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RIGID $\mathcal{OL}_p$ STRUCTURES OF NON-COMMUTATIVE L_p-SPACES ASSOCIATED WITH HYPERFINITE VON NEUMANN ALGEBRAS

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Abstract

This paper is devoted to the study of rigid local operator space structures on non-commutative L_p-spaces. We show that for $1 \leq p \neq 2 < \infty$, a non-commutative L_p-space $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space (equivalently, a rigid $\mathcal{COL}_p$ space) if and only if it is a matrix orderly rigid $\mathcal{OL}_p$ space (equivalently, a matrix orderly rigid $\mathcal{COL}_p$ space). We also show that $L_p(\mathcal{M})$ has these local properties if and only if the associated von Neumann algebra $\mathcal{M}$ is hyperfinite. Therefore, these local operator space properties on non-commutative L_p-spaces characterize hyperfinite von Neumann algebras.

1. Introduction

The aim of this paper is to study the completely positive approximation property and the rigid local operator space structures on non-commutative L_p-spaces associated with hyperfinite von Neumann algebras. Let us first recall from Banach space theory that for $1 \leq p \leq \infty$, a Banach space V is called an $\mathcal{L}_{p,\lambda}$ space for some $\lambda > 1$ if for any $x_1, \ldots, x_n \in V$, there exists a finite dimensional subspace F in V such that $x_1, \ldots, x_n \in F$ and F is λ-isomorphic to ℓ_p^m with $m = \dim F$. In this case, we let

$$\mathcal{L}_p(V) = \inf\{\lambda > 1 : V \text{ is an } \mathcal{L}_{p,\lambda} \text{ space}\}.$$

It was shown by Lindenstrass and Pelczyński [30] that for $1 \leq p < \infty$, a Banach space V is an $\mathcal{L}_{p,\lambda}$ space with $\mathcal{L}_p(V) = 1$ if and only if V is isometrically isomorphic to some $L_p(X, \mu)$ space. This provides a local characterization for classical L_p-spaces.

Classical $L_p(X, \mu)$ spaces $(1 \leq p < \infty)$ can also be characterized by a more rigid $\mathcal{L}_p$ structure, i.e. there are sufficiently many finite dimensional subspaces F in $L_p(X, \mu)$ which are isometrically isomorphic to some ℓ_p^m spaces and their union is norm dense in $L_p(X, \mu)$. Indeed, one can easily see this by

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considering the subspaces $F = \operatorname{span}\{\chi_{E_k} : 1 \leq k \leq m\}$ spanned by characteristic functions χ_{E_k} supported on mutually disjoint non-null sets E_j with finite measure. It is easy to see that F is isometrically isomorphic to ℓ_p^m under the natural injection

$$T : \sum_{j=1}^{m} \alpha_j e_j \in \ell_p^m \mapsto \sum_{j=1}^{m} \alpha_j \frac{\chi_{E_j}}{\mu(E_j)^{\frac{1}{p}}} \in F \subseteq L_p(X, \mu). \tag{1.1}$$

We note that the map T in (1.1) is actually order preserving with range space positively and contractively complemented in $L_p(X, \mu)$. Therefore, $L_p(X, \mu)$ actually has an orderly rigid $\mathscr{L}_p$ structure.

In non-commutative theory, we are interested in non-commutative L_p-spaces $L_p(\mathscr{M})$ associated with von Neumann algebras $\mathscr{M}$. In this case, the canonical matrix order and canonical operator space matrix norm play a central role in our study. For example, it is known that a von Neumann algebra $\mathscr{M}$ is hyperfinite if and only if for some (equivalently for all) $1 \leq p < \infty$, $L_p(\mathscr{M})$ is a semidiscrete non-commutative L_p-space (see [10] and [38]). This characterizes hyperfinite von Neumann algebras by the local matrix order structure on their non-commutative L_p-spaces.

The local operator space structures on non-commutative L_p-spaces have also been studied by Effros and Ruan [11], Junge, Ozawa and Ruan [19], Ng and Ozawa [32], Junge and Ruan [20], and Junge, Nielsen, Ruan and Xu [18]. Let us recall from [11] that for $1 \leq p \leq \infty$, an operator space V is called an $\mathscr{OL}_{p,\lambda}$ space for some $\lambda > 1$ if for any $x_1, \ldots, x_n \in V$, there exists a finite dimensional subspace F in V such that $x_1, \ldots, x_n \in F$ and F is λ-completely isomorphic to $L_p(\mathscr{N})$ for some finite dimensional von Neumann algebra $\mathscr{N}$. If, in addition, F is λ-completely complemented in V, then V is called a $\mathscr{COL}_{p,\lambda}$ space. To simplify our notation, we simply call V an $\mathscr{OL}_p$ space (respectively, a $\mathscr{COL}_p$ space) if it is an $\mathscr{OL}_{p,\lambda}$ space (respectively, a $\mathscr{COL}_{p,\lambda}$ space) for some $\lambda > 1$. In this case, we let

$$\mathscr{OL}_p(V) = \inf\{\lambda > 1 : V \text{ is an } \mathscr{OL}_{p,\lambda} \text{ space}\}$$

and

$$\mathscr{COL}_p(V) = \inf\{\lambda > 1 : V \text{ is an } \mathscr{COL}_{p,\lambda} \text{ space}\}.$$

An operator space V is called a *rigid* $\mathscr{OL}_p$ space if given elements $y_1, \ldots, y_n \in V$ and $\varepsilon > 0$, there exists a finite dimensional von Neumann algebra $\mathscr{N}$ and a completely isometric injection

$$T : L_p(\mathscr{N}) \to V \tag{1.2}$$

such that

$$\mathrm{dist}(T(L_p(\mathcal{N})), y_j) < \varepsilon$$

for all $j = 1, \dots, n$. If $V = L_p(\mathcal{M})$ is a non-commutative L_p-space, we can consider the *matrix orderly rigid* $\mathcal{OL}_p$ structure by requiring that T in (1.2) be a completely positive and completely isometric injection.

We note that for $1 \leq p < +\infty$ if $T : L_p(\mathcal{N}) \to L_p(\mathcal{M})$ is a completely isometric (respectively, a completely positive and completely isometric) injection, then its range space $T(L_p(\mathcal{N}))$ is completely contractively (respectively, completely positively and completely isometrically) complemented in $L_p(\mathcal{M})$. This is obvious when $p = 2$. The $p = 1$ case is known by Ng and Ozawa [32] and the $1 < p \neq 2 < \infty$ case is due to a recent work of Junge, Ruan and Sherman [21]. Therefore, the rigid $\mathcal{OL}_p$ structure and the rigid $\mathcal{COL}_p$ structure (respectively, the matrix orderly rigid $\mathcal{OL}_p$ structure and the matrix orderly rigid $\mathcal{COL}_p$ structure) are equivalent on non-commutative L_p-spaces.

It is known from [11] and [32] that if $V = L_1(\mathcal{M})$ is a non-commutative L_1-space then it satisfies any of the above discussed $\mathcal{OL}_1$ structures if and only if $\mathcal{M}$ is a hyperfinite von Neumann algebra. For $p = \infty$, Junge, Ozawa and Ruan proved in [19] that a C^*-algebra A is an $\mathcal{OL}_\infty$ space if and only if A is a nuclear C^*-algebra, and A is a rigid $\mathcal{OL}_\infty$ space if and only if A is matrix orderly rigid $\mathcal{OL}_\infty$ space. In the latter case, A is called a *strong NF-algebra* by Blackadar and Kirchberg [3]. General $\mathcal{OL}_p$ and $\mathcal{COL}_p$ structures on non-commutative L_p-spaces (for $1 < p < \infty$) have been intensively studied in [20] and [18]. It was shown that there exist non-hyperfinite (discrete) group von Neumann algebras $VN(G)$ such that $L_p(VN(G))$ are $\mathcal{COL}_p$ spaces. Therefore the general $\mathcal{OL}_p$ or $\mathcal{COL}_p$ structure on $L_p(\mathcal{M})$ can not characterize the hyperfiniteness of $\mathcal{M}$.

The aim of this paper is to show that the hyperfiniteness of a von Neumann algebra $\mathcal{M}$ can be fully characterized by the rigid $\mathcal{OL}_p$ structure (respectively, the matrix orderly rigid $\mathcal{OL}_p$ structure) on its non-commutative L_p-spaces. We can state our main results in the following theorem.

THEOREM 1.1. *Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:*

(i) *$\mathcal{M}$ is hyperfinite;*

(ii) *for some (equivalently for all) $1 < p < \infty$, $L_p(\mathcal{M})$ is a semidiscrete non-commutative L_p-space;*

(iii) *for some (equivalently for all) $1 < p < \infty$, $L_p(\mathcal{M})$ has the CPAP;*

(iv) *for some (equivalently for all) $1 < p < \infty$, $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{OL}_p$ space (or a matrix orderly rigid $\mathcal{COL}_p$ space);*

(v) *for some (equivalently for all) $1 < p \neq 2 < \infty$, $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space (or a rigid $\mathcal{COL}_p$ space).*

The paper is organized as follows. We first briefly recall the Haagerup's construction of non-commutative L_p-spaces in §2 and recall matrix order on $L_p(\mathcal{M})$ in §3. Schmitt has shown in [38] that a von Neumann algebra $\mathcal{M}$ is hyperfinite if and only if $L_p(\mathcal{M})$ is semidiscrete ($1 \leq p < \infty$). We show in Theorem 3.2 that this is also equivalent to $L_p(\mathcal{M})$ having the completely positive approximation property. In §4, we recall Kosaki's construction of non-commutative L_p-spaces and recall Pisier's construction of canonical operator space matrix norm on these spaces. We show in Theorem 5.2 that rigid $\mathcal{OL}_p$ structure and matrix orderly rigid $\mathcal{OL}_p$ structure are equivalent on $L_p(\mathcal{M})$ spaces. Therefore, we can conclude that these rigid structures on $L_p(\mathcal{M})$ imply the semidiscreteness of $L_p(\mathcal{M})$ and thus the hyperfiniteness of $\mathcal{M}$. We also give a direct proof in Theorem 5.5 that if $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space for some $1 < p < \infty$, then $L_1(\mathcal{M})$ is a rigid $\mathcal{OL}_1$ space and thus $\mathcal{M}$ is hyperfinite. In §6 and §7, we show that if $\mathcal{M}$ is a hyperfinite von Neumann algebra then $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space. This is quite easy and known in the semifinite case (see, for instance, [38, Chapter 4] or [35, Chapter 3]). The main difficulty is the type *III* case. The key point is that we need to use the direct integral theory to obtain an increasing sequence of normal conditional expectations from $\mathcal{M}$ onto type I von Neumann subalgebras.

To end this section, we note that if $\mathcal{M}$ is a hyperfinite von Neumann algebra then for any $1 < p < \infty$ we can conclude from Theorem 1.1 that $L_p(\mathcal{M})$ is an $\mathcal{OL}_{p,\lambda}$ space for all $\lambda > 1$, i.e. we have $\mathcal{OL}_p(L_p(\mathcal{M})) = 1$. We conjecture that the converse is still true, i.e. if $\mathcal{OL}_p(L_p(\mathcal{M})) = 1$ for some $1 < p \neq 2 < \infty$, then $\mathcal{M}$ must be hyperfinite. Unfortunately, we can not prove this conjecture at this moment. The difficulty is that we do not have a good representation theorem for general completely bounded maps on $L_p(\mathcal{M})$ spaces.

