PRESTABLE IDEALS AND SAGBI BASES

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Abstract

In order to find a reasonable class of squarefree monomial ideals I for which the toric ideal of the Rees algebra of I has a quadratic Gröbner basis, the concept of prestable ideals will be introduced. Prestable ideals arising from finite pure posets together with their application to Sagbi bases will be discussed.

Introduction

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and let $I \subset R$ be an ideal which is generated by monomials $u_1, \ldots, u_m$ with $\deg u_1 = \cdots = \deg u_m$. The Rees algebra of I is the subalgebra $\mathcal{R}(I) = K[x_1, \ldots, x_n, u_1 t, \ldots, u_m t]$ of $R[t]$. Let $A = K[x_1, \ldots, x_n, y_1, \ldots, y_m] = R[y_1, \ldots, y_m]$ denote the polynomial ring over K and define the surjective homomorphism $\pi : A \to \mathcal{R}(I)$ by setting $\pi(x_i) = x_i$ and $\pi(y_j) = u_j t$. The toric ideal $J_{\mathcal{R}(I)}$ of $\mathcal{R}(I)$ is the kernel of π. Blum [1] proved that if $\mathcal{R}(I)$ is Koszul, then all powers of I have linear resolutions. Thus in particular if $J_{\mathcal{R}(I)}$ has a quadratic Gröbner basis, then all powers of I have linear resolutions. However, the existence of a quadratic Gröbner basis of $J_{\mathcal{R}(I)}$ is a rather strong condition which guarantees that all powers of I have linear resolutions. In [8] a much weaker condition, called the x-condition, for $J_{\mathcal{R}(I)}$ is introduced and it is proved that if $J_{\mathcal{R}(I)}$ satisfies the x-condition, then all powers of I have linear resolutions.

In the present paper, a new class of monomial ideals, the class of prestable ideals, which contains the stable ideals [4] is introduced. If I is prestable, then $J_{\mathcal{R}(I)}$ satisfies the x-condition and all powers of I have linear resolutions. See Corollary 1.5. We then discuss a class of prestable squarefree monomial ideals I arising from finite pure posets (partially ordered sets) such that $J_{\mathcal{R}(I)}$ has a quadratic Gröbner basis. See Theorem 2.1 and Corollary 2.2. Finally, as one of the applications of such prestable ideals coming from finite pure posets, Sagbi bases of the algebras studied in [5] will be determined. See Theorem 3.2.

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1. Prestable sets

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$, and write $V_n^{(d)}$ for the set of all monomials of R of degree d.

A nonempty subset $\mathcal{N} = \{u_1, u_2, \ldots, u_m\} \subset V_n^{(d)}$ is said to be a *prestable set* if $\mathcal{N}$ possesses the exchange property $(*)$ as follows:

($*$) For all $N = 1, 2, \ldots$ and for any two monomials $\prod_{j=1}^N u_{i_j}$ and $\prod_{\ell=1}^N u_{k_\ell}$ with $\prod_{j=1}^N u_{i_j} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\prod_{\ell=1}^N u_{k_\ell} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ such that $a_1 = b_1, \ldots, a_{q-1} = b_{q-1}$ and $a_q < b_q$ for some $1 \le q < n$, there exist $1 \le j \le N$ and $q < p \le n$ such that $x_q(u_{i_j}/x_p) \in \mathcal{N}$.

One of the most fundamental classes of prestable sets is

EXAMPLE 1.1. Recall that a set of monomials $\mathcal{N} \subset V_n^{(d)}$ is stable if, for all $u \in \mathcal{N}$ and for all $i < m(u)$, one has $x_i(u/x_{m(u)}) \in \mathcal{N}$. Here $m(u)$ is the largest i for which x_i divides u. A stable set $\mathcal{N}$ is prestable. In fact, if $\prod_{j=1}^N u_{i_j} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\prod_{\ell=1}^N u_{k_\ell} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, with each $u_{i_j}, u_{k_\ell} \in \mathcal{N}$ such that $a_1 = b_1, \ldots, a_{q-1} = b_{q-1}$ and $a_q < b_q$, then there is $p > q$ with $a_p > b_p$. In particular x_p divides some u_{i_j}. Since $m(u_{i_j}) \ge p > q$ and since $\mathcal{N}$ is stable, one has $x_q(u_{i_j}/x_{m(u_{i_j})}) \in \mathcal{N}$. We will give a prestable set which is not stable. See Example 2.3.

