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ON THE AUTOMORPHISM GROUP OF CERTAIN SIMPLE C^*-ALGEBRAS

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Abstract

We show that the information contained in $KL(A, B)$ is determined by other invariants when A and B are certain simple unital projectionless C^*-algebras. This allows us to compute the group of automorphisms modulo the group of approximately inner automorphisms in terms of the Elliott invariant.

1. Introduction

For a unital C^*-algebra A the Elliott invariant $\mathscr{E}_A$ consists of the ordered group $K_0(A)$ with order unit, the group $K_1(A)$, the compact convex set $T(A)$ of tracial states, and the restriction map $r_A : T(A) \to SK_0(A)$, where $SK_0(A)$ denotes the state space of $K_0(A)$. In [3] it was proved that the Elliott invariant is a classifying invariant for the class of unital simple infinite dimensional inductive limits of sequences of finite direct sums of building blocks. A building block is a C^*-algebra of the form

$$A(n, d_1, d_2, \ldots, d_N) = \{f \in C(\mathsf{T}) \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\},$$

where $x_1, x_2, \ldots, x_N$ are (different) points in T, and $d_1, d_2, \ldots, d_N$ are integers dividing n. The points $x_1, x_2, \ldots, x_N$ will be called the exceptional points of A. By allowing $d_i = n$ we may always assume that $N \geq 2$.

The following calculation of the group of automorphisms modulo the group of approximately inner automorphisms is the main result of this paper.

Theorem 1.1. *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks.*

(i) *If $K_0(A)$ is non-cyclic then* $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ *is isomorphic to the semi-direct product*

$$\big(\mathrm{Hom}(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}) \times \mathrm{ext}(K_1(A), K_0(A))\big) \rtimes \mathrm{Aut}(\mathscr{E}_A),$$

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where the action of $(\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathscr{E}_A)$ *is given by*

$$(\eta, e) \mapsto \big(\widetilde{\varphi_T}^{-1} \circ \eta \circ \varphi_1^{-1}, \varphi_{0*} \circ \varphi_1^{-1*}(e)\big)$$

for $\eta \in \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big)$, $e \in \mathrm{ext}(K_1(A), K_0(A))$.

(ii) *If* $K_0(A) \cong \mathsf{Z}$ *then* $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ *is isomorphic to the semi-direct product*

$$\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big) \rtimes \mathrm{Aut}(\mathscr{E}_A),$$

where the action of $(\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathscr{E}_A)$ *is given by*

$$\eta \mapsto \widetilde{\varphi_T}^{-1} \circ \eta \circ \varphi_1^{-1}, \qquad \eta \in \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big).$$

Here $\mathrm{Aut}(\mathscr{E}_A)$ denotes the group of automorphisms of $\mathscr{E}_A$, i.e. the group of triples $(\varphi_0, \varphi_1, \varphi_T)$ where φ_0 is an automorphism of the ordered group $K_0(A)$ with order unit, φ_1 is an automorphism of $K_1(A)$, and φ_T is an affine homeomorphism of $T(A)$ such that

$$r_A \circ \varphi_T^{-1}(\omega) = r_A(\omega) \circ \varphi_0 \qquad \text{on} \quad K_0(A)$$

for every $\omega \in T(A)$. $\mathrm{Aff}\, T(A)$ denotes the continuous real-valued affine functions on $T(A)$ and $\rho_A : K_0(A) \to \mathrm{Aff}\, T(A)$ is the group homomorphism $\rho_A(x)(\omega) = r_A(\omega)(x)$, $x \in K_0(A)$, $\omega \in T(A)$.

It follows from [3, Theorem 12.2] that the algebras considered under (i) are exactly the class considered by Thomsen in [9]. Therefore part (i) of the above theorem follows from Thomsen's calculation, [9, Theorem 8.4]. Note that the term $\mathrm{ext}(K_1(A), K_0(A))$ is not present in case (ii). This is not because it is zero. As we shall demonstrate, it is the existence of a natural map

$$KL(A, B)_e \to \mathrm{Hom}\big(\mathrm{Tor}(U(A)/\overline{DU(A)}), \mathrm{Tor}(U(B)/\overline{DU(B)})\big),$$

when A and B are simple unital inductive limits of finite direct sums of building blocks with $K_0(A) \cong K_0(B) \cong \mathsf{Z}$, which is responsible for this. $KL(A, B)_e$ denotes the subset of elements κ in the group $KL(A, B)$ defined by Rørdam in [5] for which the induced map $\kappa_* : K_0(A) \to K_0(B)$ preserves the order unit, and $U(A)/\overline{DU(A)}$ is the group of unitaries $U(A)$ in A modulo the closure of the commutator subgroup $DU(A)$. The map above was also crucial in the proof of the classification result in [3].

It is an interesting question whether a similar map exists in greater generality – including the case where A and B are arbitrary inductive limits of subhomogeneous C^*-algebras. Such a map would probably be useful in all efforts of classifying larger classes of simple C^*-algebras, and our main result suggests that it could influence the structure of the automorphism group as well.

It should be noted that the class of C^*-algebras considered under (ii) is quite large. It consists of matrix algebras over simple unital projectionless C^*-algebras, see [3, Corollary 12.5].

2. Preliminaries

The purpose of this section is to introduce the notation used in this paper and to list some results on building blocks from [3] that we will need.

Let A be a unital C^*-algebra. Let $s(A)$ be the smallest positive integer n for which there exists a unital *-homomorphism $A \to M_n$ (we set $s(A) = \infty$ if A has no non-trivial finite dimensional representations). Note that if there exists a unital *-homomorphism $A \to B$ then $s(A) \leq s(B)$.

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block and let $x_1, x_2, \dots, x_N$ be the exceptional points. Evaluation at x_i gives rise to a unital *-homomorphism $\Lambda_i : A \to M_{d_i}$. This map will sometimes be denoted Λ_i^A. For every integer $k \geq 0$ we let Λ_i^k be the direct sum of k copies of the representation Λ_i.

Let $A = A(n, d_1, d_2, \dots, d_N)$ and $B = A(m, e_1, e_2, \dots, e_M)$ be building blocks. Let $\varphi : A \to B$ be a unital *-homomorphism. As in [9, Chapter 1] we define $s^\varphi(j, i)$ to be the multiplicity of the representation Λ_i^A in the representation $\Lambda_j^B \circ \varphi$ for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$.

Let us start with the K-theory of a building block.

PROPOSITION 2.1. *Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block and let $d = \gcd(d_1, d_2, \dots, d_N)$. We have an isomorphism of ordered groups with order units*

$$\big(K_0(A), K_0(A)^+, [1]\big) \cong \big(\mathsf{Z}, \mathsf{Z}^+, d\big).$$

PROOF. This is [3, Corollary 3.6].

Let $A = A(n, d_1, d_2, \dots, d_N)$ be a building block with exceptional points $e^{2\pi i t_k}$, $k = 1, 2, \dots, N$, where $0 < t_1 < t_2 < \dots < t_N < 1$. Set $t_{N+1} = t_1 + 1$ and $t_0 = t_N$. Define continuous functions $\omega_k : \mathsf{T} \to \mathsf{T}$ for $k = 1, 2, \dots, N$, by

$$\omega_k(e^{2\pi i t}) = \begin{cases} \exp\left(2\pi i \dfrac{t - t_k}{t_{k+1} - t_k}\right) & t_k \leq t \leq t_{k+1}, \\ 1 & t_{k+1} \leq t \leq t_k + 1. \end{cases}$$

Let U_k^A be the unitary in A defined by

$$U_k^A(z) = \operatorname{diag}(\omega_k(z), 1, 1, \dots, 1), \qquad z \in \mathsf{T}.$$

Set $U_0^A = U_N^A$.