2. Non-commutative L_p-spaces

If $\mathcal{M}$ is a semifinite von Neumann algebra, then there exists a normal faithful semifinite trace τ on $\mathcal{M}$. In this case, the non-commutative L_p-space $L_p(\mathcal{M})$ is defined to be the norm closure

$$L_p(\mathcal{M}, \tau) = \overline{\{x \in \mathcal{M} : \tau(|x|^p) < \infty\}}^{\|\cdot\|_p} \tag{2.1}$$

with norm given by

$$\|x\|_p = \left(\tau\left((x^*x)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}.$$

For any general (not necessarily semifinite) von Neumann algebra $\mathcal{M}$ (acting on a Hilbert space H), we fix a normal faithful semifinite weight φ on $\mathcal{M}$ and let σ_t^φ denote the one-parameter modular automorphism group associated with φ.

We consider the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$, which is a von Neumann subalgebra of $B(L_2(\mathsf{R}, H))$ generated by

$$\pi(x)(\xi(t)) = \sigma^{\varphi}_{-t}(x)(\xi(t)) \qquad \text{and} \qquad \lambda(s)(\xi(t)) = \xi(t-s)$$

for $t \in \mathsf{R}$ and $\xi \in L_2(\mathsf{R}, H)$. Let $W(s)$ be the unitary operator on $L_2(\mathsf{R}, H)$ defined by

$$W(s)(\xi(t)) = e^{-ist}\xi(t).$$

The dual action θ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ is given by

$$\theta_s(x) = W(s)xW(s)^*, \qquad x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}.$$

With this dual action, we have

$$\pi(\mathcal{M}) = \{x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R} : \theta_s(x) = x \text{ for all } s \in \mathsf{R}\},$$

i.e. $\pi(\mathcal{M})$ is the fixed point subalgebra of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ under θ. We identify $\mathcal{M}$ with $\pi(\mathcal{M})$.

We may define a normal faithful operator valued weight

$$T(x) = \int_{\mathsf{R}} \theta_s(x)\, ds$$

from $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R})^+$ into the extended positive part $\hat{\mathcal{M}}^+$ of $\mathcal{M}$ (see [14] and [15]). Since any normal weight ψ on $\mathcal{M}$ extends to a normal weight $\hat{\psi}$ on $\hat{\mathcal{M}}^+$, we can obtain a normal weight

$$\tilde{\psi} = \hat{\psi} \circ T$$

on $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R})^+$, which satisfies

$$\tilde{\psi} \circ \theta_s = \tilde{\psi}, \qquad s \in \mathsf{R}. \tag{2.2}$$

Actually , this gives a bijection $\psi \leftrightarrow \tilde{\psi}$ between the normal semifinite weights ψ on $\mathcal{M}$ and the normal semifinite weights $\tilde{\psi}$ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ satisfying (2.2). Since the given normal weight φ is faithful on $\mathcal{M}$, its dual weight $\tilde{\varphi}$ is also faithful on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$. It is known (see, for instance, [15, Lemma 5.2]) that there exists a unique normal faithful semifinite trace τ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ characterized by the Connes' cocycle

$$(D\tilde{\varphi} : D\tau)_t = \lambda_t$$

and τ satisfies

$$\tau(\theta_s(x)) = e^{-s}\tau(x)$$

for all $(x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R})^+$. Moreover, Pedersen and Takesaki [33] proved that there exists an invertible positive self-adjoint operator D on $L_2(\mathsf{R}, H)$ affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$, the *Radon-Nikodym derivative* of $\tilde{\varphi}$ with respect to τ, such that

$$D^{\mathrm{i}t} = (D\tilde{\varphi} : D\tau)_t = \lambda_t$$

and

$$\tilde{\varphi}(x) = \tau\big(D^{\frac{1}{2}} x D^{\frac{1}{2}}\big) = \tau(Dx) \tag{2.3}$$

for all $x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R})^+$. We also call D in (2.3) the *density operator* associated with φ, which has support $\mathrm{supp}(D) = 1$ since φ is faithful on $\mathcal{M}$. We refer the reader to Terp [44] and Strătilă [40] for more details.

The Haagerup L_p-space $L_p(\mathcal{M}, \varphi)$ associated with the normal faithful semifinite weight φ is defined to be the space of all (unbounded) τ-measurable operators affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ such that

$$\theta_s(x) = e^{-\frac{s}{p}} x$$

for all $s \in \mathsf{R}$.

It is known from Terp [44, Chapter II] that there is a one-to-one correspondence between $\psi \in \mathcal{M}_*^+$ and τ-measurable positive self-adjoint operators $h_\psi = (D\tilde{\psi} : D\tau)$ in $L_1(\mathcal{M}, \varphi)^+$ and we can define a trace linear functional $\mathrm{Tr} : L_1(\mathcal{M}, \varphi) \to \mathsf{C}$ by

$$\mathrm{Tr}(h_\psi) = \psi(1).$$

For any $x \in L_p(\mathcal{M}, \varphi)$, we have the polar decomposition $x = u|x|$, where u is a partial isometry in $\mathcal{M}$ and $|x|$ a positive self-adjoint operator in $L_p(\mathcal{M}, \varphi)$. In this case $|x|^p$ is a positive element in $L_1(\mathcal{M}, \varphi)$. Using this polar decomposition, we define a Banach space norm on $L_p(\mathcal{M}, \varphi)$ by

$$\|x\|_p = (\mathrm{Tr}(|x|^p))^{\frac{1}{p}}.$$

With this norm, it is easy to see that $L_1(\mathcal{M}, \varphi)$ is isometrically isomorphic to $\mathcal{M}_*$. We note that the non-commutative L_p-space constructed above is actually independent of the choice of normal faithful semifinite weight on $\mathcal{M}$ up to isometry. Therefore, we will simply write $L_p(\mathcal{M})$ if there is no confusion.

The usual Hölder inequality holds for non-commutative L_p-spaces. If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $yz \in L_r(\mathcal{M})$ for any $y \in L_p(\mathcal{M})$ and $z \in L_q(\mathcal{M})$ and

$$\|yz\|_r \le \|y\|_p \|z\|_q.$$

In particular, if p' denotes the conjugate index of p we have

$$|\mathrm{Tr}(yz)| \le \|yz\|_1 \le \|y\|_p \|z\|_{p'}$$

for all $y \in L_p(\mathcal{M})$ and $z \in L_{p'}(\mathcal{M})$. From this we can deduce the isometric isomorphism $L_p(\mathcal{M})^* = L_{p'}(\mathcal{M})$ for $1 \leq p < \infty$ given by the *trace duality*

$$\langle x, y\rangle = \mathrm{Tr}(xy) = \mathrm{Tr}(yx)$$

for all $x \in L_p(\mathcal{M})$ and $y \in L_{p'}(\mathcal{M})$.

It should be pointed out that Haagerup L_p-spaces have a drawback that for $p \neq q$, $L_p(\mathcal{M})$ and $L_q(\mathcal{M})$ have only trivial intersection, i.e. $L_p(\mathcal{M}) \cap L_q(\mathcal{M}) = \{0\}$. Therefore, if $\mathcal{M}$ is a semifinite von Neumann algebra, we would like to consider the original (classical) definition of non-commutative L_p-spaces $L_p(\mathcal{M}, \tau)$ defined in (2.1), which is isometrically isomorphic to the Haagerup L_p-space $L_p(\mathcal{M})$ under the isometric isomorphism

$$L_p(\mathcal{M}) \cong L_p(\mathcal{M}, \tau) \otimes \exp((\cdot)/p).$$

3. Semidiscreteness and CPAP of $L_p(\mathcal{M})$

Let $\mathcal{M}$ be a von Neumann algebra. For each $n \in \mathsf{N}$, the space $M_n(\mathcal{M}) = M_n \bar{\otimes} \mathcal{M}$ of all $n \times n$ matrices with entries in $\mathcal{M}$ is again a von Neumann algebra and thus there exists a canonical order on $M_n(\mathcal{M})$ determined by the cone $M_n(\mathcal{M})^+$ of all positive operators in $M_n(\mathcal{M})$. Then $\mathcal{M}$, together with these cones $\{M_n(\mathcal{M})^+\}$, is a *matricially ordered* space (in the sense of Choi and Effros [4]). Any linear map $u : \mathcal{N} \to \mathcal{M}$ induces a linear map

$$\mathrm{id}_{M_n} \otimes u : [x_{ij}] \in M_n(\mathcal{N}) \mapsto [u(x_{ij})] \in M_n(\mathcal{M}).$$

A map u is called *completely positive* if each $\mathrm{id}_{M_n} \otimes u$ is positive, i.e. $\mathrm{id}_{M_n} \otimes u$ maps the positive cone $M_n(\mathcal{N})^+$ into the positive cone $M_n(\mathcal{M})^+$. A von Neumann algebra $\mathcal{M}$ is called *semidiscrete* if there exist contractive normal completely positive maps $u_\alpha : \mathcal{M} \to M_{n(\alpha)}$ and $v_\alpha : M_{n(\alpha)} \to \mathcal{M}$ such that $v_\alpha \circ u_\alpha \to \mathrm{id}_{\mathcal{M}}$ in the point-weak* topology. Considering a slightly weaker condition, we say that $\mathcal{M}$ has the *completely positive approximation property* (CPAP) if there exists a net of contractive normal completely positive finite rank maps $u_\alpha : \mathcal{M} \to \mathcal{M}$ such that $u_\alpha \to \mathrm{id}_{\mathcal{M}}$ in the point-weak* topology. It is known (see Effros and Lance [10]) that a von Neumann algebra $\mathcal{M}$ is semidiscrete if and only if it has the CPAP. It is also well-known (due to the deep work of Connes [7]) that a von Neumann $\mathcal{M}$ is semidiscrete if and only if it is hyperfinite.

Semidiscreteness for non-commutative L_p-spaces has been studied by Schmitt [38]. He proved that a von Neumann algebra $\mathcal{M}$ is hyperfinite if and only if for any $1 \leq p < \infty$, $L_p(\mathcal{M})$ is a semidiscrete L_p-space. Our goal of this section is to show that this is also equivalent to the CPAP of $L_p(\mathcal{M})$. Let us first get ready by recalling some necessary notions and definitions.

Let tr_n denote the canonical trace on M_n. If φ is a normal faithful semifinite weight on $\mathcal{M}$, then $\mathrm{tr}_n \otimes \varphi$ is a normal faithful semifinite weight on $M_n \bar{\otimes} \mathcal{M}$. The non-commutative L_p-space $L_p(M_n \bar{\otimes} \mathcal{M})$ is linearly isomorphic to the (vector) space $M_n(L_p(\mathcal{M}))$ of all $n \times n$ matrices with entries in $L_p(\mathcal{M})$. In this case, we have a canonical Banach space norm on $L_p(M_n \bar{\otimes} \mathcal{M})$ induced by

$$\|x\|_{L_p(M_n \bar{\otimes} \mathcal{M})} = \big((\mathrm{tr}_n \otimes \mathrm{Tr})(|x|^p)\big)^{\frac{1}{p}}$$

for all $x \in L_p(M_n \bar{\otimes} \mathcal{M})$. For each $n \in \mathsf{N}$, there is a canonical order on $L_p(M_n \bar{\otimes} \mathcal{M})$ given by the positive cone $L_p(M_n \bar{\otimes} \mathcal{M})^+$ of all positive self-adjoint operators in $L_p(M_n \bar{\otimes} \mathcal{M})$. With this matrix order $\{L_p(M_n \bar{\otimes} \mathcal{M})^+\}$, $L_p(\mathcal{M})$ is a matricially ordered space. Like in the von Neumann algebra case, a map $u : L_p(\mathcal{N}) \to L_q(\mathcal{M})$ is called *completely positive* if each $\mathrm{id}_{M_n} \otimes u$ maps the positive cone $L_p(M_n \bar{\otimes} \mathcal{N})^+$ into the positive cone $L_q(M_n \bar{\otimes} \mathcal{M})^+$. We note that by a standard argument, it can be shown that every positive and thus completely positive map on a non-commutative L_p-space is automatically bounded.