Let $I \subset R$ be an ideal generated by monomials of degree d and $G(I)$ the minimal system of monomial generators of I, i.e., $G(I) = I \cap V_n^{(d)}$. We say that I is prestable if $G(I) \subset V_n^{(d)}$ is a prestable set. The *Rees algebra* of I is the semigroup ring

$$\mathcal{R}(I) = K[x_1, \ldots, x_n, \{ut\}_{u \in G(I)}] \qquad (\subset R[t] = K[x_1, \ldots, x_n, t]).$$

Let $A = K[x_1, \ldots, x_n, \{y_u\}_{u \in G(I)}] = R[\{y_u\}_{u \in G(I)}]$ denote the polynomial ring over K with each $\deg x_i = \deg y_u = 1$. The *toric ideal* of $\mathcal{R}(I)$ is the ideal $J_{\mathcal{R}(I)} \subset A$ which is the kernel of the surjective homomorphism $\pi : A \to \mathcal{R}(I)$ defined by setting $\pi(x_i) = x_i$ for each $1 \le i \le n$ and $\pi(y_u) = ut$ for each $u \in G(I)$.

Let $<^{(\sharp)}$ be an arbitrary monomial order on the polynomial ring $K[\{y_u\}_{u \in G(I)}]$ and $<_{lex}$ the lexicographic order on R induced by $x_1 > x_2 > \cdots > x_n$. We then introduce the new monomial order $<_{lex}^{(\sharp)}$ on A by setting $\mathbf{x^a y^b} <_{lex}^{(\sharp)} \mathbf{x^{a'} y^{b'}}$ if either (i) $\mathbf{x^a} <_{lex} \mathbf{x^{a'}}$ or (ii) $\mathbf{x^a} = \mathbf{x^{a'}}$ and $\mathbf{y^b} <^{(\sharp)} \mathbf{y^{b'}}$. Here each of $\mathbf{x^a}$ and $\mathbf{x^{a'}}$ is a monomial belonging to R and each of $\mathbf{y^b}$ and $\mathbf{y^{b'}}$ is a monomial belonging to $K[\{y_u\}_{u \in G(I)}]$.

We now state the reason why we are interested in prestable ideals. We refer the reader to, e.g., [3] for fundamental materials on Gröbner bases.

Since a prestable set satisfies the ℓ-exchange property [7] for all monomial orders $<^{(\sharp)}$ on $K[\{y_u\}_{u \in G(I)}]$, Theorem 1.2 below is a special case of [7, Theorem 5.1]. However, for the sake of completeness, a proof of Theorem 1.2 will be given.

THEOREM 1.2. *Work with the same notation as above. Suppose that $I \subset R$ is a prestable ideal and $\mathscr{G}$ is a Gröbner basis of $J_{\mathscr{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ with respect to $<^{(\sharp)}$. Then*

$$\mathscr{G} \cup \{x_i y_u - x_j y_v;\ u, v \in G(I),\ x_i u = x_j v\}$$

is a Gröbner basis of $J_{\mathscr{R}(I)}$ with respect to $<_{lex}^{(\sharp)}$.

