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A REMARK ON POINCARÉ INEQUALITIES ON METRIC MEASURE SPACES

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Abstract

We show that, in a complete metric measure space equipped with a doubling Borel regular measure, the Poincaré inequality with upper gradients introduced by Heinonen and Koskela [3] is equivalent to the Poincaré inequality with "approximate Lipschitz constants" used by Semmes in [9].

1. Introduction

In this note we prove the following result which seems to have been informally conjectured by Semmes [9, p. 17].

THEOREM 1. *Let $p \geq 1$. Then every complete metric measure space equipped with a Borel doubling measure admits a weak $(1, p)$-Poincaré inequality with upper gradients if and only if it admits a weak $(1, p)$-Poincaré inequality with the "approximate Lipschitz constant operator" D_ϵ, quantitatively.*

The terminology of the above theorem is explained in Section 2, the proof given in Section 3, and the necessity of the hypotheses on the given metric measure space discussed in Remark 2. For now we note that, roughly speaking, the above theorem claims that a Poincaré inequality holds for all functions on a metric measure space, with the gradient replaced by an infinitesimal measurement of oscillation, if a discretized version of the Poincaré inequality holds for all functions and at all scales, with bounds independent of the scale of the discretization. The former condition has been widely studied and is rich in application; see [1], [2], [3], [4], [8]. The latter condition is often easier to verify, especially when the metric measure space is studied using discrete approximations. For example, the weak p-Poincaré inequality with D_ϵ, $p \geq 1$, is easily seen to persist under measured Gromov-Hausdorff convergence. Consequently, Theorem 1 provides another proof of the fact that the $(1, p)$-Poincaré inequality on complete metric measure spaces equipped with Borel doubling measures persists under measured Gromov-Hausdorff convergence, as long as all relevant constants are uniformly controlled; see [5].

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2. Terminology

Throughout this paper (X, d, μ) denotes a metric measure space where μ is Borel regular. We denote the ball with center $x \in X$ and radius $r > 0$ by

$$B(x, r) = \{y \in X : d(x, y) < r\},$$

and use the notation

$$u_A = \frac{1}{\mu(A)} \int_A u \, d\mu = \fint_A u \, d\mu,$$

for every $A \subset X$ and measurable function $u : X \longrightarrow [-\infty, \infty]$. The measure μ is said to be *doubling* if there is a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)),$$

for every $x \in X$ and $r > 0$. A function $u : X \longrightarrow \mathsf{R}$ is *Lipschitz* if there exists a constant $L > 0$ such that

$$|u(x) - u(y)| \leq Ld(x, y),$$

for every $x, y \in X$.

We now recall the definitions of upper gradients and Poincaré inequalities on metric measure spaces as given by Heinonen and Koskela [3]. A Borel function $g : X \longrightarrow [0, \infty]$ is said to be an *upper gradient* of some measurable function $u : X \longrightarrow [-\infty, \infty]$ if

$$|u(x) - u(y)| \leq \int_\gamma g \, ds,$$

for every $x, y \in X$ and every locally rectifiable curve γ joining x and y. A metric measure space (X, d, μ) is said to admit a *weak* $(1, p)$*-Poincaré inequality with upper gradients*, $p \geq 1$, if there exist constants $C > 0$ and $\lambda \geq 1$ so that for every measurable $u : X \longrightarrow [-\infty, \infty]$ and every upper gradient g of u, we have

$$\fint_{B(a,r)} |u - u_{B(a,r)}| \, d\mu \leq Cr \left(\fint_{B(a,\lambda r)} g^p \, d\mu \right)^{\frac{1}{p}}, \tag{1}$$

for every $a \in X$ and $r > 0$. The $(1, p)$-Poincaré inequality is sometimes called strong if $\lambda = 1$.