For $1 \leq p < \infty$ we let $S_p^n = L_p(M_n)$ denote the space of all $n \times n$ matrices equipped with the Schatten p-norm. It is clear that for any $1 \leq p < \infty$, S_p^n has a canonical matrix order, which coincides with the matrix order on M_n. For $1 \leq p < \infty$, a non-commutative L_p-space $L_p(\mathcal{M})$ is called *semidiscrete* if there exist completely positive maps $u_\alpha : L_p(\mathcal{M}) \to S_p^{n(\alpha)}$ and $v_\alpha : S_p^{n(\alpha)} \to L_p(\mathcal{M})$ such that $v_\alpha \circ u_\alpha \to \mathrm{id}_{L_p(\mathcal{M})}$ in the point-norm topology. We can also define *completely positive approximation property* (CPAP) for non-commutative L_p-spaces by assuming that there exists a net of completely positive finite rank maps $u_\alpha : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ such that $u_\alpha \to \mathrm{id}_{L_p(\mathcal{M})}$ in the point-norm topology. In these definitions, the completely positive maps under consideration are not assumed to be uniformly bounded.

THEOREM 3.1. *Let $\mathcal{N}$ be a σ-finite von Neumann algebra with a normal faithful state φ and let $D \in L_1(\mathcal{N})$ denote the density operator of φ. If $1 \leq p, q < \infty$ and $u : L_p(\mathcal{N}) \to L_q(\mathcal{M})$ is a completely positive map, then there exists a normal completely positive contraction $v : \mathcal{N} \to \mathcal{M}$ such that*

$$u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = u(D^{\frac{1}{p}})^{\frac{1}{2}} v(x) u(D^{\frac{1}{p}})^{\frac{1}{2}}$$

for all $x \in \mathcal{N}$.

If u is of finite rank, then v is also of finite rank with $\mathrm{rank}(v) = \mathrm{rank}(u)$.

PROOF. Let us first assume that $\mathrm{supp}\big(u(D^{\frac{1}{p}})\big) = 1_{\mathcal{M}}$. Given $x \in \mathcal{N}^+$, $u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}})$ is a positive element in $L_q(\mathcal{M})^+$ such that

$$0 \leq u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) \leq \|x\| u(D^{\frac{1}{p}}).$$

It is known from [39, Lemma 2.2] that there exists a unique positive element $v(x) \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ such that $\|v(x)\| \le \|x\|$ and

$$u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = u(D^{\frac{1}{p}})^{\frac{1}{2}} v(x) u(D^{\frac{1}{p}})^{\frac{1}{2}}.$$

Since $u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) \le \|x\| u(D^{\frac{1}{p}})$ we conclude that v is a well-defined map from the positive cone of $\mathcal{N}$ into the positive cone of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{R}$ such that $\|v(x)\| \le \|x\|$. Since for any $s \in \mathsf{R}$,

$$\begin{aligned} e^{-\frac{s}{q}} u(D^{\frac{1}{p}})^{\frac{1}{2}} v(x) u(D^{\frac{1}{p}})^{\frac{1}{2}} &= e^{-\frac{s}{q}} u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) \\ &= \theta_s\big(u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}})\big) \\ &= \theta_s\big(u(D^{\frac{1}{p}})^{\frac{1}{2}} v(x) u(D^{\frac{1}{p}})^{\frac{1}{2}}\big) \\ &= e^{-\frac{s}{q}} u(D^{\frac{1}{p}})^{\frac{1}{2}} \theta_s(v(x)) u(D^{\frac{1}{p}})^{\frac{1}{2}}, \end{aligned}$$

we deduce that $\theta_s(v(x)) = v(x)$ and thus $v(x) \in \mathcal{M}^+$.

Now if x is a self-adjoint element in $\mathcal{N}_{s.a.}$, there exist positive elements x_1 and x_2 in $\mathcal{N}^+$ such that $x = x_1 - x_2$. We define

$$v(x) = v(x_1) - v(x_2).$$

If $x = x_1' - x_2'$ for some other $x_i' \in \mathcal{N}_+$, we have

$$\begin{aligned} u(D^{\frac{1}{p}})^{\frac{1}{2}} (v(x_1') - v(x_2')) u(D^{\frac{1}{p}})^{\frac{1}{2}} &= u(D^{\frac{1}{2p}} (x_1' - x_2') D^{\frac{1}{2p}}) \\ &= u(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = u(D^{\frac{1}{2p}} (x_1 - x_2) D^{\frac{1}{2p}}) \\ &= u(D^{\frac{1}{p}})^{\frac{1}{2}} (v(x_1) - v(x_2)) u(D^{\frac{1}{p}})^{\frac{1}{2}}; \end{aligned}$$

whence

$$v(x_1') - v(x_2') = v(x_1) - v(x_2).$$

Therefore v is a well-defined map from $\mathcal{N}_{s.a.}$ into $\mathcal{M}_{s.a.}$. In particular, if we consider the orthogonal decomposition $x = x^+ - x^-$, we obtain

$$\begin{aligned} \|v(x)\| = \|v(x^+) - v(x^-)\| &= \sup\{|\langle v(x^+)\xi|\xi\rangle - \langle v(x^-)\xi|\xi\rangle| : \|\xi\| \le 1\} \\ &\le \max\{\|v(x^+)\|, \|v(x^-)\|\} \le \|x\|. \end{aligned}$$

Therefore, v is a real contractive linear map from $\mathcal{N}_{s.a}$ to $\mathcal{M}_{s.a.}$. Furthermore, we extend v to a complex linear map

$$v(x + \mathrm{i}y) = v(x) + \mathrm{i}v(y)$$

from $\mathcal{N}$ into $\mathcal{M}$. By the complete positivity of u, we conclude that v is a completely positive contraction such that

$$u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}}) = u(D^{\frac{1}{p}})^{\frac{1}{2}}v(x)u(D^{\frac{1}{p}})^{\frac{1}{2}}.$$

Since $1 \leq p, q < \infty$, we may consider $u^* : L_{q'}(\mathcal{M}) \to L_{p'}(\mathcal{N})$. We define $\xi = u(D^{\frac{1}{p}})^q$ and consider the densely defined map $w : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ as follows

$$w(\xi^{\frac{1}{2}}y\xi^{\frac{1}{2}}) = D^{\frac{1}{2p}}u^*(\xi^{\frac{1}{2q'}}y\xi^{\frac{1}{2q'}})D^{\frac{1}{2p}}, \qquad y \in \mathcal{M}.$$

Note that

$$\begin{aligned}
\mathrm{Tr}(w(\xi^{\frac{1}{2}}y\xi^{\frac{1}{2}})x) &= \mathrm{Tr}(D^{\frac{1}{2p}}u^*(\xi^{\frac{1}{2q'}}y\xi^{\frac{1}{2q'}})D^{\frac{1}{2p}}x)\\
&= \mathrm{Tr}(u^*(\xi^{\frac{1}{2q'}}y\xi^{\frac{1}{2q'}})D^{\frac{1}{2p}}xD^{\frac{1}{2p}})\\
&= \mathrm{Tr}(\xi^{\frac{1}{2q'}}y\xi^{\frac{1}{2q'}}u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}}))\\
&= \mathrm{Tr}(\xi^{\frac{1}{2q'}}y\xi^{\frac{1}{2q'}}\xi^{\frac{1}{2q}}v(x)\xi^{\frac{1}{2q}})\\
&= \mathrm{Tr}(\xi^{\frac{1}{2}}y\xi^{\frac{1}{2}}v(x)).
\end{aligned}$$

This implies

$$|\,\mathrm{Tr}(w(\xi^{\frac{1}{2}}y\xi^{\frac{1}{2}})x)| \leq \|\xi^{\frac{1}{2}}y\xi^{\frac{1}{2}}\|\|v\|\|x\|.$$

Therefore w extends to a contraction from $L_1(\mathcal{M})$ into $L_1(\mathcal{N})$ and it is clear that $w^* = v$. Hence v is indeed normal.

Assume that u is a finite rank map. Since

$$J_{D^{\frac{1}{2p}}} : x \in \mathcal{N} \mapsto D^{\frac{1}{2p}}xD^{\frac{1}{2p}} \in L_p(\mathcal{N})$$

is a contractive map with dense range in $L_p(\mathcal{N})$, we can easily conclude that

$$\mathrm{rank}(u) = \mathrm{rank}(u \circ J_{D^{\frac{1}{2p}}}).$$

Moreover, since we assume that $\mathrm{supp}(u(D^{\frac{1}{p}})) = 1_{\mathcal{M}}$,

$$J_{u(D^{\frac{1}{p}})^{\frac{1}{2}}} : y \in \mathcal{M} \mapsto u(D^{\frac{1}{p}})^{\frac{1}{2}}yu(D^{\frac{1}{p}})^{\frac{1}{2}}$$

is an injective inclusion from $\mathcal{M}$ into $L_p(\mathcal{M})$. Then it is easy to show that

$$\mathrm{rank}(v) = \mathrm{rank}(J_{u(D^{\frac{1}{p}})^{\frac{1}{2}}} \circ v) = \mathrm{rank}(u \circ J_{D^{\frac{1}{2p}}}) = \mathrm{rank}(u).$$

In general we let e be the support projection of $u(D^{\frac{1}{p}})$. Then e is a projection in $\mathcal{M}$ and for any $x \in \mathcal{N}^+$, we have

$$u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}}) \leq \|x\|u(D^{\frac{1}{p}}).$$

This implies

$$(1-e)u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}})(1-e)=0.$$

Since $u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}})$ is positive, we also have

$$eu(D^{\frac{1}{2p}}xD^{\frac{1}{2p}})(1-e)=0=(1-e)u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}})e.$$

By continuity, we deduce that

$$u(x)=eu(x)e$$

for all $x\in L_p(\mathcal{N})$. So we actually have $u: L_p(\mathcal{N})\to eL_q(\mathcal{M})e=L_q(e\mathcal{M}e)$, and hence we can repeat the above argument by replacing $\mathcal{M}$ by $e\mathcal{M}e$.

We note that if $q=\infty$ and $u: L_p(\mathcal{N})\to\mathcal{M}$ is a completely positive map, then we can obtain a completely positive map $v:\mathcal{N}\to\mathcal{M}$ given by

$$v(x)=u(D^{\frac{1}{2p}}xD^{\frac{1}{2p}}).$$

But in this case, we can not claim that v is a normal map.

The following improves a result of Schmitt [38], who showed the equivalence between (i) and (ii) below. Note that in the case $p=1$, this is due to Effros and Lance [10].