PROOF. It is known, e.g., [12] that the reduced Gröbner basis of the toric ideal $J_{\mathscr{R}(I)}$ consists of binomials. Let $f = x_{i_1} \cdots x_{i_p} y_{u_1} \cdots y_{u_q} - x_{j_1} \cdots x_{j_p} y_{v_1} \cdots y_{v_q}$ be an irreducible binomial belonging to $J_{\mathscr{R}(I)}$ with $p \geq 1$, where $i_1 \leq i_2 \leq \cdots$ and $j_1 \leq j_2 \leq \cdots$ with $i_1 < j_1$. Thus the initial monomial of f is $in_{<_{lex}^{(\sharp)}}(f) = x_{i_1} \cdots x_{i_p} y_{u_1} \cdots y_{u_q}$. Write $u_1 u_2 \cdots u_q = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v_1 v_2 \cdots v_q = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. Then $a_1 = b_1, \ldots, a_{i_1-1} = b_{i_1-1}$ and $a_{i_1} < b_{i_1}$. Since I is prestable, it follows that there exist i and j with $1 \leq i \leq q$ and $i_1 < j \leq n$ such that $x_{i_1}(u_i/x_j) \in G(I)$. Let $w = x_{i_1}(u_i/x_j)$. Then $g = x_{i_1} y_{u_i} - x_j y_w \in J_{\mathscr{R}(I)}$. Since $x_j y_w <_{lex}^{(\sharp)} x_{i_1} y_{u_i}$, the initial monomial of g is $in_{<_{lex}^{(\sharp)}}(g) = x_{i_1} y_{u_i}$. Thus $in_{<_{lex}^{(\sharp)}}(g)$ divides $in_{<_{lex}^{(\sharp)}}(f)$.

The elimination property of the lexicographic order together with Theorem 1.2 guarantees that

COROLLARY 1.3. *Suppose that $I \subset R$ is a prestable ideal and $\mathscr{G}$ is a Gröbner basis of $J_{\mathscr{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ with respect to $<^{(\sharp)}$. Then for each $1 \leq \ell \leq n$*

$$\mathscr{G} \cup \{\, x_i y_u - x_j y_v;\ u, v \in G(I),\ \ell \leq i, j \leq n,\ x_i u = x_j v \,\}$$

is a Gröbner basis of

$$J_{\mathscr{R}(I)} \cap K[x_\ell, x_{\ell+1}, \ldots, x_n, \{y_u\}_{u \in G(I)}]$$

with respect to $<_{lex}^{(\sharp)}$ on $K[x_\ell, x_{\ell+1}, \ldots, x_n, \{y_u\}_{u \in G(I)}]$.

COROLLARY 1.4. *Let $I \subset R$ be a prestable ideal and suppose that $J_{\mathscr{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ possesses a quadratic Gröbner basis. Then the toric ideal $J_{\mathscr{R}(I)}$ has a quadratic Gröbner basis. Thus in particular the Rees algebra $\mathscr{R}(I)$ is Koszul.*

A monomial ideal I is called *strongly stable* if, for all $u \in G(I)$ and for all x_j that divides u, one has $x_i(u/x_j) \in G(I)$ for all $i < j$. Conca and De Negri [2] discovered an example of a strongly stable ideal $I \subset R$ for which $J_{\mathcal{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ possesses *no* quadratic Gröbner basis.

If $f \in A$ is a homogeneous polynomial, then its *x-degree* is the degree $\deg_x(f)$ of f as a polynomial in the variables $x_1, \ldots, x_n$ with coefficients in $K[\{y_u\}_{u \in G(I)}]$. For example, if $f = x_1^2 x_2 y_1 y_2 - x_3^3 y_3 y_4$, then its x-degree is $\deg_x(f) = 3$. We say that $J_{\mathcal{R}(I)}$ satisfies the *x-condition* [8] if there exists a Gröbner basis $\mathcal{G}$ of $J_{\mathcal{R}(I)}$ such that $\deg_x(g) \leq 1$ for each $g \in \mathcal{G}$.

COROLLARY 1.5. *Let $I \subset R$ be a prestable ideal. Then all powers of I have linear resolutions.*

PROOF. If $I \subset R$ is a prestable ideal, then the toric ideal $J_{\mathcal{R}(I)}$ satisfies the x-condition. Thus [8, Corollary 1.2] says that all powers of I have linear resolutions.

2. Prestable ideals arising from pure posets

In the present section we are interested in finding a reasonable class of prestable ideals $I \subset R$ for which $J_{\mathcal{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ has a quadratic Gröbner basis.