THEOREM 2.2. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Set for $k = 1, 2, \ldots, N-1$,*

$$s_k = \operatorname{lcm}\left(\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k}\right),$$

and

$$r_k = \gcd\left(s_k, \frac{n}{d_{k+1}}\right) = \gcd\left(\operatorname{lcm}\left(\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k}\right), \frac{n}{d_{k+1}}\right).$$

Choose integers α_k and β_k such that

$$r_k = \alpha_k s_k + \beta_k \frac{n}{d_{k+1}}, \qquad k = 1, 2, \ldots, N-1.$$

Then

$$K_1(A) \cong \mathsf{Z} \oplus \mathsf{Z}_{r_1} \oplus \mathsf{Z}_{r_2} \oplus \cdots \oplus \mathsf{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $k = 1, 2, \ldots, N-1$, a generator of the direct summand Z_{r_k} is mapped to

$$[U_k^A] - \frac{\beta_k n}{r_k d_{k+1}}[U_{k+1}^A] - \frac{\alpha_k s_k}{r_k}[U_N^A],$$

and such that a generator of the direct summand Z is mapped to $[U_N^A]$.

PROOF. See [3, Theorem 3.2].

Let A and B be unital C^*-algebras. A unital $*$-homomorphism $\varphi : A \to B$ induces morphisms $\varphi_* : K_0(A) \to K_0(B)$, $\varphi_* : K_1(A) \to K_1(B)$, $\varphi^* : T(B) \to T(A)$, $\widehat{\varphi} : \operatorname{Aff} T(A) \to \operatorname{Aff} T(B)$, $\widetilde{\varphi} : \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \to \operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))}$, and $\varphi^\# : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$. Let $q_A' : U(A) \to U(A)/\overline{DU(A)}$ and $q_A : \operatorname{Aff} T(A) \to \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}$ be the canonical maps.

PROPOSITION 2.3. *Let A be a unital inductive limit of a sequence of finite direct sums of building blocks. There exists a group homomorphism*

$$\lambda_A : \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \to U(A)/\overline{DU(A)},$$
$$\lambda_A(q_A(\widehat{a})) = q_A'(e^{2\pi i a}), \qquad a \in A_{sa}.$$

Let $\pi_A : U(A)/\overline{DU(A)} \to K_1(A)$ be the map induced by the canonical map $U(A) \to K_1(A)$. We have a short exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \xrightarrow{\pi_A} K_1(A) \longrightarrow 0.$$

This sequence is natural in A and splits unnaturally.

PROOF. See [3, Proposition 5.2].

PROPOSITION 2.4. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Let $u \in A$ be a unitary. Assume that*

$$\begin{aligned} \mathrm{Det}(u(z)) &= 1, \qquad z \in \mathsf{T}, \\ \mathrm{Det}(\Lambda_i(u)) &= 1, \qquad i = 1, 2, \ldots, N. \end{aligned}$$

Then $u \in \overline{DU(A)}$.

PROOF. See [3, Propostion 5.3].

LEMMA 2.5. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and adopt the notation of Theorem 2.2. For $k = 1, 2, \ldots, N - 1$, there exists a unitary $v_k^A \in A$ such that* $\mathrm{Det}(v_k^A(z)) = 1$, *$z \in \mathsf{T}$, and*

$$\mathrm{Det}(\Lambda_j(v_k^A)) = \begin{cases} \exp\left(2\pi i \dfrac{\alpha_k s_k}{r_k} \dfrac{d_j}{n}\right) & j = 1, 2, \ldots, k, \\ \exp\left(-2\pi i \dfrac{\beta_k}{r_k}\right) & j = k + 1, \\ 1 & j = k + 2, k + 3, \ldots, N. \end{cases}$$

Furthermore, $[v_k^A]$ has order r_k in $K_1(A)$, and $[v_1^A], [v_2^A], \ldots, [v_{N-1}^A]$ generate the torsion subgroup of $K_1(A)$. There is a group homomorphism σ_A : $\mathrm{Tor}(K_1(A)) \to \mathrm{Tor}(U(A)/\overline{DU(A)})$ *given by $\sigma_A([v_k^A]) = q'_A(v_k^A)$, $k =$* $1, 2, \ldots, N - 1$.

PROOF. The existence of v_k^A follows from [3, Lemma 5.4].

Fix $k = 1, 2, \ldots, N$. By [3, Lemma 5.4] there is a unitary $u \in A$ with $\mathrm{Det}(u(z)) = 1$, $z \in \mathsf{T}$, and

$$\mathrm{Det}(\Lambda_j(u)) = \begin{cases} 1 & j \neq k, \\ \exp\left(2\pi i \dfrac{d_k}{n}\right) & j = k. \end{cases}$$

Set $w = u\, U_{k-1}^A U_k^{A*}$. By Proposition 2.4 we have that w modulo $\overline{DU(A)}$

equals the unitary $z \mapsto e^{2\pi i \lambda(z)}$, where $\lambda : \mathsf{T} \to \mathsf{R}$ is the continuous function

$$\lambda(e^{2\pi i t}) = \begin{cases} \dfrac{1}{n}\dfrac{t - t_{k-1}}{t_k - t_{k-1}} & t_{k-1} \le t \le t_k, \\ \dfrac{1}{n}\dfrac{t_{k+1} - t}{t_{k+1} - t_k} & t_k \le t \le t_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, w is trivial in $K_1(A)$, i.e. $[u] = [U_k^A] - [U_{k-1}^A]$ in $K_1(A)$.

As a consequence of this observation,

$$\begin{aligned} [v_k^A] &= \sum_{j=1}^{k} \frac{\alpha_k s_k}{r_k}([U_j^A] - [U_{j-1}^A]) - \frac{\beta_k n}{r_k d_{k+1}}([U_{k+1}^A] - [U_k^A]) \\ &= [U_k^A] - \frac{\beta_k n}{r_k d_{k+1}}[U_{k+1}^A] - \frac{\alpha_k s_k}{r_k}[U_N^A] \end{aligned}$$

in $K_1(A)$. Hence by Theorem 2.2 we see that $[v_k^A]$ has order r_k in $K_1(A)$ and that the elements $[v_1^A], [v_2^A], \ldots, [v_{N-1}^A]$ generate $\mathrm{Tor}(K_1(A))$. Since $r_k q'_A(v_k^A) = 0$ and $\pi_A(q'_A(v_k^A)) = [v_k^A]$ it follows that $q'_A(v_k^A)$ has order r_k in $U(A)/\overline{DU(A)}$. The existence of σ_A follows.

We remark that the map σ_A is neither natural nor unique, and that $\pi_A \circ \sigma_A$ is the identity map on $\mathrm{Tor}(K_1(A))$.

In [5] Rørdam defined the bifunctor KL to be a certain quotient of KK. Recall from [5] that the Kasparov product yields a product $KL(B, C) \times KL(A, B) \to KL(A, C)$ which we will denote by $\cdot$. Furthermore, if $K_*(A)$ is finitely generated then $KK(A, \cdot) \cong KL(A, \cdot)$. If φ is a unital *-homomorphism, we let $[\varphi]$ denote the induced element in $KL(A, B)$. For unital C^*-algebras A and B we let $KL(A, B)_e$ denote the elements of $KL(A, B)$ for which the induced map $K_0(A) \to K_0(B)$ preserves the order unit.

Let A and B be building blocks. $KL(A, B)$ is conveniently described in terms of the K-homology groups $K^0(A) = KK(A, \mathsf{C}) \cong KL(A, \mathsf{C})$ and $K^0(B)$. Recall that the Kasparov product gives rise to a group homomorphism $\kappa^* : K^0(B) \to K^0(A)$ for every $\kappa \in KL(A, B)$.