THEOREM 3.2. *Let $\mathcal{M}$ be a von Neumann algebra and let $1\le p<\infty$. Then the following are equivalent:*

(i) *$\mathcal{M}$ is semidiscrete;*

(ii) *$L_p(\mathcal{M})$ is semidiscrete;*

(iii) *$L_p(\mathcal{M})$ has the CPAP.*

PROOF. We first recall the following elementary fact that given an arbitrary von Neumann algebra $\mathcal{M}$, we can always find a normal faithful semifinite weight φ on $\mathcal{M}$ and an increasing net of projections $e_i\to 1_{\mathcal{M}}$ in $\mathcal{M}$ such that for every i, $\sigma_t^{\varphi}(e_i)=e_i$ for all $t\in\mathsf{R}$ and such that the reduced von Neumann subalgebra $e_i\mathcal{M}e_i$ is σ-finite. In this case, we can completely (orderly) identify $L_p(e_i\mathcal{M}e_i)$ with a subspace of $L_p(\mathcal{M})$ and there exists a completely positive projection from $L_p(\mathcal{M})$ onto $L_p(e_i\mathcal{M}e_i)$ (given by $x\mapsto e_ixe_i$). Moreover the union of these spaces is norm dense in $L_p(\mathcal{M})$. Then $\mathcal{M}$ and $L_p(\mathcal{M})$ are semidiscrete if and only if $e_i\mathcal{M}e_i$ and $L_p(e_i\mathcal{M}e_i)$ are semidiscrete for each i. Therefore, it suffices to prove Theorem 3.2 in the σ-finite case, and so in the following we assume that $\mathcal{M}$ is σ-finite and equipped with a normal faithful state φ.

(i) ⇒ (ii) This was proved by Schmitt in his thesis [38]. According to our knowledge, this result has not been published in print. For the completeness and the convenience of the reader, we include his argument here.

Given positive elements $x_1, \ldots, x_k \in L_p(\mathcal{M})^+$ with $\|x_i\| \leq 1$, there exists a positive operator $\tilde{x} \in L_p(\mathcal{M})^+$ such that $\mathrm{supp}(\tilde{x}) = 1$ and $0 \leq \sum_{i=1}^k x_i \leq \tilde{x}$. It follows from [39, Lemma 2.2] that there exists $r_i \in \mathcal{M}$ such that $0 \leq r_i \leq 1$ and $x_i = \tilde{x}^{\frac{1}{2}} r_i \tilde{x}^{\frac{1}{2}}$. For any positive elements $y_1, \ldots, y_l \in L_{p'}(\mathcal{M})^+$ and $\varepsilon > 0$, we have $\tilde{x}^{\frac{1}{2}} y_j \tilde{x}^{\frac{1}{2}} \in L_1(\mathcal{M})^+$. Since $\mathcal{M}$ is semidiscrete, there exist normal completely positive maps $\tilde{u} : \mathcal{M} \to M_n$ and $\tilde{v} : M_n \to \mathcal{M}$ such that

$$|\langle \tilde{v} \circ \tilde{u}(r_i) - r_i,\ \tilde{x}^{\frac{1}{2}} y_j \tilde{x}^{\frac{1}{2}} \rangle| < \varepsilon.$$

Since M_n and S_p^n have the same matrix order structure we can regard $v(\cdot) = \tilde{x}^{\frac{1}{2}} \tilde{v}(\cdot) \tilde{x}^{\frac{1}{2}}$ as a completely positive map from S_p^n into $L_p(\mathcal{M})$.

The map $\tilde{u} : \mathcal{M} \to M_n$ can be identified with a positive element $[\tilde{u}_{ij}]$ in $L_1(M_n \bar{\otimes} \mathcal{M})$ under the trace duality

$$\langle [\tilde{u}_{ij}],\ [r_{ij}] \rangle = \sum_{i,j=1}^{n} \mathrm{Tr}(\tilde{u}_{ij} r_{ji})$$

for all $[r_{ij}] \in M_n(\mathcal{M})$. Since $\mathrm{supp}(\tilde{x}) = 1$, $\tilde{x}^{\frac{1}{2}} L_{p'}(\mathcal{M}) \tilde{x}^{\frac{1}{2}}$ is norm dense in $L_1(\mathcal{M})$ and its matrix space

$$(\tilde{x}^{\frac{1}{2}} \oplus \cdots \oplus \tilde{x}^{\frac{1}{2}}) L_{p'}(M_n \bar{\otimes} \mathcal{M}) (\tilde{x}^{\frac{1}{2}} \oplus \cdots \oplus \tilde{x}^{\frac{1}{2}}) = \{[\tilde{x}^{\frac{1}{2}} z_{ij} \tilde{x}^{\frac{1}{2}}] : z_{ij} \in L_{p'}(\mathcal{M})\}$$

(respectively, its positive cone) is norm dense in $L_1(M_n \bar{\otimes} \mathcal{M})$ (respectively, norm dense in its positive cone). Then we can choose a positive element $z = [z_{ij}]$ in $L_{p'}(M_n \bar{\otimes} \mathcal{M})$ such that

$$\left\| [\tilde{x}^{\frac{1}{2}} z_{ij} \tilde{x}^{\frac{1}{2}} - \tilde{u}_{ij}] \right\|_{L_1(M_n \bar{\otimes} \mathcal{M})} < \frac{\varepsilon}{K}, \tag{3.1}$$

where $K = \max\{\|\tilde{v}\|\ \|\tilde{x}^{\frac{1}{2}} y_j \tilde{x}^{\frac{1}{2}}\|\}$. Since $[z_{ij}]$ is a positive element in $L_{p'}(M_n \bar{\otimes} \mathcal{M})$ it induces a completely positive map u from $L_p(\mathcal{M})$ into S_p^n by letting

$$u(x) = [\mathrm{Tr}(z_{ij} x)]$$

for $x \in L_p(\mathcal{M})$. In particular, if we restrict u to $\tilde{x}^{\frac{1}{2}} \mathcal{M} \tilde{x}^{\frac{1}{2}}$, we have

$$u(\tilde{x}^{\frac{1}{2}} r \tilde{x}^{\frac{1}{2}}) = [\mathrm{Tr}(z_{ij} (\tilde{x}^{\frac{1}{2}} r \tilde{x}^{\frac{1}{2}}))] = [\mathrm{Tr}((\tilde{x}^{\frac{1}{2}} z_{ij} \tilde{x}^{\frac{1}{2}}) r)]. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}|\langle v \circ u(x_i) - x_i,\ y_j\rangle| &= |\langle v \circ u(\tilde{x}^{\frac{1}{2}} r_i \tilde{x}^{\frac{1}{2}}) - \tilde{x}^{\frac{1}{2}} r_i \tilde{x}^{\frac{1}{2}},\ y_j\rangle| \\ &= |\langle \tilde{v} \circ u(\tilde{x}^{\frac{1}{2}} r_i \tilde{x}^{\frac{1}{2}}) - r_i,\ \tilde{x}^{\frac{1}{2}} y_j \tilde{x}^{\frac{1}{2}}\rangle| \\ &\le |\langle \tilde{v} \circ \tilde{u}(r_i) - r_i,\ \tilde{x}^{\frac{1}{2}} y_j \tilde{x}^{\frac{1}{2}}\rangle| + \varepsilon < 2\varepsilon.\end{aligned}$$

This shows that we can find completely positive finite rank maps $u_\alpha\colon L_p(\mathcal{M}) \to S^{n(\alpha)}$ and $v_\alpha : S^{n(\alpha)} \to L_p(\mathcal{M})$ such that $v_\alpha \circ u_\alpha \to \mathrm{id}_{L_p(\mathcal{M})}$ in the point-weak topology. By a standard convexity argument, we can find such maps converging to $\mathrm{id}_{L_p(\mathcal{M})}$ in the point-norm topology. This shows that $L_p(\mathcal{M})$ is semidiscrete.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Let $u_\alpha : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a net of completely positive finite rank maps such that $u_\alpha \to \mathrm{id}_{L_p(\mathcal{M})}$ in the point-norm topology. It is known from Theorem 3.1 that for each α there exists a normal completely positive finite rank contractions $v_\alpha : \mathcal{M} \to \mathcal{M}$ such that

$$u_\alpha(D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = u_\alpha(D^{\frac{1}{p}})^{\frac{1}{2}} v_\alpha(x) u_\alpha(D^{\frac{1}{p}})^{\frac{1}{2}}.$$

We claim that $v_\alpha \to \mathrm{id}_{\mathcal{M}}$ in the point-weak* topology. To see this we note that $D^{\frac{1}{2p}} L_{p'}(\mathcal{M}) D^{\frac{1}{2p}}$ is norm dense in $L_1(\mathcal{M})$. We also note that since $D^{\frac{1}{p}}$ is an element in $L_p(\mathcal{M})$ we have $u_\alpha(D^{\frac{1}{p}}) \to D^{\frac{1}{p}}$ in $L_p(\mathcal{M})$ and thus $u_\alpha(D^{\frac{1}{p}})^{\frac{1}{2}} \to D^{\frac{1}{2p}}$ in $L_{2p}(\mathcal{M})$. Then for any $x \in \mathcal{M}$ and $y \in L_{p'}(\mathcal{M})$, we have

$$D^{\frac{1}{2p}} y D^{\frac{1}{2p}} = \lim u_\alpha(D^{\frac{1}{p}})^{\frac{1}{2}} y u_\alpha(D^{\frac{1}{p}})^{\frac{1}{2}}.$$

It is routine to verify that

$$|\langle v_\alpha(x) - x, D^{\frac{1}{2p}} y D^{\frac{1}{2p}}\rangle| \to 0.$$

This shows that $v_\alpha \to \mathrm{id}_{\mathcal{M}}$ in the point-weak* topology. Therefore $\mathcal{M}$ is a semidiscrete von Neumann algebra.

4. Canonical operator space structure on $L_p(\mathcal{M})$

As we have discussed in the previous section, for each $n \in \mathsf{N}$, there is a canonical norm on the non-commutative L_p-space $L_p(M_n \bar{\otimes} \mathcal{M})$. However, this family of norms $\{\|\cdot\|_{L_p(M_n \bar{\otimes} \mathcal{M})}\}$ on the matrix spaces over $L_p(\mathcal{M})$ only satisfies the L_p-matricial condition

$$\|x \oplus y\|_{L_p(M_{m+n} \bar{\otimes} \mathcal{M})} = (\|x\|^p_{L_p(M_m \bar{\otimes} \mathcal{M})} + \|x\|^p_{L_p(M_n \bar{\otimes} \mathcal{M})})^{\frac{1}{p}}$$

for all $x \in L_p(M_m \bar{\otimes} \mathcal{M})$, $y \in L_p(M_n \bar{\otimes} \mathcal{M})$. Therefore, it is not an operator space matrix norm on $L_p(\mathcal{M})$. In [35], Pisier succeeded in constructing a canonical operator space matrix norm on non-commutative L_p-spaces by using the complex interpolation method. In the following let us first recall this construction for σ-finite von Neumann algebras, which on Banach space level is due to Kosaki [28]. The construction for general von Neumann algebras can be obtained similarly by applying Terp's contruction developed in [45].