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_d$ be a finite pure poset [11, p. 99] of rank $d - 1$, where each Ω_i is the set of rank $i - 1$ elements of Ω. Recall that a poset of rank $d - 1$ is called pure if the length of every maximal chain is equal to $d - 1$. Let $\Omega_i = \{x_1^{(i)}, x_2^{(i)}, \cdots, x_{p_i}^{(i)}\}$, and let $R = K[\{x_j^{(i)}\}_{\substack{1 \leq i \leq d \\ 1 \leq j \leq p_i}}]$ denote the polynomial ring over a field K with each $\deg x_j^{(i)} = 1$. We will associate each maximal chain

$$C : x_{q_1}^{(1)} < x_{q_2}^{(2)} < \cdots < x_{q_d}^{(d)}$$

of Ω with the squarefree monomial

$$x_C = x_{q_1}^{(1)} x_{q_2}^{(2)} \cdots x_{q_d}^{(d)}$$

of R of degree d. Let $\mathcal{M}(\Omega)$ denote the set of maximal chains of Ω.

THEOREM 2.1. *Work with the same notation as above. Suppose that Ω satisfies the condition that if $x_j^{(i)} > x_k^{(i-1)}$, then $x_{j'}^{(i)} > x_k^{(i-1)}$ for all j' with $j' < j$ and $x_j^{(i)} > x_{k'}^{(i-1)}$ for all k' with $k < k'$. Then $\{x_C; C \in \mathcal{M}(\Omega)\}$ is a prestable set.*

PROOF. We work with the ordering

$$x_1^{(d)}, x_2^{(d)}, \cdots, x_{p_d}^{(d)}, x_1^{(d-1)}, \cdots, x_{p_{d-1}}^{(d-1)}, x_1^{(d-2)}, \cdots, x_{p_2}^{(2)}, x_1^{(1)}, \cdots, x_{p_1}^{(1)}.$$

Let $\prod_{\lambda=1}^{N} x_{C_{p_\lambda}} = \prod_{i,j} x_j^{(i)a_j^{(i)}}$ and $\prod_{\mu=1}^{N} x_{C_{q_\mu}} = \prod_{i,j} x_j^{(i)b_j^{(i)}}$. Suppose that $a_{j_0}^{(i_0)} < b_{j_0}^{(i_0)}$ and $a_j^{(i)} = b_j^{(i)}$ for all i and j with either (i) $i > i_0$ or (ii) $i = i_0$ and $j < j_0$. Let $A_j = \{p_\lambda; x_j^{(i_0)} \in C_{p_\lambda}\}$ and $B_j = \{q_\mu; x_j^{(i_0)} \in C_{q_\mu}\}$. Then $\sum_{j=1}^{j_0} |A_j| < \sum_{j=1}^{j_0} |B_j|$, where $|A_j|$ stands for the cardinality of the finite set A_j. It then follows that there exist C_{p_λ} and C_{q_μ} such that $C_{p_\lambda} \cap \Omega_{i_0+1} = C_{q_\mu} \cap \Omega_{i_0+1}$, $C_{p_\lambda} \cap \Omega_{i_0} \subset \{x_{j_0+1}^{(i_0)}, \ldots, x_{p_{i_0}}^{(i_0)}\}$ and $C_{q_\mu} \cap \Omega_{i_0} \subset \{x_1^{(i_0)}, \ldots, x_{j_0}^{(i_0)}\}$. Let $C_{p_\lambda} \cap \Omega_{i_0+1} = \{x_{j'}^{(i_0+1)}\}$, $C_{p_\lambda} \cap \Omega_{i_0} = \{x_{j'}^{(i_0)}\}$ and $C_{q_\mu} \cap \Omega_{i_0} = \{x_{j''}^{(i_0)}\}$. Then $j'' \leq j_0 < j'$. Since $x_{j''}^{(i_0)} < x_r^{(i_0+1)}$ in Ω, one has $x_{j_0}^{(i_0)} < x_r^{(i_0+1)}$. Let $C_{p_\lambda} \cap \Omega_{i_0-1} = \{x_k^{(i_0-1)}\}$. Since $x_k^{(i_0-1)} < x_{j'}^{(i_0)}$ in Ω, one has $x_k^{(i_0-1)} < x_{j_0}^{(i_0)}$. Hence $x_k^{(i_0-1)} < x_{j_0}^{(i_0)} < x_r^{(i_0+1)}$. Thus $(C_{p_\lambda} \setminus \{x_{j'}^{(i_0)}\}) \cup \{x_{j_0}^{(i_0)}\} \in \mathcal{M}(\Omega)$, as required.