THEOREM 2.6. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks such that $s(B) \ge Nn$.*

(i) *If $\nu \in KL(A, B)_e$ then there exists a unital *-homomorphism $\varphi : A \to B$ such that $[\varphi] = \nu$ in $KL(A, B)$.*

(ii) *Let $\varphi : A \to B$ be a unital *-homomorphism and let $\kappa \in KL(A, B)_e$. If $\varphi^* = \kappa^*$ on $K^0(B)$ and $\varphi_*([U_N^A]) = \kappa_*([U_N^A])$ in $K_1(B)$ then $[\varphi] = \kappa$ in $KL(A, B)$.*

PROOF. This follows from [3, Theorem 4.7].

The next result says that $K^0(\cdot)$ and the torsion subgroup of $U(\cdot)/\overline{DU(\cdot)}$ are related for building blocks.

PROPOSITION 2.7. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. There exists a finite set $F \subseteq \mathrm{Tor}(U(A)/\overline{DU(A)})$ such that if B is a building block and $\varphi, \psi : A \to B$ are unital *-homomorphisms with $\varphi^\#(x) = \varphi^\#(x)$, $x \in F$, then $\varphi^* = \psi^*$ on $K^0(B)$.*

PROOF. This is part of [3, Theorem 5.5].

We also need a description of the structure of the group $K^0(\cdot)$ for a building block.

PROPOSITION 2.8. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Then $K^0(A)$ is generated by $[\Lambda_1], [\Lambda_2], \ldots, [\Lambda_N]$. Furthermore, for $a_1, a_2, \ldots, a_N \in \mathbb{Z}$ we have that*

$$a_1[\Lambda_1] + a_2[\Lambda_2] + \cdots + a_N[\Lambda_N] = 0$$

if and only if there exist $b_1, b_2, \ldots, b_N \in \mathbb{Z}$ such that $\sum_{i=1}^N b_i = 0$ and

$$a_i = b_i \frac{n}{d_i}, \qquad i = 1, 2, \ldots, N.$$

PROOF. This is [3, Proposition 4.2].

We conclude with a technical proposition which is needed in the next section.

PROPOSITION 2.9. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks with $s(B) \geq Nn$. Let $\chi \in K_1(B)$ and let $h : K^0(B) \to K^0(A)$ be a homomorphism of the form*

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1N} \\ h_{21} & h_{22} & \ldots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \ldots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}$$

*with $\sum_{i=1}^N h_{ji} d_i = e_j$ for $j = 1, 2, \ldots, M$. There exists a unital *-homomorphism $\varphi : A \to B$ such that $\varphi^* = h$ on $K^0(B)$ and $\varphi_*([U_N^A]) = \chi$ in $K_1(B)$.*

PROOF. By Proposition 2.8 we have that $\frac{n}{d_i}[\Lambda_i^A] = \frac{n}{d_N}[\Lambda_N^A]$ in $K^0(A)$. Hence we may assume that $0 \le h_{ji} < \frac{n}{d_i}$ for $i \ne N$ and still have that $\sum_{i=1}^N h_{ji} d_i = e_j$ for $j = 1, 2, \ldots, M$. Note that for $j = 1, 2, \ldots, M$,

$$Nn \le \sum_{i=1}^{N} h_{ji} d_i < (N-1)n + h_{jN} d_N$$

and hence $h_{jN} > \frac{n}{d_N}$. The conclusion follows from [3, Proposition 4.4].

3. KL and other invariants

Let A and B be unital C^*-algebras and let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism. Assume that $K_0(A) \cong \mathsf{Z}$. Then $\rho_A(K_0(A))$ is closed in Aff $T(A)$, and we have a well-defined map

$$\widetilde{\varphi_0} : \operatorname{Tor}\big(\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}\big) \to \operatorname{Tor}\big(\operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))}\big)$$

given by $\widetilde{\varphi_0}\big(q_A\big(\frac{1}{k}\rho_A(x)\big)\big) = q_B\big(\frac{1}{k}\rho_B(\varphi_0(x))\big)$ for $x \in K_0(A)$ and every positive integer k.

THEOREM 3.1. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks with $s(B) \ge Nn$. If $\varphi_0 : K_0(A) \to K_0(B)$ is an order unit preserving group homomorphism, if $\Phi : \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big) \to \operatorname{Tor}\big(U(B)/\overline{DU(B)}\big)$ is a group homomorphism such that the diagram*

$$\begin{array}{ccc}
\operatorname{Tor}\big(\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}\big) & \xrightarrow{\lambda_A} & \operatorname{Tor}\big(U(A)/\overline{DU(A)}\big) \\
\widetilde{\varphi_0}\downarrow & & \downarrow\Phi \\
\operatorname{Tor}\big(\operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))}\big) & \xrightarrow{\lambda_B} & \operatorname{Tor}\big(U(B)/\overline{DU(B)}\big)
\end{array}$$

*commutes, and if $\chi \in K_1(B)$, then there is a unital *-homomorphism $\psi : A \to B$ such that $\psi^{\#} = \Phi$ on $\operatorname{Tor}\big(U(A)/\overline{DU(A)}\big)$ and $\psi_*[U_N^A] = \chi$ in $K_1(B)$.*

PROOF. We adopt the notation of Theorem 2.2. Set $\alpha_0 = 1$. If i and k are integers, $1 \le i \le k \le N$, we define an integer c_i^k by

$$c_i^k = \alpha_{i-1}\beta_i\beta_{i+1}\ldots\beta_{k-1}.$$

We claim that

$$\frac{1}{s_k} = \sum_{i=1}^{k} c_i^k \frac{d_i}{n}, \qquad k = 1, 2, \ldots, N. \tag{1}$$

As $c_1^1 = 1$, this is clear for $k = 1$. Assume it is correct for k, $1 \leq k \leq N-1$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} c_i^{k+1}\frac{d_i}{n} &= c_{k+1}^{k+1}\frac{d_{k+1}}{n} + \sum_{i=1}^{k} c_i^{k+1}\frac{d_i}{n} = c_{k+1}^{k+1}\frac{d_{k+1}}{n} + \beta_k \sum_{i=1}^{k} c_i^{k}\frac{d_i}{n} \\
&= \alpha_k \frac{d_{k+1}}{n} + \beta_k \frac{1}{s_k} = \frac{d_{k+1}}{n}\frac{1}{s_k}\left(\alpha_k s_k + \beta_k \frac{n}{d_{k+1}}\right) \\
&= \frac{r_k}{s_k \frac{n}{d_{k+1}}} = \frac{1}{s_{k+1}},
\end{aligned}$$

proving (1).

Choose a unitary $u_k \in B$ such that $\Phi(q'_A(v_k^A)) = q'_B(u_k), k = 1, 2, \ldots, N-1$. Let $q_k^j \in \mathsf{R}$ be numbers such that

$$\mathrm{Det}(\Lambda_j(u_k)) = e^{2\pi i q_k^j}, \qquad k = 1, 2, \ldots, N-1, \quad j = 1, 2, \ldots, M.$$

Set $q_0^j = 0$ and set $d = \gcd(d_1, d_2, \ldots, d_N)$. By Proposition 2.1 d divides e_j, $j = 1, 2, \ldots, M$. Define for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$,

$$h_{ji} = c_i^N \frac{e_j}{d} - \frac{n}{d_i} q_{i-1}^j + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l} q_{N-l}^j.$$

Since $r_k q'_A(v_k^A) = 0$ in $U(A)/\overline{DU(A)}$ by Lemma 2.5, we see that $u_k^{r_k} \in \overline{DU(B)}$ and hence $r_k q_k^j \in \mathsf{Z}$, $k = 1, 2, \ldots, N-1$. It follows that $h_{ji} \in \mathsf{Z}$ for every i and j. Since $q'_B(u_k)$ has finite order, $\mathrm{Det}(u_k(\cdot))$ is constantly equal to $e^{2\pi i a_k}$ for some $a_k \in \mathsf{R}$. Note that

$$e^{2\pi i a_k} = e^{2\pi i q_k^j \frac{m}{e_j}}, \qquad j = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, N-1.$$