Let $\mathcal{M}$ be a σ-finite von Neumann algebra with a normal faithful positive linear functional (or state) φ. We may obtain a standard representation $(H_\varphi, \xi_\varphi, J_\varphi, P)$ of $\mathcal{M}$ induced by φ, where $\xi_\varphi = 1_\varphi$ is a cyclic and separating vector for $\mathcal{M}$ such that $\varphi(x) = \langle x\xi_\varphi \mid \xi_\varphi\rangle$ for all $x \in \mathcal{M}$ and J_φ is the conjugate linear isomorphism on H_φ obtained from the polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{\frac{1}{2}}$. The map S_φ is the closure of the involution

$$x\xi_\varphi \mapsto x^*\xi_\varphi.$$

The operator $\Delta_\varphi = S_\varphi^* S_\varphi$ is a self-adjoint positive unbounded operator for non-tracial φ. If φ is tracial, then $\Delta_\varphi = \mathrm{id}_{H_\varphi}$. We have

$$J_\varphi \Delta_\varphi^{\frac{1}{2}} = \Delta_\varphi^{-\frac{1}{2}} J_\varphi, \qquad \Delta_\varphi \xi_\varphi = \xi_\varphi \qquad \text{and} \qquad J_\varphi \xi_\varphi = \xi_\varphi, \tag{4.1}$$

and a one-parameter $*$-automorphism group σ_t^φ on $\mathcal{M}$ given by

$$\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$$

for all $t \in \mathbb{R}$. Kosaki showed in [28, Theorem 2.5] that for every $x \in \mathcal{M}$, the map

$$t \in \mathbb{R} \mapsto \sigma_t^\varphi(x) \cdot \varphi = \varphi(\,\cdot\,\sigma_t^\varphi(x)) \in \mathcal{M}_*$$

extends to a bounded and continuous $\mathcal{M}_*$-valued function

$$f_x(z) = \begin{cases} g(z) = \langle \cdot\, \Delta_\varphi^{iz} x\xi_\varphi \mid \xi_\varphi\rangle & \text{if } -\frac{1}{2} \le \operatorname{Im} z \le 0 \\ h(z) = \langle \cdot\, \xi_\varphi \mid \Delta_\varphi^{1+\mathrm{i}\bar{z}} x^*\xi_\varphi\rangle & \text{if } -1 \le \operatorname{Im} z \le -\frac{1}{2} \end{cases} \tag{4.2}$$

on the strip $\{z \in \mathbb{C} : -1 \le \operatorname{Im} z \le 0\}$, which is analytic in the interior of the strip and satisfies

$$f_x(-\mathrm{i} + t) = \varphi \cdot \sigma_t^\varphi(x) = \varphi(\sigma_t^\varphi(x) \cdot\,).$$

Therefore, for any $\eta \in [0, 1]$, we obtain a bounded injective embedding

$$\mathcal{I}_\eta^\varphi : x \in \mathcal{M} \mapsto f_x(-\mathrm{i}\eta) \in \mathcal{M}_*. \tag{4.3}$$

With this notation, the left embedding $x \in \mathcal{M} \mapsto x \cdot \varphi = \varphi(\,\cdot\, x) \in \mathcal{M}_*$ and the right embedding $x \in \mathcal{M} \mapsto \varphi \cdot x = \varphi(x\,\cdot\,) \in \mathcal{M}_*$ are given by $\mathcal{I}_0^\varphi$ and $\mathcal{I}_1^\varphi$, respectively.

For any $1 \leq p \leq \infty$ and $\eta \in [0, 1]$, we let

$$L_p(\mathcal{M}, \mathcal{I}_\eta^\varphi) = (\mathcal{I}_\eta^\varphi(\mathcal{M}), \mathcal{M}_*)_{\frac{1}{p}}$$

denote the complex interpolation space associated with the embedding $\mathcal{I}_\eta^\varphi$. Kosaki [28] proved that these spaces $L_p(\mathcal{M}, \mathcal{I}_\eta^\varphi)$ are all isometric to the Haagerup L_p-space $L_p(\mathcal{M})$.

If $\eta = \frac{1}{2}$, we obtain from (4.1) and (4.2) that

$$\begin{aligned}\mathcal{I}_{\frac{1}{2}}^\varphi(x)(y) &= \langle \Delta_\varphi^{\frac{1}{2}} x\xi_\varphi \mid y^*\xi_\varphi\rangle = \langle \Delta_\varphi^{\frac{1}{2}}(\Delta_\varphi^{-\frac{1}{2}} J_\varphi)x^*\xi_\varphi \mid y^*\xi_\varphi\rangle \\ &= \langle J_\varphi x^*\xi_\varphi \mid y^*\xi_\varphi\rangle = \langle J_\varphi y^*\xi_\varphi \mid x^*\xi_\varphi\rangle = \langle x J_\varphi y^* J_\varphi \xi_\varphi \mid \xi_\varphi\rangle\end{aligned}$$

for all $x, y \in \mathcal{M}$. In this case, we can regard $\mathcal{I}_{\frac{1}{2}}^\varphi$ as a map

$$\mathcal{I}_{\frac{1}{2}}^\varphi : x \in \mathcal{M} \mapsto \mathcal{I}_{\frac{1}{2}}^\varphi(x) \in (\mathcal{M}')_*, \tag{4.4}$$

where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$. This is of particular interest since we may regard $\mathcal{M}'$ as a concrete representation of $\mathcal{M}^{\mathrm{op}}$ on H_φ with the $*$-isomorphism given by

$$\pi : x^{\mathrm{op}} \in \mathcal{M}^{\mathrm{op}} \mapsto J_\varphi x^* J_\varphi \in \mathcal{M}'.$$

The operator predual $(\mathcal{M}')_*$ of $\mathcal{M}'$ can be (completely isometrically) identified with the operator predual $(\mathcal{M}^{\mathrm{op}})_*$ of $\mathcal{M}^{\mathrm{op}}$, where the latter space can be identified with $\mathcal{M}_*$ as a Banach space but is equipped with the opposite operator space matrix norm

$$\|[\omega_{ij}^{\mathrm{op}}]\| = \|[\omega_{ji}]\|,$$

i.e. the matrix norm on $M_n((\mathcal{M}^{\mathrm{op}})_*)$ is given by the isometric identification

$$M_n \check{\otimes} (\mathcal{M}^{\mathrm{op}})_* = M_n^{\mathrm{op}} \check{\otimes} \mathcal{M}_*.$$

Then we can regard (4.4) as a canonical embedding

$$\mathcal{I}_{\frac{1}{2}}^\varphi : \mathcal{M} \to (\mathcal{M}_*)^{\mathrm{op}} = (\mathcal{M}^{\mathrm{op}})_*.$$

With this embedding, we can define a canonical operator space matrix norm on $L_p(\mathcal{M})$ by the complex interpolation

$$M_n(L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^\varphi)) = ((M_n(\mathcal{I}_{\frac{1}{2}}^\varphi(\mathcal{M}))), M_n(\mathcal{M}_*^{\mathrm{op}})))_{\frac{1}{p}} \tag{4.5}$$

(see Pisier [35] and [36]).

The reason we should consider the opposite space $\mathcal{M}_*^{op}$ in (4.5) is because the canonical duality between M_n and its operator dual $T_n = M_n^*$ is given by the *parallel duality*

$$\langle x, \omega \rangle = \sum_{i,j=1}^{n} \langle x_{ij}, \omega_{ij} \rangle.$$

With this duality, we obtain the complete isometry

$$(M_n \bar{\otimes} \mathcal{M})_* = T_n \hat{\otimes} \mathcal{M}_*$$

(see [12]). However if we wish to use the trace duality

$$\langle x, \omega \rangle^{\mathrm{tr}_n} = \sum_{i,j=1}^{n} \langle x_{ij}, \omega_{ji} \rangle,$$

(which corresponds to the parallel duality between $M_n \otimes \mathcal{M}$ and $T_n^{\mathrm{op}} \otimes \mathcal{M}_*$) we obtain the complete isometries

$$(M_n \bar{\otimes} \mathcal{M})_*^{\mathrm{tr}_n} = T_n^{op} \hat{\otimes} \mathcal{M}_*.$$

Therefore, it is more appropriate to define $L_1(\mathcal{M}, \varphi) = \mathcal{M}_*^{\mathrm{op}}$ in the operator space setting. With this notation, we have the complete isometries

$$L_1(M_n \bar{\otimes} \mathcal{M}, \mathrm{tr}_n \otimes \varphi) = (T_n \hat{\otimes} \mathcal{M}_*)^{op} = T_n^{op} \hat{\otimes} \mathcal{M}_*^{op} = L_1(M_n, \mathrm{tr}_n) \hat{\otimes} L_1(\mathcal{M}, \varphi).$$

Let R^n and C^n denote the n-dimensional row and column Hilbert spaces, respectively. For $1 < p < \infty$, Pisier [35] showed that for every operator space V, there is a canonical non-commutative S_p^n integral defined by the Haagerup tensor product

$$S_p^n[V] = (C^n, R^n)_{\frac{1}{p}} \otimes^h V \otimes^h (R^n, C^n)_{\frac{1}{p}}.$$

Then for each $n \in \mathsf{N}$, we claim the following isometric identification

$$S_p^n[L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi})] = L_p(M_n \bar{\otimes} \mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\mathrm{tr}_n \otimes \varphi}). \tag{4.6}$$

Indeed, by Pisier [34, Theorem 2.3] and the preceding discussion we obtain the isometries

$$\begin{aligned} S_p^n[L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi})] &= (C^n, R^n)_{\frac{1}{p}} \otimes^h (\mathcal{I}_{\frac{1}{2}}^{\varphi}(\mathcal{M}), (\mathcal{M}^{\mathrm{op}})_*)_{\frac{1}{p}} \otimes^h (R^n, C^n)_{\frac{1}{p}} \\ &= (C^n \otimes^h \mathcal{I}_{\frac{1}{2}}^{\varphi}(\mathcal{M}) \otimes^h R^n, R^n \otimes^h (\mathcal{M}^{\mathrm{op}})_* \otimes^h C^n)_{\frac{1}{p}} \\ &= ((\mathrm{id}_{M_n} \otimes \mathcal{I}_{\frac{1}{2}}^{\varphi})(M_n \bar{\otimes} \mathcal{M}),\ T_n \hat{\otimes} \mathcal{M}_*^{\mathrm{op}})_{\frac{1}{p}} \\ &= (\mathcal{I}_{\frac{1}{2}}^{\mathrm{tr}_n \otimes \varphi}(M_n \bar{\otimes} \mathcal{M}),\ T_n^{\mathrm{op}} \hat{\otimes} \mathcal{M}_*^{\mathrm{op}})_{\frac{1}{p}} \\ &= L_p(M_n \bar{\otimes} \mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\mathrm{tr}_n \otimes \varphi}). \end{aligned}$$

On the other hand, Pisier proved in [35] that the operator space matrix norms on $M_n(L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi}))$ can be recovered from the norms on $S_p^n[L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi})]$, i.e. for every $x \in M_n(L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi}))$ we have

$$\|x\|_{M_n(L_p(\mathcal{M}, \mathcal{I}_{\frac{1}{2}}^{\varphi}))} = \sup\{\|\alpha x \beta\|_{L_p(M_n \bar{\otimes} \mathcal{M})} : \|\alpha\|_{S_{2p}^n}, \|\beta\|_{S_{2p}^n} \le 1\}. \tag{4.7}$$

Moreover, a linear map $T : V \to W$ between operator spaces V and W is a complete contraction (respectively, a complete isometry, or a complete quotient map) if and only if for each $n \in \mathsf{N}$, the induced map

$$\mathrm{id}_{S_p^n} \otimes T : S_p^n[V] \to S_p^n[W]$$

is a contraction (respectively, an isometry or a quotient map). Therefore, we only need to work with the Haagerup non-commutative L_p-spaces $L_p(M_n \bar{\otimes} \mathcal{M})$ in the rest of this paper.