COROLLARY 2.2. *Suppose that Ω satisfies the same condition as in Theorem 2.1 and let $I \subset R$ denote the prestable ideal with $G(I) = \{x_C; C \in \mathcal{M}(\Omega)\}$. Then all powers of I have linear resolutions. Moreover, the toric ideal $J_{\mathcal{R}(I)}$ of the Rees algebra $\mathcal{R}(I)$ possesses a squarefree and quadratic Gröbner basis. (A Gröbner basis is called squarefree if the initial monomial of each polynomial belonging to the Gröbner basis is squarefree.)*

PROOF. By virtue of Theorem 2.1 together with Corollary 1.5 it follows that all powers of I have linear resolutions. By using Corollary 1.4, in order to show that $J_{\mathcal{R}(I)}$ has a squarefree and quadratic Gröbner basis, our work is to prove that $J_{\mathcal{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ has a squarefree and quadratic Gröbner basis.

Let $<_{rev}$ denote the reverse lexicographic order on $R = K[\{x_j^{(i)}\}_{\substack{1 \leq i \leq d \\ 1 \leq j \leq p_i}}]$ induced by the same ordering as in the proof of Theorem 2.1. Thus $u_{x_1^{(d)}}$ is the biggest variable and $u_{x_{p_1}^{(1)}}$ is the smallest variable with respect to $<_{rev}$. We then introduce the reverse lexicographic order $<^{(\sharp)}$ on $K[\{y_u\}_{u \in G(I)}]$ such that $y_u <^{(\sharp)} y_v$ if $u <_{rev} v$.

Let $C_\alpha, C_\beta \in \mathcal{M}(\Omega)$. We say that (C_α, C_β) is a *noncrossing pair* if $C_\alpha \cap \Omega_i = \{x_{j_i(\alpha)}^{(i)}\}$ and $C_\beta \cap \Omega_i = \{x_{j_i(\beta)}^{(i)}\}$, then $j_i(\alpha) \leq j_i(\beta)$ for all $1 \leq i \leq d$.

Let $C_\lambda, C_\mu \in \mathcal{M}(\Omega)$ for which (C_λ, C_μ) is a crossing pair, i.e., there is $1 \leq i_0 < d$ such that if $C_\lambda \cap \Omega_{i_0+1} = \{x_j^{(i_0+1)}\}$, $C_\mu \cap \Omega_{i_0+1} = \{x_{j'}^{(i_0+1)}\}$, $C_\lambda \cap \Omega_{i_0} = \{x_k^{(i_0)}\}$ and $C_\mu \cap \Omega_{i_0} = \{x_{k'}^{(i_0)}\}$, then $j' < j$ and $k < k'$. Since Ω satisfies the same condition as in Theorem 2.1, it follows that $x_k^{(i_0)} < x_{j'}^{(i_0+1)}$ and $x_{k'}^{(i_0)} < x_j^{(i_0+1)}$. Let $C_{\lambda'} = (C_\lambda \cap (\cup_{i=i_0+1}^{d} \Omega_i)) \cup (C_\mu \cap (\cup_{i=1}^{i_0} \Omega_i))$ and $C_{\mu'} = (C_\mu \cap (\cup_{i=i_0+1}^{d} \Omega_i)) \cup (C_\lambda \cap (\cup_{i=1}^{i_0} \Omega_i))$. Then $C_{\lambda'}, C_{\mu'} \in \mathcal{M}(\Omega)$ and $x_{C_\lambda} x_{C_\mu} = x_{C_{\lambda'}} x_{C_{\mu'}}$. Repeated applications of such the technique guarantees the existence of $C_\alpha, C_\beta \in \mathcal{M}(\Omega)$ such that (C_α, C_β) is a noncrossing pair

with $x_{C_\lambda} x_{C_\mu} = x_{C_\alpha} x_{C_\beta}$. The existence of a noncrossing pair (C_α, C_β) with $x_{C_\lambda} x_{C_\mu} = x_{C_\alpha} x_{C_\beta}$ is unique and $u_{x_{C_\alpha}} u_{x_{C_\beta}} <^{(\sharp)} u_{x_{C_\lambda}} u_{x_{C_\mu}}$.