We now summarize a definition for a Poincaré inequality on metric measure spaces introduced by Semmes [9, Section 2.3]. For every $\epsilon > 0$ we define an operator D_ϵ so that for every measurable function $u : X \longrightarrow [-\infty, \infty]$, we have

$$D_\epsilon u(x) = \sup_{y \in B(x,\epsilon)} \frac{|u(x) - u(y)|}{\epsilon},$$

for every $x \in X$. Then (X, d, μ) is said to admit a *weak* $(1, p)$*-Poincaré inequality with* D_ϵ, $p \geq 1$, if there exist constants $C > 0$ and $\lambda \geq 1$ so that for every measurable function $u : X \longrightarrow [-\infty, \infty]$ we have

$$\fint_{B(a,r)} |u - u_{B(a,r)}| \, d\mu \leq Cr \left(\fint_{B(a,\lambda r)} (D_\epsilon u)^p \, d\mu \right)^{\frac{1}{p}}, \tag{2}$$

whenever $a \in X$ and $0 < \epsilon < r$. Note that the constants are required to be independent of ϵ.

3. Proof of Theorem 1

Let us first show that (2) implies (1). For this we define the pointwise Lipschitz constant Lip so that for a real-valued Lipschitz function u,

$$\operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)},$$

for every $x \in X$. Now by [5, Theorem 1.3.4], inequality (1) follows (for all measurable functions) as soon as we have

$$\fint_{B(a,r)} |u - u_{B(a,r)}| \, d\mu \leq Cr \left(\fint_{B(a,\lambda r)} (\operatorname{Lip} u)^p \, d\mu \right)^{\frac{1}{p}}, \tag{3}$$

for all real-valued Lipschitz functions u and all $a \in X$ and $r > 0$. So it suffices to show that (2) implies (3). For this we take a real-valued Lipschitz function u and a ball $B(a, r)$, and let $\epsilon \to 0$ in (2). Since u is Lipschitz, the functions $D_\epsilon u$ are bounded by some constant $L > 0$ not depending on ϵ, and thus we can use the Lebesgue Dominated Convergence Theorem to replace the right hand side of (2) by the right hand side of (3).

To show that (1) implies (2), we use the ϵ-partition of unity to approximate a locally p-integrable function u (note that it suffices to prove (2) for locally p-integrable functions). A similar method is used in [6] by Koskela and MacManus. Fix $\epsilon > 0$ and cover X by balls $B_i = B(x_i, \epsilon)$ so that the balls

$B(x_i, \epsilon/5)$ are disjoint. This can be done because of the existence of a doubling measure in X. Note that it then follows that the balls in $\{B_i\}$ have overlap bounded by a number that depends only on the doubling constant associated with μ, and therefore not ϵ. As in the standard construction of partition of unity, one can now define $C\epsilon^{-1}$-Lipschitz functions $0 \le \phi_i \le 1$ so that the support of ϕ_i lies inside B_i for all i and $\sum_i \phi_i(x) = 1$ for all $x \in X$. Here and below $C > 0$ is a varying constant whose value in any one usage is fixed and depends only on the doubling constant of μ and the constants in (1). Having the partition of unity for $\epsilon > 0$, we approximate u by a function u_ϵ, where

$$u_\epsilon(x) = \sum_i V_i \phi_i(x), \qquad V_i = \fint_{B_i} u(y)\, d\mu(y).$$

This approximation turns out to be useful for our purpose. Following [6], we define a function $D_\epsilon^A u$, similar to $D_\epsilon u$, by

$$D_\epsilon^A u(x) = \fint_{B(x,\epsilon)} \frac{|u(x) - u(y)|}{\epsilon}\, d\mu(x).$$

This function can be used to find an upper gradient for u_ϵ; define g_ϵ by

$$g_\epsilon(a) = \fint_{B(a,2\epsilon)} D_{5\epsilon}^A u(x)\, d\mu(x).$$

If $d(a, b) < \epsilon$, then, by [6, Lemma 4.6], we have

$$|u_\epsilon(a) - u_\epsilon(b)| \le C d(a, b) g_\epsilon(a),$$

It further follows that $C g_\epsilon$ is an upper gradient of u_ϵ.