Thus if we set $a_0 = 0$ we find that

$$\begin{aligned}
\frac{m}{e_j} h_{ji} &= c_i^N \frac{m}{d} - \frac{n}{d_i} q_{i-1}^j \frac{m}{e_j} + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l} q_{N-l}^j \frac{m}{e_j} \\
&\equiv c_i^N \frac{m}{d} - \frac{n}{d_i} a_{i-1} + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l} a_{N-l} \qquad \mod \frac{n}{d_i}
\end{aligned}$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Hence

$$\frac{m}{e_j} h_{ji} \equiv \frac{m}{e_M} h_{Mi} \quad \mod \frac{n}{d_i}. \tag{2}$$

For $k = 1, 2, \dots, N$,

$$\begin{aligned}
&\sum_{i=1}^{k} \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l} q_{N-l}^{j} \frac{d_i}{n} \\
&\quad = \sum_{l=1}^{N-1} \sum_{i=1}^{\min(k,N-l)} c_i^{N-l} s_{N-l} q_{N-l}^{j} \frac{d_i}{n} \\
&\quad = \sum_{l=1}^{N-k} \sum_{i=1}^{k} c_i^{N-l} s_{N-l} q_{N-l}^{j} \frac{d_i}{n} + \sum_{l=N-k+1}^{N-1} \sum_{i=1}^{N-l} c_i^{N-l} s_{N-l} q_{N-l}^{j} \frac{d_i}{n} \\
&\quad = \sum_{l=1}^{N-k} \beta_k \beta_{k+1} \dots \beta_{N-l-1} \sum_{i=1}^{k} c_i^{k} \frac{d_i}{n} s_{N-l} q_{N-l}^{j} + \sum_{l=N-k+1}^{N-1} q_{N-l}^{j} \\
&\quad = \sum_{l=1}^{N-k} \beta_k \beta_{k+1} \dots \beta_{N-l-1} \frac{1}{s_k} s_{N-l} q_{N-l}^{j} + \sum_{l=1}^{k-1} q_l^{j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{k} h_{ji} \frac{d_i}{n} &= \sum_{i=1}^{k} c_i^{N} \frac{e_j}{d} \frac{d_i}{n} - \sum_{i=1}^{k} q_{i-1}^{j} + \sum_{i=1}^{k} \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l} q_{N-l}^{j} \frac{d_i}{n} \\
&= \frac{e_j}{d} \beta_k \beta_{k+1} \dots \beta_{N-1} \sum_{i=1}^{k} c_i^{k} \frac{d_i}{n} + \sum_{l=1}^{N-k} \beta_k \beta_{k+1} \dots \beta_{N-l-1} \frac{1}{s_k} s_{N-l} q_{N-l}^{j} \\
&= \frac{e_j}{d} \beta_k \beta_{k+1} \dots \beta_{N-1} \frac{1}{s_k} + \sum_{l=1}^{N-k} \beta_k \beta_{k+1} \dots \beta_{N-l-1} \frac{1}{s_k} s_{N-l} q_{N-l}^{j}.
\end{aligned}$$

By setting $k = N$ we see that

$$\sum_{i=1}^{N} h_{ji} d_i = e_j \frac{n}{d} \frac{1}{s_N} = e_j.$$

Combining this equation with (2) and Proposition 2.8 it is an elementary exercise to prove that we can define a homomorphism $h : K^0(B) \to K^0(A)$ by

$$\begin{pmatrix} h([\Lambda_1^B]) \\ h([\Lambda_2^B]) \\ \vdots \\ h([\Lambda_M^B]) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{M1} & h_{M2} & \dots & h_{MN} \end{pmatrix} \begin{pmatrix} [\Lambda_1^A] \\ [\Lambda_2^A] \\ \vdots \\ [\Lambda_N^A] \end{pmatrix}.$$

By Proposition 2.9 there exists a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = h$ on $K^0(B)$ and $\psi_*([U_N^A]) = \chi$ on $K_1(B)$. Fix $j = 1, 2, \ldots, M$. Let $t_i = s^\psi(j, i)$. By [3, Lemma 2.1] there exist a unitary $w \in M_{e_j}$ and $z_1, z_2, \ldots, z_L \in \mathsf{T}$ such that

$$\Lambda_j^B \circ \psi(f) = w \operatorname{diag}\big(\Lambda_1^{t_1}(f), \Lambda_2^{t_2}(f), \ldots, \Lambda_N^{t_N}(f), f(z_1), f(z_2), \ldots, f(z_L)\big) w^*$$

for $f \in A$. Since point-evaluations are homotopic *-homomorphisms $A \to M_n$, we see that

$$\psi^*[\Lambda_j^B] = [\Lambda_j^B \circ \psi] = \sum_{i=1}^N t_i [\Lambda_i^A] + L \frac{n}{d_N} [\Lambda_N^A].$$

in $K^0(A)$. On the other hand, $\psi^*[\Lambda_j^B] = \sum_{i=1}^N h_{ji} [\Lambda_i^A]$. It follows from Proposition 2.8 that

$$s^\psi(j, i) \equiv h_{ji} \quad \operatorname{mod} \frac{n}{d_i}, \qquad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M.$$

Note that for $k = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, M$,

$$\begin{aligned} \frac{\alpha_k s_k}{r_k} \sum_{i=1}^k h_{ji} \frac{d_i}{n} &= \frac{e_j}{d} \frac{\alpha_k}{r_k} \beta_k \beta_{k+1} \ldots \beta_{N-1} + \sum_{l=1}^{N-k} \frac{\alpha_k}{r_k} \beta_k \beta_{k+1} \ldots \beta_{N-l-1} s_{N-l} q_{N-l}^j \\ &= \frac{e_j}{d} \frac{\beta_k}{r_k} c_{k+1}^N + \sum_{l=1}^{N-k-1} \frac{\beta_k}{r_k} c_{k+1}^{N-l} s_{N-l} q_{N-l}^j + \frac{\alpha_k}{r_k} s_k q_k^j \\ &= \frac{\beta_k}{r_k} \left(h_{j(k+1)} + \frac{n}{d_{k+1}} q_k^j \right) + \frac{\alpha_k}{r_k} s_k q_k^j \\ &= \frac{\beta_k}{r_k} h_{j(k+1)} + q_k^j. \end{aligned}$$

Since $\operatorname{Det}(v_k^A(z)) = 1$, $z \in \mathsf{T}$, we see that

$$\begin{aligned} \operatorname{Det}\big(\Lambda_j(\psi(v_k^A))\big) &= \prod_{i=1}^N \operatorname{Det}\big(\Lambda_i(v_k^A)\big)^{s^\psi(j,i)} = \prod_{i=1}^N \operatorname{Det}\big(\Lambda_i(v_k^A)\big)^{h(j,i)} \\ &= \exp\left(2\pi i \left(\sum_{i=1}^k \frac{\alpha_k s_k}{r_k} h_{ji} \frac{d_i}{n} - \frac{\beta_k}{r_k} h_{j(k+1)} \right)\right) \\ &= \exp\big(2\pi i q_k^j\big) = \operatorname{Det}\big(\Lambda_j(u_k)\big). \end{aligned}$$

Thus $\mathrm{Det}(\psi(v_k^A)(\cdot))$ and $\mathrm{Det}(u_k(\cdot))$ agree at the exceptional points of B, and hence they agree everywhere. It follows from Proposition 2.4 that

$$q'_B(\psi(v_k^A)) = q'_B(u_k) = \Phi(q'_A(v_k^A)), \qquad k = 1, 2, \ldots, N-1.$$

As $\widetilde{\psi} = \widetilde{\varphi}_0$ on $\mathrm{Tor}\big(\mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big)$, we conclude from Lemma 2.5 and Proposition 2.3 that $\psi^\#$ and Φ agree on all of $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$.