Hence by using the well-known technique, e.g., [10, Lemma 0.1], it turns out that the set of all binomials $u_{x_{C_\lambda}} u_{x_{C_\mu}} - u_{x_{C_\alpha}} u_{x_{C_\beta}}$, where (C_α, C_β) is a noncrossing pair, (C_λ, C_μ) is a crossing pair and $x_{C_\lambda} x_{C_\mu} = x_{C_\alpha} x_{C_\beta}$ is a Gröbner basis of $J_{\mathscr{R}(I)} \cap K[\{y_u\}_{u \in G(I)}]$ with respect to $<^{(\sharp)}$.

EXAMPLE 2.3. Let $R = K[\{x_j^{(i)}\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}]$, where $n \le m$, denote the polynomial ring in nm variables and $\mathcal{N}$ the set of all (squarefree) monomials $x_{j_1}^{(1)} x_{j_2}^{(2)} \ldots x_{j_n}^{(n)}$ with $1 \le j_1 < j_2 < \cdots < j_n \le m$. Let $I \subset R$ be the ideal with $G(I) = \mathcal{N}$. Then all powers of I have linear resolutions and the toric ideal $J_{\mathscr{R}(I)}$ of the Rees algebra $\mathscr{R}(I)$ possesses a squarefree and quadratic Gröbner basis. It follows from Corollary 1.3 that, for each $1 \le \ell \le n$, the ideal $J_{\mathscr{R}(I)} \cap K[\{x_j^{(i)}\}_{\substack{\ell \le i \le n \\ 1 \le j \le m}}, \{y_u\}_{u \in G(I)}]$ (as well as the ideal $J_{\mathscr{R}(I)} \cap K[\{x_j^{(i)}\}_{\substack{1 \le i \le \ell \\ 1 \le j \le m}}, \{y_u\}_{u \in G(I)}]$) possesses a squarefree and quadratic Gröbner basis.

3. Computations of Sagbi bases

Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ denote the polynomial ring in d variables over a field K. Given a finite set $\mathscr{F} = \{f_1, \ldots, f_n\} \subset K[\mathbf{t}]$, we write $K[\mathscr{F}] = K[f_1, \ldots, f_n]$ for the subalgebra of $K[\mathbf{t}]$ generated by $f_1, \ldots, f_n$. The *initial algebra* of $K[\mathscr{F}]$ with respect to a monomial order $<$ on $K[\mathbf{t}]$ is the subalgebra

$$in_<(K[\mathscr{F}]) = K[\{in_<(f); f \in K[\mathscr{F}]\}]$$

of $K[\mathbf{t}]$. A subset $\mathscr{S}$ of $K[\mathscr{F}]$ is said to be a *Sagbi basis* of $K[\mathscr{F}]$ with respect to $<$ if $in_<(K[\mathscr{F}])$ is generated by $\{in_<(s); s \in \mathscr{S}\}$. A Sagbi basis always exists. However, a *finite* Sagbi basis does not necessarily exist.

Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over K with each $\deg x_i = 1$ and $I_{\mathscr{F}} (\subset R)$ the defining ideal of $K[\mathscr{F}]$. Thus $I_{\mathscr{F}}$ is the kernel of the surjective ring homomorphism from R to $K[\mathscr{F}]$ defined by setting $x_i \mapsto f_i$.

