that $D - 1$ is an odd prime power, using our theorem and some known results of quartic diophantine equations (see [1], [2], [3], [6], [7], [8], [11]), we can find all solutions of (1) with ease. As an example, we prove the following corollary.

COROLLARY. *If* $D = 6$, *then* (1) *has only the solutions* $(x, y, z) = (1, 1, 1)$, $(7, 3, 2)$, $(17, 7, 3)$, $(71, 29, 6)$, $(16561, 6761, 91)$. *If* $D = 8$, *then* (1) *has only the solution* $(x, y, z) = (1, 1, 1)$, $(31, 11, 4)$.

2. Proof of the Theorem

LEMMA 4. *For any positive integer n, we have* $u_{2n} + 1 = 2u_n^2$, $u_{2n} - 1 = 2Dv_n^2$, $v_{2n} = 2u_n v_n$.

PROOF. Let

$$\alpha = u_1 + v_1\sqrt{D}, \qquad \overline{\alpha} = u_1 - v_1\sqrt{D}. \tag{8}$$

Since $u_1^2 - Dv_1^2 = 1$, we get from (8) that

$$\alpha + \overline{\alpha} = 2u_1, \qquad \alpha - \overline{\alpha} = 2v_1\sqrt{D}, \qquad \alpha\overline{\alpha} = 1. \tag{9}$$

By (4) and (8), we obtain

$$u_n = \frac{1}{2}(\alpha^n + \overline{\alpha}^n), \quad v_n = \frac{1}{2\sqrt{D}}(\alpha^n - \overline{\alpha}^n), \qquad n \in \mathsf{N}. \tag{10}$$

Therefore, by (9) and (10), we get

$$\begin{aligned}
u_{2n} + 1 &= \frac{1}{2}(\alpha^{2n} + \overline{\alpha}^{2n}) + 1 = \frac{1}{2}(\alpha^{2n} + 2(\alpha\overline{\alpha})^n + \overline{\alpha}^{2n}) \\
&= \frac{1}{2}\left(\alpha^n + \overline{\alpha}^n\right)^2 = 2u_n^2, \\
u_{2n} - 1 &= \frac{1}{2}(\alpha^{2n} + \overline{\alpha}^{2n}) - 1 = \frac{1}{2}(\alpha^{2n} - 2(\alpha\overline{\alpha})^n + \overline{\alpha}^{2n}) \\
&= \frac{1}{2}\left(\alpha^n - \overline{\alpha}^n\right)^2 = 2Dv_n^2, \\
v_{2n} &= \frac{1}{2\sqrt{D}}(\alpha^{2n} - \overline{\alpha}^{2n}) = \frac{1}{2\sqrt{D}}(\alpha^n + \overline{\alpha}^n)(\alpha^n - \overline{\alpha}^n) = 2u_n v_n.
\end{aligned}$$

The lemma is proved.

LEMMA 5. *For any odd positive integer m, we have* $u_m + 1 = 2D_1U_m^2$, $u_n - 1 = 2D_2V_m^2$, $v_m = 2U_mV_m/\varepsilon$.

PROOF. Let

$$\beta = U_1\sqrt{D_1} + V_1\sqrt{D_2}, \qquad \overline{\beta} = U_1\sqrt{D_1} - V_1\sqrt{D_2}. \tag{11}$$

Since $D_1U_1^2 - D_2V_1^2 = 1$, we get from (11) that

$$\beta + \overline{\beta} = 2U_1\sqrt{D_1}, \qquad \beta - \overline{\beta} = 2V_1\sqrt{D_2}, \qquad \beta\overline{\beta} = 1. \tag{12}$$

By (7) and (11), we obtain

$$U_m = \frac{1}{2\sqrt{D_1}}(\beta^m + \overline{\beta}^m), \qquad V_m = \frac{1}{2\sqrt{D_2}}(\beta^m - \overline{\beta}^m). \tag{13}$$