In the above construction, we have used the symmetric embedding $\mathcal{I}_{\frac{1}{2}}^{\varphi}$ from $\mathcal{M}$ into $L_1(\mathcal{M})$. It is worthy to note that we can also consider the more general embeddings $\mathcal{I}_{\eta}^{\varphi}$ given in (4.3). In virtue of (4.6) (and its analogue for $\mathcal{I}_{\eta}^{\varphi}$) and Kosaki [28], we see that the resulting non-commutative L_p-spaces are all completely isometric (by completely positive isometries). Similarly, all these non-commutative L_p-spaces are independent of φ up to complete isometry.

Finally for each non-σ-finite von Neumann algebra $\mathcal{M}$, we can define a canonical operator space structure on $L_p(\mathcal{M})$ by using the complex interpolation technique developed by Terp [45]. We can also use equation (4.7) to determine this canonical operator space matrix norm on $L_p(\mathcal{M})$.

5. Rigid $\mathcal{OL}_p$ structure

Our goal of this section is to show in Theorem 5.2 that the rigid $\mathcal{OL}_p$ structure and the matrix orderly rigid $\mathcal{OL}_p$ structure are equivalent on non-commutative L_p-spaces. A major tool needed is a generalization of Yeadon's theorem for completely isometric inclusions between non-commutative L_p-spaces. Let us first recall that the structure of isometric isomorphisms between classical L_p-spaces has been studied by Banach [2] and Lamperti [29]. Broise [9], Russo [37], Arazy [1], Katavolos [24], [25], [26], and Tam [43] developed a corresponding structure theory for isometric isomorphisms between non-commutative L_p-spaces. In 1981, Yeadon obtained the following very satisfactory result for isometric inclusions between non-commutative L_p-spaces associated with semifinite von Neumann algebras.

THEOREM 5.1 (Yeadon [46, Theorem 2]). *Let $\mathcal{N}$ and $\mathcal{M}$ be semifinite von Neumann algebras with normal faithful semifinite traces $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{M}}$, respectively. For $1 \leq p \neq 2 < \infty$, a linear map*

$$T : L_p(\mathcal{N}, \varphi_{\mathcal{N}}) \to L_p(\mathcal{M}, \varphi_{\mathcal{M}})$$

is an isometric injection if and only if there exist a weak* continuous $*$-Jordan isomorphism $J : \mathcal{N} \to J(\mathcal{N}) \subseteq \mathcal{M}$, a positive operator $B \in L_p(\mathcal{M}, \varphi_{\mathcal{M}})$ commuting with all $J(x)$ with $x \in \mathcal{N}$, and a partial isometry $W \in \mathcal{M}$ with $W^*W = J(1) = \mathrm{supp}(B)$ such that

$$T(x) = WBJ(x) \tag{5.1}$$

for all $x \in \mathcal{N} \bigcap L_p(\mathcal{N}, \varphi_{\mathcal{N}})$ and

$$\varphi_{\mathcal{N}}(x) = \varphi_{\mathcal{M}}(B^p J(x)) \tag{5.2}$$

for all $x \in \mathcal{N}^+$.

Slightly modifying Yeadon's argument we proved in [21] that Yeadon's result still holds if $\mathcal{N}$ is finite (or semifinite) and $\mathcal{M}$ is an arbitrary von Neumann algebra. In this general case, we only need to replace (5.2) by

$$\varphi_{\mathcal{N}}(x) = \mathrm{Tr}_{\mathcal{M}}(B^p J(x)), \tag{5.3}$$

where $\mathrm{Tr}_{\mathcal{M}}$ is the distinguished tracial functional on $L_1(\mathcal{M})$. It was also shown in [21] that if T is a completely isometric injection, then J in (5.1) is actually a normal injective $*$-isomorphism from $\mathcal{N}$ into $\mathcal{M}$. In this case, it is easy to see that

$$S(x) = W^*T(x) = B^{\frac{1}{2}} J(x) B^{\frac{1}{2}} \tag{5.4}$$

is a completely positive and completely isometric injection from $L_p(\mathcal{N})$ into $L_p(\mathcal{M})$. Using these facts, we can obtain the following result.

THEOREM 5.2. *Let $\mathcal{M}$ be a von Neumann algebra and let $1 \leq p \neq 2 < \infty$. Then the rigid $\mathcal{OL}_p$ structure is equivalent to the matrix orderly rigid $\mathcal{OL}_p$ structure on $L_p(\mathcal{M})$.*

PROOF. We need to prove that for $1 \leq p \neq 2 < \infty$ the rigid $\mathcal{OL}_p$ structure implies the matrix orderly rigid $\mathcal{OL}_p$ structure on $L_p(\mathcal{M})$. Assume that we are given positive operators $y_1, \ldots, y_n \in L_p(\mathcal{M})^+$. Since $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space, for any $\epsilon_k > 0$ ($\epsilon_k \to 0$ as $k \to 0$), there exist a finite dimensional von Neumann algebra $\mathcal{N}_k$ and a completely isometric injection $T^k : L_p(\mathcal{N}_k) \to L_p(\mathcal{M})$ such that

$$\mathrm{dist}(T^k(L_p(\mathcal{N}_k)), y_j) < \epsilon_k$$

for all $j = 1, \ldots, n$, i.e. there exist $x_1^k, \ldots, x_n^k \in L_p(\mathcal{N}_k)$ such that

$$\|T^k(x_j^k) - y_j\|_p < \epsilon_k.$$

Then each T^k has the Yeadon representation

$$T^k = W_k B_k J_k$$

and as we discussed in (5.4),

$$S^k = W_k^* T^k = B_k J_k$$

is a completely positive and completely isometric injection from $L_p(\mathcal{N}_k)$ into $L_p(\mathcal{M})$. If we let $x_j^k = u_j^k |x_j^k|$ denote the polar decomposition of x_j^k, we claim that

$$\|S^k(|x_j^k|) - y_j\|_p \to 0 \tag{5.5}$$

as $k \to \infty$. Therefore we can replace T^k by S^k in the approximation. This shows that the rigid $\mathcal{OL}_p$ structure implies the matrix orderly rigid $\mathcal{OL}_p$ structure on $L_p(\mathcal{M})$.

To prove (5.5), we need to apply a result of Kosaki [27, Theorem 4.4]. Let us first consider

$$T^k(x_j^k) = W_k B_k J_k(u_j^k |x_j^k|) = W_k J_k(u_j^k) B_k J_k(|x_j^k|).$$

From the construction of W_k, B_k and J_k in Yeadon [46], it is easy to check that $W_k J_k(u_j^k)$ is a partial isometry in $\mathcal{M}$ and $B_k J_k(|x_j^k|)$ a positive operator in $L_p(\mathcal{M})$ such that

$$\ker W_k J_k(u_j^k) = \ker J_k(u_j^k) = \ker J_k(|x_j^k|) = \ker B_k J_k(|x_j^k|).$$

Thus by the uniqueness of polar decomposition, we conclude that

$$|T^k(x_j^k)| = B_k J_k(|x_j^k|) = S^k(|x_j^k|).$$

Since $\|T(x_j^k) - y_j\|_p < \epsilon_k \to 0$ for each $j = 1, \ldots, n$, by Kosaki [27, Theorem 4.4],

$$\|S^k(|x_j^k|) - y_j\|_p = \||T(x_j^k)| - |y_j|\|_p \to 0.$$

This proves our claim.

We note that the case $p = 1$ of Theorem 5.2 has been implicitly proved in [11].

COROLLARY 5.3. *Let $\mathcal{M}$ be a von Neumann algebra. If $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space for some $1 < p \neq 2 < \infty$, then $\mathcal{M}$ is a hyperfinite von Neumann algebra.*

PROOF. It follows from Theorem 5.2 that if $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space, then it is a matrix orderly rigid $\mathcal{OL}_p$ space. It was shown in a recent work of Junge, Ruan and Sherman [21] that the image space of any completely positive and completely isometric injection from a finite dimensional $L_p(\mathcal{N})$ space into $L_p(\mathcal{M})$ must be completely positively and completely contractively complemented in $L_p(\mathcal{M})$. Therefore, $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space and thus is a semisdiscrete non-commutative L_p-space. It follows from Theorem 3.2 that $\mathcal{M}$ is hyperfinite.

Using the generalized Yeadon representation theorem for completely isometric injections, we may obtain the following lemma.

LEMMA 5.4. *Let $\mathcal{N}$ be a finite von Neumann algebra and $\mathcal{M}$ an arbitrary von Neumann algebra. Let $1 < p \neq 2 < \infty$. If $T : L_p(\mathcal{N}) \to L_p(\mathcal{M})$ is a completely isometric injection with Yeadon representation $T(x) = WBJ(x)$ for all $x \in \mathcal{N}$, then*

$$\delta(x) = WB^pJ(x)$$

extends to a completely isometric injection from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$ and

$$\gamma(x) = B^pJ(x)$$

extends to a completely positive and completely isometric injection from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$.

PROOF. It is clear that $\delta(x) = WB^pJ(x)$ is a well-defined linear map from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$. Let $\tau_{\mathcal{N}}$ denote the normal faithful tracial state on $\mathcal{N}$ and let $\mathrm{Tr}_{\mathcal{M}}$ denote the canonical tracial functional on $L_1(\mathcal{M})$. It follows from (5.3) that

$$\|x\|_{L_1(\mathcal{N})} = \tau_{\mathcal{N}}(|x|) = \mathrm{Tr}_{\mathcal{M}}(B^p J(|x|)) = \|\delta(x)\|_{L_1(\mathcal{M})}.$$

This shows that δ preserves the L_1-norms and thus extends to an isometric injection from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$. To show that δ is a complete isometry, for each $n \in \mathbb{N}$ we consider the induced map $\mathrm{id}_{S_1^n} \otimes \delta = \mathrm{id}_{S_1^n} \otimes WB^pJ$. Applying the above argument to this, we see that $\mathrm{id}_{S_1^n} \otimes \delta$ is an isometric injection from $S_1^n[L_1(\mathcal{N})] = L_1(M_n \bar{\otimes} \mathcal{N})$ into $S_1^n[L_1(\mathcal{M})] = L_1(M_n \bar{\otimes} \mathcal{M})$. Therefore, δ is completely isometric.

Since $\gamma = B^pJ = W^*\delta$, it is immediate that γ extends to a completely positive and completely isometric injection from $L_1(\mathcal{N})$ into $L_1(\mathcal{M})$.

Using Lemma 5.4, we can give the following direct proof for Corollary 5.3.