Given a generic weight vector $w \in \mathsf{R}_{\ge 0}^d$ on $K[\mathbf{t}]$, we introduce the new weight vector $\tilde{w} = (w \cdot \mathbf{a}_1, \ldots, w \cdot \mathbf{a}_n) \in \mathsf{R}_{\ge 0}^n$ on R, where $in_w(f_i) = \mathbf{t}^{\mathbf{a}_i} = t_1^{a_i^{(1)}} \cdots t_d^{a_i^{(d)}}$ with $\mathbf{a}_i = (a_i^{(1)}, \ldots, a_i^{(d)})$. The initial ideal $in_{\tilde{w}}(I_{\mathscr{F}})$ of $I_{\mathscr{F}}$ with respect to $\tilde{w}$ may not be a monomial ideal. Let $J_{in_w(\mathscr{F})} (\subset R)$ denote the toric ideal of the semigroup ring $K[in_w(f_1), \ldots, in_w(f_n)]$. It is known [12, Lemma 11.3] that $in_{\tilde{w}}(I_{\mathscr{F}}) \subset J_{in_w(\mathscr{F})}$. Moreover, [12, Theorem 11.4] says that

LEMMA 3.1. *A subset $\mathscr{F} \subset K[\mathscr{F}]$ is a Sagbi basis of $K[\mathscr{F}]$ with respect to a weight vector w if and only if $J_{in_w(\mathscr{F})} \subset in_{\tilde{w}}(I_{\mathscr{F}})$.*

Fix integers $1 \leq \ell < n \leq m$. Let $X = \left(x_j^{(i)}\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ be the $n \times m$ matrix of variables and $K[X] = K[\{x_j^{(i)}\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}]$ the polynomial ring over K. The notation $[j_1, j_2, \ldots, j_n]$, $1 \leq j_1 < j_2 < \cdots < j_n \leq m$, stands for the $n \times n$ submatrix

$$
\begin{pmatrix}
x_{j_1}^{(1)} & x_{j_2}^{(1)} & \cdots & x_{j_n}^{(1)} \\
x_{j_1}^{(2)} & x_{j_2}^{(2)} & \cdots & x_{j_n}^{(2)} \\
\vdots & & & \vdots \\
x_{j_1}^{(n)} & x_{j_2}^{(n)} & \cdots & x_{j_n}^{(n)}
\end{pmatrix}
$$

of X. Let $X_\ell = \{x_j^{(i)}; 1 \leq i \leq \ell, 1 \leq j \leq m\}$ and

$$
\Gamma(X) = \{\det([j_1, j_2, \ldots, j_n]); 1 \leq j_1 < j_2 < \cdots < j_n \leq m\}.
$$

In [5] the authors discuss the subalgebra $R_\ell(X) = K[\Gamma(X) \cup X_\ell]$ of $K[X]$. If $\ell = 1$, then a Sagbi basis of $R_1(X)$ is given in [9, Proposition 2.1] (when $n = 2$) and [6] (when $n \geq 2$). Using Example 2.3 we determine a Sagbi basis of $R_\ell(X)$ for all $\ell \geq 1$.

Recall that a *diagonal order* on $K[X]$ is a monomial order $<$ on $K[X]$ such that

$$
in_<(\det([j_1, j_2, \ldots, j_n])) = x_{j_1}^{(1)} x_{j_2}^{(2)} \cdots x_{j_n}^{(n)}
$$

for all $\det([j_1, j_2, \ldots, j_n]) \in \Gamma(X)$. We work with the diagonal order $<_{diag}$ on $K[X]$ defined by the weight vector $w = \left(i^j\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$.

THEOREM 3.2. *The finite set $\Gamma(X) \cup X_\ell$ is a Sagbi basis of $R_\ell(X)$ with respect to the weight vector $w = \left(i^j\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$. Moreover, the toric ideal $J_{in_w(\Gamma(X) \cup X_\ell)}$ possesses a squarefree and quadratic initial ideal. In particular the initial algebra $in_w(R_\ell(X))$ is normal and Koszul.*