By (i) of Lemma 3, we have $\alpha = \beta^2$ and $\overline{\alpha} = \overline{\beta}^2$. Hence, by (10), (12) and (13), we get

$$\begin{aligned} u_m + 1 &= \frac{1}{2}(\beta^{2m} + \overline{\beta}^{2m}) + 1 = \frac{1}{2}(\beta^m + \overline{\beta}^m)^2 = 2D_1U_m^2, \\ u_m - 1 &= \frac{1}{2}(\beta^{2m} + \overline{\beta}^{2m}) - 1 = \frac{1}{2}(\beta^m - \overline{\beta}^m)^2 = 2D_2V_m^2, \\ v_m &= \frac{1}{2\sqrt{D}}(\beta^{2m} - \overline{\beta}^{2m}) = \frac{1}{2\sqrt{D}}(\beta^m + \overline{\beta}^m)(\beta^m - \overline{\beta}^m) \\ &= 2\frac{\sqrt{D}'}{\sqrt{D}}U_mV_m = \frac{2}{\varepsilon}U_mV_m. \end{aligned}$$

The lemma is proved.

LEMMA 6. *Let D be a nonsquare positive integer, and let k be an integer with* $|k| > 1$. *If the equation*

$$X^2 - DY^2 = k, \qquad X, Y \in \mathbb{Z}, \;\; \gcd(X, Y) = 1 \tag{14}$$

has solutions (X, Y), *then all solutions* (X, Y) *of* (14) *can be classified into* $2^{\omega(k)-1}$ *classes, where* $\omega(k)$ *is the number of distinct prime divisors of k. Further, every class of solutions of* (14) *contain a unique solution* (X_1, Y_1) *such that* $X_1 > 0$, $Y_1 > 0$ *and*

$$1 < \left|\frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}}\right| < \left(u_1 + v_1\sqrt{D}\right)^2, \tag{15}$$

where $u_1 + v_1\sqrt{D}$ is the fundamental solution of (2). *Then,* (X_1, Y_1) *is called the least solution of the class. Furthermore, if* (X_1, Y_1) *is the least solution of a certain class, then every solution* (X, Y) *of the class can be expressed as*

$$(16) \qquad X + Y\sqrt{D} = \left(X_1 + \lambda_1 Y_1\sqrt{D}\right)\left(u + v\sqrt{D}\right), \qquad \lambda_1 \in \{1, -1\},$$

where (u, v) *is a solution of* (2).

PROOF. This lemma is the special case of Theorems 1 and 2 of [5] for $D_1 = 1$ and $D_2 > 0$.

LEMMA 7. *Let D be a nonsquare positive integer. If $D - 1$ is an odd prime power, then the equation*

$$(17) \qquad X^2 - DY^2 = 1 - D, \qquad X, Y \in \mathbb{Z}, \ \gcd(X, Y) = 1$$

has solutions (X, Y). *Moreover, every solution* (X, Y) *of* (17) *can be expressed as*

$$(18) \qquad X + Y\sqrt{D} = \left(1 + \lambda_1\sqrt{D}\right)\left(u + v\sqrt{D}\right), \qquad \lambda_1 \in \{1, -1\},$$

where (u, v) *is a solution of* (2).

PROOF. Since $D-1$ is an odd prime power and $(X, Y) = (1, 1)$ is a solution of (17), by Lemma 6, all solutions of (17) belong to a unique class. Since D is not a square, we have $D \geq 6$ and

$$(19) \qquad 1 < \left|\frac{1 + \sqrt{D}}{1 - \sqrt{D}}\right| = 1 + \frac{2}{\sqrt{D} - 1} < 1 + \sqrt{D} < \left(u_1 + v_1\sqrt{D}\right)^2.$$

It implies that $(X_1, Y_1) = (1, 1)$ is the least solution of the class. Thus, by Lemma 6, we obtain (18) immediately. The lemma is proved.