Our next result says that the information contained in $KL(A, B)$ can be detected by other invariants when A and B are building blocks.

PROPOSITION 3.2. *Let $A = A(n, d_1, d_2, \ldots, d_N)$ and B be building blocks with $s(B) \geq Nn$. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism, and let $\Phi : \mathrm{Tor}(U(A)/\overline{DU(A)}) \to \mathrm{Tor}(U(B)/\overline{DU(B)})$ and $\varphi_1 : K_1(A) \to K_1(B)$ be group homomorphisms such that the diagram*

$$\begin{array}{ccccc}
\mathrm{Tor}\big(\mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big) & \xrightarrow{\lambda_A} & \mathrm{Tor}\big(U(A)/\overline{DU(A)}\big) & \xrightarrow{\pi_A} & \mathrm{Tor}\big(K_1(A)\big) \\
\downarrow{\scriptstyle \widetilde{\varphi}_0} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi_1} \\
\mathrm{Tor}\big(\mathrm{Aff}\, T(B)/\overline{\rho_B(K_0(B))}\big) & \xrightarrow[\lambda_B]{} & \mathrm{Tor}\big(U(B)/\overline{DU(B)}\big) & \xrightarrow[\pi_B]{} & \mathrm{Tor}\big(K_1(B)\big)
\end{array}$$

commutes.

(i) *There exists a unital *-homomorphism $\varphi : A \to B$ such that $\varphi_* = \varphi_0$ on $K_0(A)$, $\varphi_* = \varphi_1$ on $K_1(A)$ and $\varphi^\# = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$.*

(ii) *If $\psi : A \to B$ is another unital *-homomorphism such that $\psi_* = \varphi_0$ on $K_0(A)$, $\psi_* = \varphi_1$ on $K_1(A)$ and $\psi^\# = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$, then $[\varphi] = [\psi]$ in $KL(A, B)$.*

PROOF. Choose by Theorem 3.1 a unital *-homomorphism $\varphi : A \to B$ such that $\varphi^\# = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$ and $\varphi_*[U_N^A] = \varphi_1[U_N^A]$ in $K_1(B)$. Then $\varphi_* = \varphi_1$ on $\mathrm{Tor}(K_1(A))$, and thus $\varphi_* = \varphi_1$ on all of $K_1(A)$ by Theorem 2.2. Obviously $\varphi_* = \varphi_0$ since φ is unital. This proves (i).

To prove (ii), note that since $\varphi^\# = \psi^\#$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$ we have that $\varphi^* = \psi^*$ on $K^0(B)$ by Proposition 2.7. Hence $[\varphi] = [\psi]$ by Theorem 2.6.

Let A and B be simple unital infinite dimensional inductive limits of sequences of finite direct sums of building blocks. In [3, Chapter 10] a group homomorphism

$$s_\kappa : \mathrm{Tor}\big(U(A)/\overline{DU(A)}\big) \to \mathrm{Tor}\big(U(B)/\overline{DU(B)}\big),$$

was constructed for every $\kappa \in KL(A, B)_T$ (the map was constructed for slightly different B but can be applied in our case by [3, Lemma 10.3], [3, Lemma 9.6] and [3, Theorem 9.9]). Recall from [3] that $KL(A, B)_T$ is the set of elements $\kappa \in KL(A, B)_e$ for which there exists an affine continuous map $\varphi_T : T(B) \to T(A)$ such that $r_B(\omega)(\kappa_*(x)) = r_A(\varphi_T(\omega))(x)$ for $x \in K_0(A), \omega \in T(B)$. Note that if $K_0(A) \cong \mathbb{Z}$ and $K_0(B) \cong \mathbb{Z}$ then $KL(A, B)_T = KL(A, B)_e$.

Recall furthermore from [3, Chapter 10] that $s_{[\mu]} = \mu^\#$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$ for every unital *-homomorphism $\mu : A \to B$, and that if C is a finite direct sum of building blocks, and $\varphi : C \to A$, $\psi : C \to B$ are unital *-homomorphisms such that $[\psi] = \kappa \cdot [\varphi]$ in $KL(C, B)$, then $\psi^\# = s_\kappa \circ \varphi^\#$ on $\mathrm{Tor}\big(U(C)/\overline{DU(C)}\big)$.

We can now generalize Theorem 3.2 to simple inductive limits for which $K_0(A)$ and $K_0(B)$ are cyclic:

THEOREM 3.3. *Let A and B be unital simple infinite dimensional inductive limits of sequences of finite direct sums of building blocks. Assume that $K_0(A)$ and $K_0(B)$ are cyclic groups. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism, and let $\varphi_1 : K_1(A) \to K_1(B)$ and $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be group homomorphisms such that the diagram*

$$\begin{array}{ccccc} \mathrm{Tor}\big(\mathrm{Aff}\,T(A)/\overline{\rho_A(K_0(A))}\big) & \xrightarrow{\lambda_A} & \mathrm{Tor}\big(U(A)/\overline{DU(A)}\big) & \xrightarrow{\pi_A} & \mathrm{Tor}\big(K_1(A)\big) \\ \downarrow{\scriptstyle \tilde{\varphi}_0} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi_1} \\ \mathrm{Tor}\big(\mathrm{Aff}\,T(B)/\overline{\rho_B(K_0(B))}\big) & \xrightarrow[\lambda_B]{} & \mathrm{Tor}\big(U(B)/\overline{DU(B)}\big) & \xrightarrow[\pi_B]{} & \mathrm{Tor}\big(K_1(B)\big) \end{array}$$

commutes. There exists a unique element $\kappa \in KL(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$, $\kappa_* = \varphi_1$ on $K_1(A)$ and $s_\kappa = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\big)$.

PROOF. We may by [3, Theorem 9.9] assume that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of finite direct sums of building blocks with unital and injective connecting maps. Similarly we may assume that B is the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

of finite direct sums of building blocks with unital and injective connecting maps. Since $K_0(A) \cong \mathbb{Z}$ it is easy to see that we may furthermore assume that each A_k is a building block, rather than a finite direct sum of building blocks. Similarly we may assume that each B_k is a building block. Let $\alpha_{k,\infty} : A_k \to A$ and $\beta_{k,\infty} : B_k \to B$ denote the canonical *-homomorphisms.

By passing to subsequences we may assume that for every positive integer k there exist an order unit preserving group homomorphism $\mu_k : K_0(A_k) \to$

$K_0(B_k)$ and a group homomorphism $\eta_k : K_1(A_k) \to K_1(B_k)$ such that

$$\beta_{k,\infty *} \circ \mu_k = \varphi_0 \circ \alpha_{k,\infty *} \qquad \text{on} \quad K_0(A_k),$$
$$\beta_{k,\infty *} \circ \eta_k = \varphi_1 \circ \alpha_{k,\infty *} \qquad \text{on} \quad K_1(A_k).$$

By passing to a subsequence again, we may assume that

$$\mu_{k+1} \circ \alpha_{k*} = \beta_{k*} \circ \mu_k \qquad \text{on} \quad K_0(A_k),$$
$$\eta_{k+1} \circ \alpha_{k*} = \beta_{k*} \circ \eta_k \qquad \text{on} \quad K_1(A_k).$$