THEOREM 5.5. *Let $\mathcal{M}$ be a von Neumann algebra. If $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space for some $1 < p \neq 2 < \infty$, then $L_1(\mathcal{M})$ is a rigid $\mathcal{OL}_1$ space and thus $\mathcal{M}$ is a hyperfinite von Neumann algebra.*

PROOF. Assume that for some $1 < p \neq 2 < \infty$, $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space. Given positive operators $y_1, \ldots, y_n \in L_1(\mathcal{M})^+$, $y_1^{\frac{1}{p}}, \ldots, y_n^{\frac{1}{p}}$ are positive operators in $L_p(\mathcal{M})^+$. Since $L_p(\mathcal{M})$ is a rigid $\mathcal{OL}_p$ space, for any sequence of positive numbers $\varepsilon_k \to 0$, there exist a sequence of finite dimensional von Neumann algebras $\mathcal{N}_k$, completely isometric injections $T^k = W_kB_kJ_k : L_p(\mathcal{N}_k) \to L_p(\mathcal{M})$ and operators x_j^k in $\mathcal{N}_k$ such that

$$\|T^k(x_j^k) - y_j^{\frac{1}{p}}\|_p < \varepsilon_k$$

for all $j = 1, \ldots, n$. Then for each k, it is known from Lemma 5.4 that

$$\gamma^k = B_k^p J_k : x \in L_1(\mathcal{N}_k) \mapsto B_k^p J_k(x) \in L_1(\mathcal{M})$$

is a completely positive and completely isometric injection from $L_1(\mathcal{N}_k)$ into $L_1(\mathcal{M})$ such that

$$\gamma^k(|x_j^k|^p) = B_k^p J_k(|x_j^k|^p) = |T^k(x_j^k)|^p.$$

It follows from Kosaki [27, Theorem 4.2], in which he proved that the map $y \in L_p(\mathcal{M})^+ \mapsto y^p \in L_1(\mathcal{M})$ is norm continuous, that

$$\|\gamma^k(|x_j^k|^p) - y_j\|_{L_1(\mathcal{M})} \to 0.$$

This shows that $L_1(\mathcal{M})$ is a (matrix orderly) rigid $\mathcal{OL}_1$ space. Therefore, $\mathcal{M}$ is hyperfinite by [11].

6. Conditional expectations

To complete the proof of Theorem 1.1, it remains to show that $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space for every hyperfinite von Neumann algebra $\mathcal{M}$ and for every $1 < p < \infty$. When $\mathcal{M}$ is semifinite or a type *III* factor, this result is easy to prove and is known to experts. The main difficulty is for general hyperfinite type *III* von Neumann algebras. The rest of this paper is essentially devoted to this problem. Our main tool is disintegration. We will need to integrate a Borel field of increasing normal faithful conditional expectations onto finite dimensional von Neumann subalgebras. In order to guarantee that the resulting mappings are still increasing normal faithful conditional expectations, we need some sufficient conditions for a sequence of mappings to be a sequence of increasing normal faithful conditional expectations. We do this in this section. We will consider direct integral in the next one.

We recall that if $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, i.e. a unital weak* closed $*$-subalgebra of $\mathcal{M}$, a *conditional expectation* of $\mathcal{M}$ onto $\mathcal{N}$ is a map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that $\mathcal{E}^2 = \mathcal{E}$ and $\|\mathcal{E}\| \leq 1$. A conditional expectation $\mathcal{E}$ is *normal* if it is weak* continuous, and $\mathcal{E}$ is said to be *faithful* if for any $x \in \mathcal{M}^+$, $\mathcal{E}(x) = 0$ implies that $x = 0$.

Given a normal faithful state φ on $\mathcal{M}$, it is known from Takesaki [41] that a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ is invariant with respect to σ_t^φ, i.e. $\sigma_t^\varphi(\mathcal{N}) = \mathcal{N}$ for all $t \in \mathsf{R}$, if and only if there is a (unique) normal faithful conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that $\varphi \circ \mathcal{E} = \varphi$ ($\mathcal{E}$ is then called *φ-invariant*).

If $\{\mathcal{E}_k\}$ is a sequence of φ-invariant normal conditional expectations from $\mathcal{M}$ onto an increasing sequence of von Neumann subalgebras $\{\mathcal{N}_k\}$, then $\mathcal{E}_k$ are all faithful and satisfy the ascending condition

$$\mathcal{E}_k \circ \mathcal{E}_{k+1} = \mathcal{E}_k = \mathcal{E}_{k+1} \circ \mathcal{E}_k, \qquad k \in \mathsf{N}.$$

In this case we say that $\{\mathcal{E}_k\}$ is an *increasing* sequence of φ-invariant normal conditional expectations on $\mathcal{M}$.

In the rest of this section we assume that $\mathcal{M}$ is a von Neumann algebra with a separable predual. Then there exists a countable dense subset $\{f_m\}$ in $\mathcal{M}_*$ and the weak* topology on the closed unit ball $\mathcal{M}_b$ is metrizable. It follows that there exists a countable weak* dense subset $\{r_m\}$ in $\mathcal{M}_b$. Here and in the sequel, we denote by X_b the closed unit ball of a normed space X. Similarly, we have a countable weak* dense subset $\{r'_m\}$ in the closed unit ball $\mathcal{M}'_b$ of the commutant $\mathcal{M}'$. Let us fix these dense subsets throughout this section.

Let us assume that $\mathcal{M}$ is standardly represented on $H = L_2(\mathcal{M})$. Then every contractive normal linear functional $f \in \mathcal{M}_*$ has the form $f(x) = \omega_{\xi,\eta}(x) = \langle x\xi|\eta\rangle$ for some contractive vectors $\xi, \eta \in H$. In this case, every finite rank normal map $\mathscr{E} : \mathscr{M} \to B(H)$ can be written as

$$\mathscr{E} = \sum_{i=1}^{n} \omega_{\xi_i,\eta_i} \otimes x_i$$

for some $\xi_i, \eta_i \in H$ and $x_i \in B(H)$. If $\mathscr{E}$ is contractive, then we may choose $\|\xi_i\|, \|\eta_i\| \leq 1$ and $\|x_i\| \leq 1$. In the following, we discuss the conditions under which $\mathscr{E}$ is a normal conditional expectation from $\mathscr{M}$ onto a finite dimensional von Neumann subalgebra $\mathscr{N} = \mathscr{E}(\mathscr{M})$ of $\mathscr{M}$.

The map $\mathscr{E}$ is unital and contractive if and only if it satisfies

$$\text{(6.1)} \qquad \mathscr{E}(1) = 1 \quad \text{and} \quad \|\mathscr{E}(r_m)\| \leq 1, \qquad \forall\, m \in \mathsf{N}.$$

It is an idempotent, i.e. $\mathscr{E}^2 = \mathscr{E}$ if and only if

$$\text{(6.2)} \qquad \mathscr{E}(\mathscr{E}(r_m)) = \mathscr{E}(r_m), \qquad \forall\, m \in \mathsf{N}.$$

The range space $\mathscr{E}(\mathscr{M})$ is contained in $\mathscr{M}$ if and only if

$$\text{(6.3)} \qquad \mathscr{E}(r_m)r'_l = r'_l\mathscr{E}(r_m), \qquad \forall\, m, l \in \mathsf{N}.$$

To ensure that $\mathscr{E}(\mathscr{M})$ is a von Neumann subalgebra of $\mathscr{M}$, we need to assume that $\mathscr{E}$ satisfies

$$\text{(6.4)} \qquad \mathscr{E}(\mathscr{E}(r_m)\mathscr{E}(r_l)) = \mathscr{E}(r_m)\mathscr{E}(r_l), \qquad \forall\, m, l \in \mathsf{N}.$$

If we are given a sequence of finite rank normal conditional expectations

$$\mathscr{E}_k = \sum_{i=1}^{n_k} \omega_{\xi_i^k,\eta_i^k} \otimes x_i^k,$$

then $\{\mathscr{E}_k\}$ is an increasing sequence if and only if

$$\text{(6.5)} \qquad \mathscr{E}_k \circ \mathscr{E}_{k+1}(r_m) = \mathscr{E}_k(r_m) = \mathscr{E}_{k+1} \circ \mathscr{E}_k(r_m), \qquad \forall\, m, k \in \mathsf{N}.$$

The following lemmas provide a density condition for the range spaces of such an increasing sequence of normal conditional expectations. The first one is well-known. For the convenience of the reader, we include a proof.

LEMMA 6.1. *Let $\{\mathscr{E}_k\}$ be an increasing sequence of normal faithful conditional expectations from $\mathscr{M}$ onto von Neumann subalgebras $\{\mathscr{N}_k\}$. Then the $*$-subalgebra $\bigcup_{k\in\mathsf{N}} \mathscr{N}_k$ is weak* dense in $\mathscr{M}$ if and only if*

$$\|f \circ \mathscr{E}_k - f\| \to 0, \qquad \forall\, f \in \mathscr{M}_*.$$

PROOF. $\Leftarrow$ This is obvious.

$\Rightarrow$ Let ψ be a normal faithful state on $\mathcal{N}_1$. Then $\varphi = \psi \circ \mathcal{E}_1$ is a normal faithful state on $\mathcal{M}$ such that $\varphi \circ \mathcal{E}_k = \varphi$ for all $k \in \mathbb{N}$. If we let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the cyclic representation of $\mathcal{M}$ related to φ, then $\mathcal{M}$ is *standard* on H_φ and ξ_φ is a separating and cyclic vector of $\mathcal{M}$. In this case, every contractive normal linear functional f on $\mathcal{M}$ has the form $f = \omega_{\xi,\eta}$ for some contractive vectors ξ and $\eta \in H_\varphi$. Since the weak* closure coincides with the weak operator closure on the $*$-subalgebras $\bigcup_{k\in\mathbb{N}} \mathcal{N}_k$, $\bigcup_{k\in\mathbb{N}} \mathcal{N}_k \xi_\varphi$ is norm dense in H_φ. Then we may find an integer k_0 and elements x, y in $\mathcal{N}_{k_0}$ such that $\|x\xi_\varphi\|, \|y\xi_\varphi\| \le 1$ and

$$\|f - \omega_{x\xi_\varphi, y\xi_\varphi}\| < \frac{\varepsilon}{2}.$$

Then for any $k \ge k_0$, we have $x, y \in \mathcal{N}_k$ and thus

$$\omega_{x\xi_\varphi, y\xi_\varphi} \circ \mathcal{E}_k = \omega_{x\xi_\varphi y\xi_\varphi},$$

since for any $r \in \mathcal{M}$,

$$\omega_{x\xi_\varphi, y\xi_\varphi}(r) = \varphi(y^* r x) = \varphi(\mathcal{E}_k(y^* r x)) = \varphi(y^* \mathcal{E}_k(r) x) = \omega_{x\xi_\varphi, x\xi_\varphi} \circ \mathcal{E}_k(r).$$

It follows that

$$\|f - f \circ \mathcal{E}_k\| \le \|f - \omega_{x\xi_\varphi, y\xi_\varphi}\| + \|\omega_{x\xi_\varphi, y\xi_\varphi} \circ \mathcal{E}_k - f \circ \mathcal{E}_k\| < \varepsilon.$$

LEMMA 6.2. *Let $\{\mathcal{E}_k\}$ be an increasing sequence of normal conditional expectations onto von Neumann subalgebras $\{\mathcal{N}_k\}$ of $\mathcal{M}$. Then the $*$-subalgebra $\bigcup_{k\in\mathbb{N}} \mathcal{N}_k$ is weak* dense in $\mathcal{M}$ if and only if for any $m, l \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that*

$$\|f_m \circ \mathcal{E}_k - f_m\| < \frac{1}{l} \tag{6.6}$$

for all $k \ge k_0$.

We will also need the following elementary lemma, which provides a faithfulness condition for a normal state φ on $\mathcal{M}$.