PROOF OF THE THEOREM. Let (x, y, z) be a solution of (1). We first consider the case that D is a square. Since $D-1$ is an odd prime power, we have $D = 4$. Then, by the first equation of (1), we get $2y+x = 3$ and $2y-x = 1$. It follows that $x = y = 1$. Hence, by the second equation of (1), we get $z = 1$. Therefore, (1) has only the solution $(x, y, z) = (1, 1, 1)$ for $D = 4$.

We next consider the case that D is not a square. Then D is an even integer with $D \geq 6$. By Lemma 7, we get from the first equation of (1) that

$$(20) \qquad x + y\sqrt{D} = \left(1 + \lambda_1\sqrt{D}\right)\left(u + v\sqrt{D}\right), \qquad \lambda_1 \in \{1, -1\},$$

where (u, v) is a solution of (2).

If $v = 0$, then $u = \pm 1$ and $x = y = 1$ by (20). It implies that $(x, y, z) = (1, 1, 1)$ and the solution is of the shape (i).

If $v \neq 0$, then $(|u|, |v|)$ is a positive integer solution of (2). Hence, by (ii) of Lemma 1, we get from (20) that

$$x+y\sqrt{D}=\left(1+\lambda_1\sqrt{D}\right)\left(\lambda_2 u_r+\lambda_3 v_r\sqrt{D}\right), \qquad \lambda_1, \lambda_2, \lambda_3 \in \{1, -1\}. \tag{21}$$

where r is a suitable positive integer. Since $D \geq 6$, we have

$$Dv_r > v_r\sqrt{D}+1 > v_r\sqrt{D}+\frac{1}{u_r+v_r\sqrt{D}} = u_r > v_r\sqrt{D} > v_r. \tag{22}$$

Therefore, by (21) and (22), we obtain either

$$x = u_r + Dv_r, \qquad y = u_r + v_r \tag{23}$$

or

$$x = -u_r + Dv_r, \qquad y = u_r - v_r. \tag{24}$$

If (23) holds and r is even, then $r = 2n$, where n is a positive integer. By Lemma 4, we get from (23) and the second equation of (1) that

$$2z^2 = x+1 = (u_{2n}+1)+Dv_{2n} = 2u_n^2+2Du_nv_n = 2u_n(u_n+Dv_n). \tag{25}$$

We see from (23) and (25) that the solution is of the shape (ii). By the same argument, we can prove that if (24) holds and r is even, then the solution is of the shape (iii).

If (23) holds and r is odd, let $r = m$, where m is an odd positive integer. By Lemma 5, we get from (23) and the second equation of (1) that

$$\begin{aligned} 2z^2 = x+1 = (u_m+1)+Dv_m &= 2D_1U_m^2+2\varepsilon D_1D_2U_mV_m \\ &= 2D_1U_m(U_m+\varepsilon D_2V_m). \end{aligned} \tag{26}$$

We see from (23) and (26) that the solution is of the shape (iv). Similarly, if (24) holds and r is odd, then the solution is of the shape (v). The theorem is proved.

3. Proof of the Corollary

LEMMA 8 ([3]). *If $D \neq 2^{2r} \cdot 1785$, where $r \in \{0, 1, 2\}$, then the equation*

$$X^4 - DY^2 = 1, \qquad X, Y \in \mathbb{N} \tag{27}$$

has at most one solution. Further, if (X, Y) is a solution of (27), then either $(X, Y) = \left(\sqrt{u_1}, v_1\right)$ or $(X, Y) = \left(\sqrt{u_2}, v_2\right)$.

LEMMA 9 ([11]). . *Let* $D \neq 2^{4s} \cdot 1785$, *where* $s \in \{0, 1\}$. *If* v_1*and* $2u_1$ *are both squares, then the equation*

$$X^2 - DY^4 = 1, \qquad X, Y \in \mathbb{N} \tag{28}$$

has exactly two solutions $(X, Y) = \left(u_1, \sqrt{v_1}\right)$ *and* $\left(u_2, \sqrt{v_2}\right)$. *Otherwise,* (28) *has at most one solution (X,Y).*

LEMMA 10 ([6]). *Let* D_1, D_2 *be positive integers with* $\min(D_1, D_2) > 1$. *The equation*

$$D_1X^4 - D_2Y^2 = 1, \qquad x, y \in \mathbb{N} \tag{29}$$

has solutions (X, Y) *if and only if* (6) *has solutions* (U, V) *and* U_1 *is a square, where* (U_1, V_1) *is the least solution of* (6).