Let $A_k = A(n_k, d_1^k, d_2^k, \dots, d_{N_k}^k)$. By Proposition 2.3, Lemma 2.5, and [3, Lemma 10.8], we may also assume that for every positive integer k, there exists a group homomorphism $\Phi_k \colon \mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\big) \to \mathrm{Tor}\big(U(B_k)/\overline{DU(B_k)}\big)$ such that

$$\lambda_{B_k} \circ \widetilde{\mu_k} = \Phi_k \circ \lambda_{A_k}$$

on $\mathrm{Tor}\big(\mathrm{Aff}\, T(A_k)/\overline{\rho_{A_k}(K_0(A_k))}\big)$ and

$$\beta^{\#}_{k,\infty} \circ \Phi_k\big(q'_{A_k}(v_j^{A_k})\big) = \Phi \circ \alpha^{\#}_{k,\infty}\big(q'_{A_k}(v_j^{A_k})\big)$$

for $j = 1, 2, \dots, N_k - 1$. Since for every positive integer k,

$$\begin{aligned}\beta^{\#}_{k,\infty} \circ \Phi_k \circ \lambda_{A_k} &= \beta^{\#}_{k,\infty} \circ \lambda_{B_k} \circ \widetilde{\mu_k} = \lambda_B \circ \widetilde{\beta_{k,\infty}} \circ \widetilde{\mu_k} \\ &= \lambda_B \circ \widetilde{\varphi}_0 \circ \widetilde{\alpha_{k,\infty}} = \Phi \circ \lambda_A \circ \widetilde{\alpha_{k,\infty}} = \Phi \circ \alpha^{\#}_{k,\infty} \circ \lambda_{A_k}\end{aligned}$$

on $\mathrm{Tor}\big(\mathrm{Aff}\, T(A_k)/\overline{\rho_{A_k}(K_0(A_k))}\big)$, we conclude from Proposition 2.3 and Lemma 2.5 that

$$\beta^{\#}_{k,\infty} \circ \Phi_k = \Phi \circ \alpha^{\#}_{k,\infty}$$

on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\big)$.

It follows from the above equation and [3, Lemma 10.4] that by passing to subsequences we may assume that for every positive integer k,

$$\beta^{\#}_{k} \circ \Phi_k\big(q'_{A_k}(v_j^{A_k})\big) = \Phi_{k+1} \circ \alpha^{\#}_{k}\big(q'_{A_k}(v_j^{A_k})\big)$$

for $j = 1, 2, \dots, N_k - 1$. Since for every positive integer k,

$$\begin{aligned}\beta^{\#}_{k} \circ \Phi_k \circ \lambda_{A_k} &= \beta^{\#}_{k} \circ \lambda_{B_k} \circ \widetilde{\mu_k} = \lambda_{B_{k+1}} \circ \widetilde{\beta}_k \circ \widetilde{\mu_k} \\ &= \lambda_{B_{k+1}} \circ \widetilde{\mu_{k+1}} \circ \widetilde{\alpha}_k = \Phi_{k+1} \circ \lambda_{A_{k+1}} \circ \widetilde{\alpha}_k = \Phi_{k+1} \circ \alpha^{\#}_{k} \circ \lambda_{A_k}\end{aligned}$$

on $\mathrm{Tor}\big(\mathrm{Aff}\, T(A_k)/\overline{\rho_{A_k}(K_0(A_k))}\big)$, we see that

$$\beta^{\#}_{k} \circ \Phi_k = \Phi_{k+1} \circ \alpha^{\#}_{k}$$

on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\,\big)$.

Note that for every positive integer k,

$$\begin{aligned}\beta_{k,\infty*} \circ \eta_k \circ \pi_{A_k} &= \varphi_1 \circ \alpha_{k,\infty*} \circ \pi_{A_k} = \varphi_1 \circ \pi_A \circ \alpha^{\#}_{k,\infty} \\ &= \pi_B \circ \Phi \circ \alpha^{\#}_{k,\infty} = \pi_B \circ \beta^{\#}_{k,\infty} \circ \Phi_k = \beta_{k,\infty*} \circ \pi_{B_k} \circ \Phi_k\end{aligned}$$

on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\,\big)$. By passing to subsequences again we may assume that

$$\eta_k \circ \pi_{A_k}\big(q'_{A_k}(v_j^{A_k})\big) = \pi_{B_k} \circ \Phi_k\big(q'_{A_k}(v_j^{A_k})\big)$$

in $K_1(B)$ for $j = 1, 2, \ldots, N_k - 1$. Since $\pi_{B_k} \circ \Phi_k \circ \lambda_{A_k} = 0$ on the torsion subgroup of $\mathrm{Aff}\, T(A_k)/\overline{\rho_{A_k}(K_0(A_k))}$, it follows that $\eta_k \circ \pi_{A_k} = \pi_{B_k} \circ \Phi_k$ on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\,\big)$. Thus the diagram

$$\begin{array}{ccccc}\mathrm{Tor}(\mathrm{Aff}\, T(A_k)/\overline{\rho_{A_k}(K_0(A_k))}) & \xrightarrow{\lambda_{A_k}} & \mathrm{Tor}(U(A_k)/\overline{DU(A_k)}) & \xrightarrow{\pi_{A_k}} & \mathrm{Tor}(K_1(A_k)) \\ \Big\downarrow{\scriptstyle \widetilde{\mu_k}} & & \Big\downarrow{\scriptstyle \Phi_k} & & \Big\downarrow{\scriptstyle \eta_k} \\ \mathrm{Tor}(\mathrm{Aff}\, T(B_k)/\overline{\rho_{B_k}(K_0(B_k))}) & \xrightarrow[\lambda_{B_k}]{} & \mathrm{Tor}(U(B_k)/\overline{DU(B_k)}) & \xrightarrow[\pi_{B_k}]{} & \mathrm{Tor}(K_1(B_k))\end{array}$$

commutes. Finally we may by [3, Lemma 9.6] assume that $s(B_k) \geq N_k n_k$.

It follows from Proposition 3.2 that for every positive integer k, there exists a unital *-homomorphism $\psi_k : A_k \to B_k$ such that $\psi_{k*} = \mu_k$ on $K_0(A_k)$, $\psi_{k*} = \eta_k$ on $K_1(A_k)$, and $\psi_k^{\#} = \Phi_k$ on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\,\big)$. By the uniqueness part of the same proposition, $[\beta_k] \cdot [\psi_k] = [\psi_{k+1}] \cdot [\alpha_k]$ in $KL(A_k, B_{k+1})$. By [6, Theorem 1.12] and [7, Theorem 7.1] there exists an element $\kappa \in KL(A, B)$ such that $\kappa \cdot [\alpha_{k,\infty}] = [\beta_{k,\infty}] \cdot [\psi_k]$ in $KL(A_k, B)$ for every positive integer k. Then $\kappa_* = \varphi_0$ on $K_0(A)$, $\kappa_* = \varphi_1$ on $K_1(A)$, and

$$s_\kappa \circ \alpha^{\#}_{k,\infty} = (\beta_{k,\infty} \circ \psi_k)^{\#} = \beta^{\#}_{k,\infty} \circ \Phi_k = \Phi \circ \alpha^{\#}_{k,\infty} \quad \text{on} \quad \mathrm{Tor}(U(A_k)/\overline{DU(A_k)}).$$

By [3, Lemma 10.8] we see that $s_\kappa = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\,\big)$.