Now we estimate the left hand side of the Poincaré inequality by using u_ϵ. First, we have

$$\begin{aligned}
&\fint_{B(a,r)} |u(x) - u_B|\, d\mu(x) \\
&\quad \le \fint_{B(a,r)} |u_\epsilon(x) - u(x)|\, d\mu(x) + \fint_{B(a,r)} |u_\epsilon(x) - u_{\epsilon,B}|\, d\mu(x) \\
&\qquad + \fint_{B(a,r)} |u_{\epsilon,B} - u_B|\, d\mu(x) \\
&\quad \le 2 \fint_{B(a,r)} |u_\epsilon(x) - u(x)|\, d\mu(x) + \fint_{B(a,r)} |u_\epsilon(x) - u_{\epsilon,B}|\, d\mu(x).
\end{aligned}$$

The first term can be estimated by using the definition of u_ϵ as follows:

$$\begin{aligned}
&\fint_{B(a,r)} |u_\epsilon(x) - u(x)|\, d\mu(x) \\
&\qquad = \fint_{B(a,r)} \left| \sum_i \fint_{B_i} u(y)\, d\mu(y)\phi_i(x) - u(x) \right| d\mu(x) \\
&\qquad \le \fint_{B(a,r)} \sum_i \phi_i(x) \fint_{B_i} |u(y) - u(x)|\, d\mu(y)\, d\mu(x) \\
&\qquad \le 7C\epsilon \fint_{B(a,r)} \sum_i \phi_i(x) D_{7\epsilon}u(x)\, d\mu(x) = 7C\epsilon \fint_{B(a,r)} D_{7\epsilon}u(x)\, d\mu(x) \\
&\qquad \le 7Cr \left(\fint_{B(a,\lambda r)} D_{7\epsilon}u(x)^p\, d\mu(x) \right)^{\frac{1}{p}}.
\end{aligned}$$

Here we used Hölder's inequality, the doubling property of μ, and the assumption $\epsilon \le r$. The second term can now be estimated by the Poincaré inequality (1), to get

$$\begin{aligned}
&\fint_{B(a,r)} |u_\epsilon(x) - u_{\epsilon,B}|\, d\mu(x) \\
&\qquad \le Cr \left(\fint_{B(a,\lambda r)} g_\epsilon(x)^p\, d\mu(x) \right)^{\frac{1}{p}} \\
&\qquad = Cr \left(\fint_{B(a,\lambda r)} \left(\fint_{B(x,2\epsilon)} D^A_{5\epsilon}u(y)\, d\mu(y) \right)^p d\mu(x) \right)^{\frac{1}{p}} \\
&\qquad = Cr \left(\fint_{B(a,\lambda r)} \left(\fint_{B(x,2\epsilon)} \fint_{B(y,5\epsilon)} \frac{|u(y) - u(z)|}{5\epsilon}\, d\mu(z)\, d\mu(y) \right)^p d\mu(x) \right)^{\frac{1}{p}} \\
&\qquad \le 3Cr \left(\fint_{B(a,\lambda r)} D_{7\epsilon}u(x)^p\, d\mu(x) \right)^{\frac{1}{p}}.
\end{aligned}$$

Combining the estimates we have (2) with ϵ replaced by 7ϵ. This completes the proof of Theorem 1.

REMARK 2. The proof that the weak $(1, p)$-Poincaré inequality with upper gradients, $p \ge 1$, implies the weak $(1, p)$-Poincaré inequality with D_ϵ did not use the assumption that (X, d) is complete. However, the converse implication requires this hypothesis. To see this, consider a non-complete metric measure space (X, d, μ) equipped with a Borel doubling measure, that admits (1) for all

Lipschitz functions u and their upper gradients g, but that does not admit (1) for all measurable functions u and their upper gradients g. Such metric measure spaces have been constructed by Koskela [7]. By the above argument that (1) implies (2), it follows that (2) holds on (X, d, μ) for all Lipschitz functions u. We can then use the density of Lipschitz functions amongst p-integrable functions to conclude that (2) holds on (X, d, μ) for all p-integrable functions, and therefore for all measurable functions. Thus (X, d, μ) is a (non-complete) metric measure space equipped with a Borel doubling measure, that admits a $(1, p)$-Poincaré inequality with D_ϵ, but that does not admit a $(1, p)$-Poincaré inequality with upper gradients.

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