LEMMA 11 ([2], [7]). *Let* $D_2 = 1$. *If* $D_1 = 2$, *then* (29) *has exactly two solutions* $(X, Y) = (1, 1)$ *and* $(239, 13)$. *For* $D_1 \neq 2$, (29) *has at most one solution* (X, Y).

LEMMA 12 ([8]). *Let* D_1, D_2 *be positive integers with* $D_1 > 1$. *Then the equation*

$$D_1X^2 - D_2Y^4 = 1, \qquad X, Y \in \mathbb{N} \tag{30}$$

has at most one solution (X, Y). *Further, if* (X, Y) *is a solution of* (30), *then* (6) *has solutions* (U, V), $V_1 = ln^2$ *and* $(X, Y) = \left(U_l, \sqrt{V_l}\right)$, *where* (U_1, V_1) *is the least solution of* (6), l *and* t *are odd positive integers with* l *is square free.*

PROOF OF THE COROLLARY. For $D = 6$, we have the parameters in Lemmas 1–3 as follows:

$$(u_1, v_1) = (5, 2), \quad D' = D = 6, \quad \varepsilon = 1, \quad (D_1, D_2) = (3, 2), \quad (U_1, V_1) = (1, 1). \tag{31}$$

Let (x, y, z) be a solution of (1). If (x, y, z) has the shape (ii), then we have

$$u_n(u_n + 6v_n) = z^2, \qquad z \in \mathbb{N}. \tag{32}$$

Since $\gcd(u_n, 6v_n) = \gcd(u_n, u_n + 6v_n) = 1$, we get from (32) that

$$z = ab, \quad u_n = a^2, \quad u_n + 6v_n = b^2, \qquad a, b \in \mathbb{N}. \tag{33}$$

We see from the second equality of (33) that the equation

$$X^4 - 6Y^2 = 1, \qquad X, Y \in \mathbb{N} \tag{34}$$

has a solution $(X, Y) = \left(\sqrt{u_n}, v_n\right)$. Since $u_1 = 5$ and $u_2 = 49 = 7^2$, by Lemma 8, we get $n = 2$. Further, by (33), we get $z = 91$. Therefore, by the Theorem, the only solution of the system (1) of shape (ii) is given by $(x, y, z) = (16561, 6761, 91)$.

If (x, y, z) has the shape (iii), then we have

$$6v_n(u_n - v_n) = z^2, \qquad z \in \mathsf{N}. \tag{35}$$

Since $v_1 = 2$, v_n is even, u_n and $u_n - v_n$ are both odd. Hence, we get from (35) that

$$z = 6ab, \quad v_n = \begin{cases} 6a^2, \\ 2a^2, \end{cases} \quad u_n - v_n = \begin{cases} b^2, \\ 3b^2, \end{cases} \qquad a, b \in \mathsf{N}. \tag{36}$$

When $v_n = 6a^2$ and n is even, we have $n = 2t$, $v_{2t} = 2u_t v_t$ and

$$a = cd, \quad u_t = c^2, \quad v_t = 3d^2, \qquad c, d, t \in \mathsf{N}. \tag{37}$$

We see from the second equality of (37) that $t = 2$. But, since $v_2 = 20$, the third equality of (37) is false. When $v_n = 6a^2$ and n is odd, by Lemma 5, we have $v_n = 2U_n V_n$ and

$$U_n V_n = 3a^2. \tag{38}$$

Since $\gcd(3U_n, V_n) = 1$, we get from (38) that

$$a = cd, \quad U_n = 3c^2, \quad V_n = d^2, \qquad c, d \in \mathsf{N}. \tag{39}$$

We see from the third equality of (39) that the equation

$$3X^2 - 2Y^4 = 1, \qquad X, Y \in \mathsf{N} \tag{40}$$

has a solution $(X, Y) = \left(U_n, \sqrt{V_n}\right)$. By Lemma 12, (40) has only the solution $(X, Y) = (1, 1)$. So we have $n = 1$ and $d = 1$. But, then the second equality of (39) is false.