To prove uniqueness, let $\nu \in KL(A, B)$ be another element such that $\nu_* = \varphi_0$ on $K_0(A)$, $\nu_* = \varphi_1$ on $K_1(A)$ and $s_\nu = \Phi$ on $\mathrm{Tor}\big(U(A)/\overline{DU(A)}\,\big)$. By passing to a subsequence, we may assume that there is an element ν_k in $KL(A_k, B_k)$ such that $[\beta_{k,\infty}] \cdot \nu_k = \nu \cdot [\alpha_{k,\infty}]$. By passing to a subsequence again we may assume that $\psi_{k*} = \nu_{k*}$ on $K_0(A)$ as well as on $K_1(A)$. By Theorem 2.6 there exists a unital *-homomorphism $\xi_k : A_k \to B_k$ such that $[\xi_k] = \nu_k$ in $KL(A_k, B_k)$. Then

$$\beta^{\#}_{k,\infty} \circ \xi_k^{\#} = s_\nu \circ \alpha^{\#}_{k,\infty} = s_\kappa \circ \alpha^{\#}_{k,\infty} = \beta^{\#}_{k,\infty} \circ \psi_k^{\#}$$

on $\mathrm{Tor}\big(U(A_k)/\overline{DU(A_k)}\,\big)$. By passing to subsequences again, we may by [3, Lemma 10.4] assume that $\xi_k^{\#} = \psi_k^{\#}$ on any given finite subset of $\mathrm{Tor}\big(U(A_k)/$

$\overline{DU(A_k)}\big)$. Hence, we can arrange that $\xi_k{}^* = \psi_k{}^*$ on $K^0(B_k)$ by Proposition 2.7. It follows from Theorem 2.6 that $[\xi_k] = [\psi_k]$ in $KL(A_k, B_k)$. Thus, $\kappa \cdot [\alpha_{k,\infty}] = \nu \cdot [\alpha_{k,\infty}]$ for all k. It follows that $\kappa = \nu$ by [5, Lemma 5.8].

4. Main results

In [3] the existence result [3, Theorem 11.2] was subsequently simplified in the case where $K_0(A)$ is non-cyclic. The following theorem shows that a similar simplification is possible when $K_0(A)$ and $K_0(B)$ are cyclic, but this time without KL.

THEOREM 4.1. *Let A and B be unital simple inductive limits of sequences of finite direct sums of building blocks and assume that $K_0(A) \cong \mathbb{Z}$, $K_0(B) \cong \mathbb{Z}$ and that B is infinite dimensional. Let $\varphi_T : T(B) \to T(A)$ be an affine continuous map, let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism, let $\varphi_1 : K_1(A) \to K_1(B)$ be a group homomorphism, and let $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be a homomorphism such that the diagram*

$$\begin{array}{ccccc} \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\ \downarrow{\scriptstyle \widetilde{\varphi_T}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi_1} \\ \operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B) \end{array}$$

*commutes. There exists a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$, $\psi^\# = \Phi$ on $U(A)/\overline{DU(A)}$, and $\psi_* = \varphi_0$ on $K_0(A)$.*

PROOF. We may assume that A is infinite dimensional. By Theorem 3.3 there exists an element $\kappa \in KL(A, B)$ such that $\kappa_* = \varphi_0$ on $K_0(A)$, $\kappa_* = \varphi_1$ on $K_1(A)$, and $s_\kappa = \Phi$ on $\operatorname{Tor}\big(U(A)/\overline{DU(A)}\big)$. By [3, Theorem 11.2] there exists a unital *-homomorphism $\psi : A \to B$ such that $[\psi] = \kappa$ in $KL(A, B)$, $\psi^* = \varphi_T$ on $T(B)$, and $\psi^\# = \Phi$ on $U(A)/\overline{DU(A)}$.

The next result says that KL can also be removed from the uniqueness theorem, [3, Theorem 11.5], when $K_0(B)$ is cyclic.

THEOREM 4.2. *Let A and B be simple unital inductive limit of sequences of finite direct sums of building blocks such that $K_0(A) \cong \mathbb{Z}$ and $K_0(B) \cong \mathbb{Z}$. Let $\varphi, \psi : A \to B$ be unital *-homomorphisms with $\varphi^\# = \psi^\#$ on $U(A)/\overline{DU(A)}$. Then φ and ψ are approximately unitarily equivalent.*

PROOF. We may assume that A is infinite dimensional. As in the proof of Theorem 3.3 we see that A is the inductive limit of a sequence

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \dots$$

of building blocks with unital and injective connecting maps. Similarly B is the inductive limit of a sequence

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \dots$$

of building blocks with unital and injective connecting maps. By [3, Lemma 8.5] we have that $s(B_k) \to \infty$. Obviously $\varphi_* = \psi_*$ on $K_0(A)$ and $\varphi_* = \psi_*$ on $K_1(A)$, such that $[\varphi] = [\psi]$ in $KL(A, B)$ by Theorem 3.3. Finally note that $\varphi^\# = \psi^\#$ implies $\widetilde{\varphi} = \widetilde{\psi}$. Thus the linear map $\widehat{\varphi} - \widehat{\psi}$ takes values in $\overline{\rho_B(K_0(B))}$, and hence it must be 0. Therefore $\varphi^* = \psi^*$ on $T(B)$. It follows from [3, Theorem 11.5] that φ and ψ are approximately unitarily equivalent.

We need the following isomorphism version of Theorem 4.1.

THEOREM 4.3. *Let A and B be simple unital infinite dimensional inductive limits of sequences of finite direct sums of building blocks with $K_0(A) \cong \mathsf{Z}$. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism of ordered groups with order units, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism of groups, let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism, and let $\Phi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$ be an isomorphism of groups, such that the diagram*

$$\begin{array}{ccccc}
\operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) \\
\Big\downarrow{\scriptstyle \widetilde{\varphi_T}} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \varphi_1} \\
\operatorname{Aff} T(B)/\overline{\rho_B(K_0(B))} & \xrightarrow[\lambda_B]{} & U(B)/\overline{DU(B)} & \xrightarrow[\pi_B]{} & K_1(B)
\end{array}$$

commutes. Then there exists an isomorphism $\psi : A \to B$ such that $\psi_* = \varphi_1$ on $K_1(A)$, $\psi^* = \varphi_T$ on $T(B)$, and $\psi^\# = \Phi$ on $U(A)/\overline{DU(A)}$.

PROOF. By Theorem 4.1 there exists a unital *-homomorphism $\mu : A \to B$ such that $\mu^\# = \Phi$ on $U(A)/\overline{DU(A)}$, $\mu^* = \varphi_T$ on $T(B)$, and $\mu_* = \varphi_1$ on $K_1(A)$. Similarly, there exists a unital *-homomorphism $\xi : B \to A$ such that $\xi^\# = \Phi^{-1}$ on $U(B)/\overline{DU(B)}$, $\xi^* = \varphi_T^{-1}$ on $T(A)$, and $\xi_* = \varphi_1^{-1}$ on $K_1(B)$. By Theorem 4.2 we see that $\mu \circ \xi$ and $\xi \circ \mu$ are approximately inner. Hence by [4, Proposition A] μ is approximately unitarily equivalent to an isomorphism $\psi : A \to B$.

We are now in a position to prove part (ii) of Theorem 1.1.

THEOREM 4.4. *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks with $K_0(A) \cong \mathsf{Z}$. Then*

$$\operatorname{Aut}(A)/\overline{\operatorname{Inn}(A)} \cong \operatorname{Hom}\big(K_1(A), \operatorname{Aff} T(A)/\overline{\rho_A(K_0(A))}\big) \rtimes \operatorname{Aut}(\mathscr{E}_A),$$

where the action of $(\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathscr{E}_A)$ *is given by*

$$\eta \mapsto \widetilde{\varphi_T}^{-1} \circ \eta \circ \varphi_1^{-1}, \qquad \eta \in \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big).$$

Proof. We may assume that A is infinite dimensional. By Proposition 2.3 we may identify $U(A)/\overline{DU(A)}$ with $G_1 \oplus G_2$, where $G_1 = \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}$ and $G_2 = K_1(A)$. Thus an endomorphism ψ of the group $U(A)/\overline{DU(A)}$ can be identified with a 2×2 matrix

$$\begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

where $\psi_{ij} : G_j \to G_i$ is a homomorphism, $i, j = 1, 2$. Note that if ψ is induced by an automorphism of A then $\psi_{21} = 0$ since the short exact sequence of Proposition 2.3 is natural.