PROOF. Example 2.3 says that the toric ideal $J_{in_w(\Gamma(X))}$ possesses a square-free quadratic Gröbner basis $\mathcal{G}$ and that

$$
\mathcal{G} \cup \{x_j^{(i)} y_u - x_{j'}^{(i)} y_v; u, v \in \mathcal{N}, x_j^{(i)} u = x_{j'}^{(i)} v, 1 \leq i \leq \ell\}
$$

is a squarefree quadratic Gröbner basis of $J_{in_w(\Gamma(X) \cup X_\ell)}$. Since $\Gamma(X)$ is a Sagbi basis of $K[\Gamma(X)]$ with respect to the weight vector $w = \left(i^j\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ ([12, Theorem 11.8]), it follows that $\mathcal{G} \subset in_{\tilde{w}}(I_{\Gamma(X)}) \subset in_{\tilde{w}}(I_{\Gamma(X) \cup X_\ell})$. By virtue of Lemma 3.1, our work is to show that each binomial $x_j^{(i)} y_u - x_{j'}^{(i)} y_v \in J_{in_w(\Gamma(X) \cup X_\ell)}$ belongs to $in_{\tilde{w}}(I_{\Gamma(X) \cup X_\ell})$.

Let $x_j^{(i)} y_u - x_{j'}^{(i)} y_v \in J_{in_w(\Gamma(X) \cup X_\ell)}$. Then there exist

$$
1 \leq k_1 < \cdots < k_{i-1} < j < j' < k_{i+1} < \cdots < k_n \leq m
$$

such that u (resp. v) is the main diagonal monomial of $\det(U)$ (resp. $\det(V)$), where $U = [k_1, \ldots, k_{i-1}, j', k_{i+1}, \ldots, k_n]$ (resp. $V = [k_1, \ldots, k_{i-1}, j, k_{i+1}, \ldots, k_n]$). Now, we introduce the $(n+1) \times (n+1)$ matrix

$$
M = \begin{pmatrix}
x_{k_1}^{(1)} & \cdots & x_j^{(1)} & x_{j'}^{(1)} & \cdots & x_{k_n}^{(1)} \\
\vdots & & \vdots & \vdots & & \vdots \\
x_{k_1}^{(i)} & \cdots & x_j^{(i)} & x_{j'}^{(i)} & \cdots & x_{k_n}^{(i)} \\
x_{k_1}^{(i)} & \cdots & x_j^{(i)} & x_{j'}^{(i)} & \cdots & x_{k_n}^{(i)} \\
\vdots & & \vdots & \vdots & & \vdots \\
x_{k_1}^{(n)} & \cdots & x_j^{(n)} & x_{j'}^{(n)} & \cdots & x_{k_n}^{(n)}
\end{pmatrix}
$$

with $\det(M) = 0$. Since

$$
\begin{aligned}
\det(M) = & \sum_{p=1}^{i-1} (-1)^p x_{k_p}^{(i)} \det(k_1, \ldots, k_{p-1}, k_{p+1}, \ldots, j, j', \ldots, k_n) \\
& + (-1)^i x_j^{(i)} \det(U) + (-1)^{i+1} x_{j'}^{(i)} \det(V) \\
& - \sum_{p=i+1}^{n} (-1)^p x_{k_p}^{(i)} \det(k_1, \ldots, j, j', \ldots, k_{p-1}, k_{p+1}, \ldots, k_n),
\end{aligned}
$$

the polynomial

$$
f = (-1)^i (x_j^{(i)} y_u - x_{j'}^{(i)} y_v) + \sum_{p=1}^{i-1} (-1)^p x_{k_p}^{(i)} y_{u_p} - \sum_{p=i+1}^{n} (-1)^p x_{k_p}^{(i)} y_{u_p},
$$

where each of the u_p's is equal to either $\det[k_1, \ldots, k_{p-1}, k_{p+1}, \ldots, j, j', \ldots, k_n]$ or $\det[k_1, \ldots, j, j', \ldots, k_{p-1}, k_{p+1}, \ldots, k_n]$, belongs to the defining ideal $I_{\Gamma(X) \cup X_\ell}$. Since the main diagonal monomial of $\det(M)$ is $in_w(x_j^{(i)} \det(U)) = in_w(x_{j'}^{(i)} \det(V))$, it follows that $in_{\tilde{w}}(f) = x_j^{(i)} y_u - x_{j'}^{(i)} y_v$, as desired.

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