When $v_n = 2a^2$, the equation

$$X^2 - 24Y^4 = 1, \qquad X, Y \in \mathsf{N} \tag{41}$$

has a solution $(X, Y) = (u_n, a)$. By Lemma 9, (41) has only the solution $(X, Y) = (5, 1)$. Hence, by (36), (1) has only the solution $(x, y, z) = (71, 29, 6)$ which is of the shape (iii).

If (x, y, z) has the shape (iv), then we have

$$3U_m(U_m + 2V_m) = z^2, \qquad z \in \mathsf{N}. \tag{42}$$

Since $\gcd(U_m, U_m + 2V_m) = 1$, we get from (42) that

$$(43) \qquad z = 3ab, \quad U_m = \begin{cases} 3a^2, \\ a^2, \end{cases} \quad U_m + 2V_m = \begin{cases} b^2, \\ 3b^2, \end{cases} \qquad a, b \in \mathsf{N}.$$

When $U_m = 3a^2$, the equation

$$(44) \qquad 27X^4 - 2Y^2 = 1, \qquad X, Y \in \mathsf{N}$$

has a solution $(X, Y) = (a, V_m)$. But, since the least solution of the equation

$$(45) \qquad 27A^2 - 2B^2 = 1, \qquad A, B \in \mathsf{N}$$

is $(A_1, B_1) = (3, 11)$, by Lemma 10, (44) has no solutions (X, Y). When $U_m = a^2$, the equation

$$(46) \qquad 3X^4 - 2Y^2 = 1, \qquad X, Y \in \mathsf{N}$$

has a solution $(X, Y) = (\sqrt{U_m}, V_m)$. By [1], (46) has only two solutions $(X, Y) = (1, 1)$ and $(3, 11)$. Therefore, by (43), (1) has only the solution $(x, y, z) = (17, 7, 3)$ is the of shape (iv).

If (x, y, z) has the shape (v), then we have

$$(47) \qquad 2V_m(3U_m - V_m) = z^2, \qquad z \in \mathsf{N}.$$

Since U_m and V_m are odd integers with $\gcd(V_m, 3U_m - V_m) = 1$, we get from (47) that

$$(48) \qquad z = 2ab, \quad V_m = a^2, \quad 3U_m - V_m = 2b^2, \qquad a, b \in \mathsf{N}.$$

We see from the second equality of (48) that the equation

$$(49) \qquad 3X^2 - 2Y^4 = 1, \qquad X, Y \in \mathsf{N}$$

has a solution $(X, Y) = \left(U_m, \sqrt{V_m}\right)$. By Lemma 12, (49) has only the solution $(X, Y) = (1, 1)$. Thus, by (48), (1) has only the solution $(x, y, z) = (7, 3, 2)$ is of the shape (v). To sum up, we determine all solutions of (1) for $D = 6$.

For $D = 8$, the parameters in Lemmas 1–3 are

$$(50) \quad (u_1, v_1) = (3, 1), \quad D' = \frac{D}{4} = 2, \quad \varepsilon = 2,$$
$$(D_1, D_2) = (2, 1), \quad (U_1, V_1) = (1, 1).$$

By the same argument as in the case $D = 6$, we can prove that if $D = 8$, then (1) has only the solutions $(x, y, z) = (1, 1, 1)$ and $(31, 11, 4)$. The latter solution is of shape (ii) and arises for the value $n = 1$. The Corollary is proved.

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