Let $H = \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\big)$. Let $\eta \in H$. Choose by Theorem 4.3 an element $\psi \in \mathrm{Aut}(A)$ such that $\psi^* = id$ on $T(A)$, $\psi_* = id$ on $K_1(A)$, and

$$\psi^{\#} = \begin{pmatrix} id & \eta \\ 0 & id \end{pmatrix}$$

on $U(A)/\overline{DU(A)}$. By Theorem 4.2 we obtain a well-defined group homomorphism

$$\iota : H \to \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$$

by setting $\iota(\eta) = p(\psi)$, where $p : \mathrm{Aut}(A) \to \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ denotes the canonical map. Let $\pi : \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \to \mathrm{Aut}(\mathscr{E}_A)$ be the homomorphism

$$\pi(p(\psi)) = (\psi_*, \psi_*, (\psi^*)^{-1}).$$

We have a short exact sequence

$$0 \longrightarrow H \xrightarrow{\ \iota\ } \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \xrightarrow{\ \pi\ } \mathrm{Aut}(\mathscr{E}_A) \longrightarrow 0$$

of groups. Let $(\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathscr{E}_A)$. Choose by Theorem 4.1 an element ψ in $\mathrm{Aut}(A)$ such that $\psi_* = \varphi_1$, $\psi^* = \varphi_T^{-1}$, and

$$\psi^{\#} = \begin{pmatrix} \widetilde{\varphi_T}^{-1} & 0 \\ 0 & \varphi_1 \end{pmatrix}.$$

By Theorem 4.2 we obtain a well-defined map $\sigma : \mathrm{Aut}(\mathscr{E}_A) \to \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ by setting $\sigma(\varphi_0, \varphi_1, \varphi_T) = p(\psi)$. Note that σ splits the sequence above. Hence $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is isomorphic to a semi-direct product $H \rtimes \mathrm{Aut}(\mathscr{E}_A)$. Since

$$\iota(\widetilde{\varphi_T}^{-1} \eta \varphi_1^{-1}) = \sigma(\varphi_0, \varphi_1, \varphi_T)\, \iota(\eta)\, \sigma(\varphi_0, \varphi_1, \varphi_T)^{-1},$$

it follows that the action of $\mathrm{Aut}(\mathscr{E}_A)$ on H is the desired one.

Let us finally show that our main result can be simplified when $K_1(A)$ is a torsion group. Recall that $\mathrm{ext}(G, H)$ is defined as $\mathrm{Ext}(G, H)/\mathrm{Pext}(G, H)$ for abelian groups G and H, where $\mathrm{Pext}(G, H)$ is the subgroup of pure (i.e. locally trivial) extensions in $\mathrm{Ext}(G, H)$, see [5, Chapter 5].

COROLLARY 4.5. *Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks such that $K_1(A)$ is a torsion group. Then*

$$\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \cong \mathrm{ext}(K_1(A), K_0(A))\big) \rtimes \mathrm{Aut}(\mathscr{E}_A),$$

where the action of $(\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathscr{E}_A)$ is given by $e \mapsto \varphi_{0*} \circ \varphi_1^{-1*}(e)$ for $e \in \mathrm{ext}(K_1(A), K_0(A))$.

PROOF. If $K_0(A)$ is non-cyclic, then $\mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}$ is torsion-free by [3, Lemma 10.3], and hence the result follows in this case from (i) in Theorem 1.1. Therefore we may assume that $K_0(A) \cong \mathbf{Z}$. Then ρ_A is injective and has closed range, and hence we have a short exact sequence

$$0 \longrightarrow K_0(A) \xrightarrow{\rho_A} \mathrm{Aff}\, T(A) \xrightarrow{q_A} \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} \longrightarrow 0.$$

Let E denote the corresponding class in $\mathrm{Ext}(\mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}, K_0(A))$. Note that $\mathrm{Aff}\, T(A)$ is divisible, and therefore $\mathrm{Ext}(K_1(A), \mathrm{Aff}\, T(A)) = 0$. Hence by applying [2, Theorem III.3.4] we get an isomorphism

$$E_* : \mathrm{Hom}(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}) \;\to\; \mathrm{Ext}(K_1(A), K_0(A)),$$

where $E_*(\eta) = \eta^*(E)$. By a result of C. U. Jensen, see e.g. [8, Theorem 6.1], we have that $\mathrm{Pext}(K_1(A), K_0(A)) = 0$. Thus $\mathrm{Ext}(K_1(A), K_0(A)) = \mathrm{ext}(K_1(A), K_0(A))$. To see that the two actions of $\mathrm{Aut}(\mathscr{E}_A)$ can be identified as well, note that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\, T(A) & \xrightarrow{q_A} & \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_T^{-1*}} & & \downarrow{\scriptstyle \widetilde{\varphi_T}^{-1}} & & \\
0 & \longrightarrow & K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}\, T(A) & \xrightarrow{q_A} & \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))} & \longrightarrow & 0
\end{array}$$

commutes, such that $\widetilde{\varphi_T}^{-1*}(E) = \varphi_{0*}(E)$ by [2, Proposition III.1.8]. The corollary follows.

We mention without proof that the C^*-algebras considered in the above corollary are exactly the simple unital inductive limits of sequences of finite direct sums of building blocks of the form

$$\{f \in C[0, 1] \otimes M_n : f(x_i) \in M_{d_i},\ i = 1, 2, \ldots, N\}.$$

Corollary 4.5 suggests that only KL, and not $U(\cdot)/\overline{DU(\cdot)}$, is needed in an approximate intertwining argument to show that the Elliott invariant is a classifying invariant for these C^*-algebras. This was demonstrated by Jiang and Su [1] for a large subclass of this class of C^*-algebras.

Let us finally emphasize the following surprising consequence of the corollary above. Let A be a simple unital inductive limit of a sequence of finite direct sums of building blocks. If $K_0(A)$ is non-cyclic then $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is isomorphic to a semi-direct product

$$\big(\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\,\big) \times \mathrm{ext}(K_1(A), K_0(A))\big) \rtimes \mathrm{Aut}(\mathscr{E}_A).$$

The term $\mathrm{Hom}\big(K_1(A), \mathrm{Aff}\, T(A)/\overline{\rho_A(K_0(A))}\,\big)$ vanishes e.g. if A has real rank zero, whereas the term $\mathrm{ext}(K_1(A), K_0(A))$ vanishes e.g. if A is an inductive limit of a sequence of finite direct sums of circle algebras. When $K_0(A) \cong \mathsf{Z}$ and $K_1(A)$ is a torsion group, however, these two terms agree, but only one of them appear in the expression for $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$.

REFERENCES

1. Jiang, X., and Su, H., *On a simple unital projectionless C^*-algebra*, Amer. J. Math. 121 (1999), 359–413.
2. Mac Lane, S., *Homology*, Springer Verlag, 1963.
3. Mygind, J., *Classification of certain simple C^*-algebras with torsion in K_1*, Canad. J. Math. 53 (2001), 1223–1308.
4. Rørdam, M., *A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$*, C. R. Math. Rep. Acad. Sci. Canada 16 (1994), 31–36.
5. Rørdam, M., *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal. 131 (1995), 415–458.
6. Rosenberg, J., and Schochet, C., *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. 55 (1987), 431–474.
7. Schochet, C., *Topological methods for C^*-algebras. III. Axiomatic homology*, Pacific J. Math. 114 (1984), 399–445.
8. Schochet, C., *A* Pext *primer: Pure extensions and* $\lim^1$ *for infinite abelian groups*, New York J. Math. Monographs 1 (2003).
9. Thomsen, K., *Limits of certain subhomogeneous C^*-algebras*, Mém. Soc. Math. Fr. 71 (1997).

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