LEMMA 6.3. *Let $\{f_n\}$ be a countable dense subset in $\mathcal{M}_*$. A normal state $\varphi \in \mathcal{M}_*$ is faithful if and only if for any $n, l \in \mathbb{N}$, there exists r_m such that*

$$\|\varphi \cdot r_m - f_n\| < \frac{1}{l}. \tag{6.7}$$

PROOF. $\Leftarrow$ Suppose that we are given $b \in \mathcal{M}^+$ such that $\varphi(b) = 0$. We claim that $b^{\frac{1}{2}} = 0$ and thus $b = 0$. To see this, let us first fix an arbitrary $n \in \mathsf{N}$. For any $l \in \mathsf{N}$, there exists r_m such that

$$\|\varphi \cdot r_m - f_n\| < \frac{1}{l}.$$

Since

$$|\varphi \cdot r_m(b^{\frac{1}{2}})| = |\varphi(r_m b^{\frac{1}{2}})| \leq \|r_m\| \varphi(b)^{\frac{1}{2}} = 0,$$

we have $\varphi \cdot r_m(b^{\frac{1}{2}}) = 0$. This implies that

$$|f_n(b^{\frac{1}{2}})| \leq \|(\varphi \cdot r_m - f_n)(b^{\frac{1}{2}})\| < \frac{1}{l}\|b^{\frac{1}{2}}\|.$$

Letting $l \to \infty$, we get $f_n(b^{\frac{1}{2}}) = 0$. Since $\{f_n\}$ is norm dense in $\mathcal{M}_*$, we can conclude that $b^{\frac{1}{2}} = 0$, and thus $b = 0$. This shows that φ is faithful.

$\Rightarrow$ Let V be the norm closed subspace of $\mathcal{M}_*$ spanned by $\{\varphi \cdot r_m\}$. It suffices to show $V = \mathcal{M}_*$. Let $a \in V^\perp \subseteq \mathcal{M}$ with $\|a\| \leq 1$. Then we have $\varphi(r_m a) = 0$ for all $m \in \mathsf{N}$. Since $a^* \in \mathcal{M}_b$ and $\{r_m\}$ is weak* dense in $\mathcal{M}_b$, we can conclude that $\varphi(a^*a) = 0$. By the faithfulness of φ, we must have $a^*a = 0$, and thus $a = 0$. This yields the desired equality $V = \mathcal{M}_*$.

7. Direct Integrals

The key to the implication (i) $\Rightarrow$ (iv) in Theorem 1.1 is the following result.

THEOREM 7.1. *Let $\mathcal{M}$ be a hyperfinite von Neumann algebra with a separable predual. Then there exists a normal faithful state φ on $\mathcal{M}$ and an increasing sequence of φ-invariant normal faithful conditional expectations $\{\mathcal{E}_k\}$ from $\mathcal{M}$ onto type I von Neumann subalgebras $\{\mathcal{N}_k\}$ of $\mathcal{M}$ such that $\bigcup \mathcal{N}_k$ is weak* dense in $\mathcal{M}$.*

In the case where $\mathcal{M}$ is semifinite, this result is due to Pisier [35, Theorem 3.4]. To treat the type *III* case, we need to develop a direct integral theory for increasing sequences of normal conditional expectations from $\mathcal{M}$ onto finite dimensional (or type I) von Neumann subalgebras. Let us assume that $\mathcal{M}$ is a hyperfinite type *III* von Neumann algebra on a separable Hilbert space H. Then there exists a standard Borel space (Z, μ) and a measurable field of hyperfinite type *III* factors $\mathcal{M}(z)$ such that

$$\mathcal{M} = \int_Z^{\oplus} \mathcal{M}(z)\, d\mu(z)$$

(see details in Takesaki [42] and Kadison-Ringrose [23] vol. II). Since each $\mathcal{M}(z)$ is a hyperfinite type *III* factor, there exists a normal faithful state $\varphi(z)$

on $\mathcal{M}(z)$ and an increasing sequence of $\varphi(z)$-invariant normal faithful conditional expectations $\{\mathcal{E}_k(z)\}$ of $\mathcal{M}(z)$ onto finite dimensional von Neumann subalgebras $\{\mathcal{N}_k(z)\}$ such that $\bigcup_{k=1}^{\infty} \mathcal{N}_k(z)$ is weak* dense in $\mathcal{M}(z)$ (see [5], [6] [7], [8], and [16], or [40] §28–30 and [23] §13.4).

What we need to do is to "select" a measurable field of normal faithful states $\varphi(z)$ and for each fixed $k \geq 0$, a measurable field of normal faithful conditional expectations $\{\mathcal{E}_k(z)\}$ on finite dimensional subalgebras $\{\mathcal{N}_k(z)\}$ of $\mathcal{M}(z)$ such that

$$\varphi = \int_Z^{\oplus} \varphi(z)\, d\mu(z)$$

is a normal faithful state on $\mathcal{M}$ and

$$\mathcal{E}_k = \int_Z^{\oplus} \mathcal{E}_k(z)\, d\mu(z)$$

give an increasing sequence of φ-invariant normal faithful conditional expectations of $\mathcal{M}$ onto type I subalgebras

$$\mathcal{N}_k = \int_Z^{\oplus} \mathcal{N}_k(z)\, d\mu(z)$$

with $\bigcup_{k=0}^{\infty} \mathcal{N}_k$ weak* dense in $\mathcal{M}$.

We may assume that for each $z \in Z$, $\mathcal{M}(z)$ acts standardly on $H(z)$ and assume that $H(z)$ is isometric to a fixed Hilbert space $H_0 = \ell_2$. Then we can write

$$H = \int_Z^{\oplus} H(z)\, d\mu(z) = L_2(Z, \mu) \otimes H_0.$$

Up to a null-set equivalence, each vector $\xi \in H$ corresponds to a Borel function $z \in Z \mapsto \xi(z) \in H_0$ such that

$$\|\xi\| = \left(\int_Z \|\xi(z)\|^2\, d\mu(z) \right)^{\frac{1}{2}}.$$

We will use the notation

$$\xi = \int_Z^{\oplus} \xi(z)\, d\mu(z)$$

for the equivalence class of ξ. Similarly, up to a null-set equivalence, every operator $r \in \mathcal{M}$ corresponds to a Borel function $z \in Z \mapsto r(z) \in \mathcal{M}(z)$ (relative to the weak* topology) and we have

$$\|r\| = \operatorname*{ess\,sup}_{z \in Z} \|r(z)\|.$$

For every normal linear functional $f \in \mathcal{M}_*$, there exist two measurable fields of vectors $\xi = \int_Z^{\oplus} \xi(z)\, d\mu(z)$ and $\eta = \int_Z^{\oplus} \eta(z)\, d\mu(z)$ in H such that $f = \omega_{\xi,\eta}$. Therefore, we may obtain a measurable field of normal linear functionals

$$z \in Z \mapsto f(z) = \omega_{\xi(z),\eta(z)} \in \mathcal{M}(z)_*$$

such that $z \in Z \mapsto \langle r(z), f(z)\rangle$ is Borel on Z. In this case, $z \in Z \mapsto \|f(z)\| \in \mathbb{R}$ is a Borel function and

$$\|f\| = \int_Z \|f(z)\|\, d\mu(z).$$

We simply write

$$f = \int_Z^{\oplus} f(z)\, d\mu(z).$$

Since $\mathcal{M}_*$ is separable, $\mathcal{M}_*$ contains a countable norm dense subset $\{f_n\}$, for which there exist Borel measurable fields of normal linear functionals such that

$$f_n = \int_Z^{\oplus} f_n(z)\, d\mu(z).$$

It can be shown that for almost all $z \in Z$, $\{f_n(z)\}$ is norm dense in $\mathcal{M}(z)_*$ (left to the reader). Similarly, if we let $\{r_m\}$ be a countable weak* dense subset in the unit ball $\mathcal{M}_b$, then we can write

$$r_m = \int_Z^{\oplus} r_m(z)\, d\mu(z)$$

such that for almost all $z \in Z$, $\{r_m(z)\}$ is weak* dense in $\mathcal{M}(z)_b$. Accordingly, we can also choose a similar subset $\{r'_m\}$ in the unit ball $\mathcal{M}'_b$ of the commutant $\mathcal{M}'$ such that

$$r'_m = \int_Z^{\oplus} r'_m(z)\, d\mu(z).$$

We will fix these countable dense Borel measurable fields in the sequel.

Let us set $\Gamma_0 = \{\xi \in H_0 : \|\xi\| = 1\}$. Then Γ_0 is a separable metric space and thus is a standard Borel space. It is clear that the mapping $\xi \in \Gamma_0 \mapsto \omega_{\xi,\xi} \in \mathcal{M}_*$ is continuous with respect to the norm topology on $\mathcal{M}_*$. Since $(H_0)_b$ and $B(H)_b$ are standard Borel spaces with respect to the norm topology and the weak* topology, respectively, the Cartesian products

$$\Gamma = (H_0)_b \times (H_0)_b \times \mathcal{B}(H_0)_b \qquad \text{and} \qquad \Gamma^k = \overbrace{\Gamma \times \cdots \times \Gamma}^{k}$$

are standard Borel spaces. Now each element $[(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}] \in \Gamma^k$ determines a finite rank normal map

$$\mathscr{E}_k = \sum_{i=1}^{k} \omega_{\xi_i^k, \eta_i^k} \otimes x_i^k : B(H_0) \to B(H_0).$$

It is clear that the mapping

$$[(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}] \in \Gamma^k \mapsto \mathscr{E}_k \in B\big(B(H_0), B(H_0)\big)$$

is continuous with respect to the point-weak* topology on $B\big(B(H_0), B(H_0)\big)$.

Now let us put $\Gamma^{\mathscr{F}} = \prod_{k\in\mathsf{N}} \Gamma^k$. We let S be the subset of $Z \times \Gamma_0 \times \Gamma^{\mathscr{F}}$ consisting of all points $\big(z, \xi, \big\{[(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}]\big\}_{k\in\mathsf{N}}\big)$ such that the normal state $\varphi(z) = \omega_{\xi,\xi}$ and the finite rank normal mappings

$$\mathscr{E}_k(z) = \sum_{i=1}^{k} \omega_{\xi_i^k, \eta_i^k} \otimes x_i^k : \mathscr{M}(z) \to \mathscr{B}(H_0)$$

satisfy (6.1)–(6.7) discussed in section §6 with respect to the dense subsets $\{f_m(z)\} \subset \mathscr{M}(z)_*$, $\{r_m(z)\} \subset \mathscr{M}(z)$ and $\{r'_m(z)\} \subset \mathscr{M}(z)'$. By the previous discussion, one can easily see that for each fixed $z \in Z$, there exists a point $\big(z, \xi, \big\{[(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}]\big\}_{k\in\mathsf{N}}\big)$ verifying these conditions (the corresponding components ξ and $\big\{[(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}]\big\}_{k\in\mathsf{N}}$ depend on z). Therefore, S is non-empty. Moreover, S is a Borel subset since the mappings

$$\xi \mapsto \omega_{\xi,\xi} \qquad \text{and} \qquad [(\xi_i^k, \eta_i^k, x_i^k)_{i=1,2,\ldots,k}] \mapsto \mathscr{E}_k$$

are continuous (actually independent of z), and there are at most countably many restrictions involved in the conditions (6.1)–(6.7). Thus we may use the "measurable axiom of choice" (see [31, Mackey Theorem 6.3]) to select a Borel field of vectors

$$z \in Z \mapsto \big(\xi(z), \big\{[(\xi_i^k(z), \eta_i^k(z), x_i^k(z))_{i=1,2,\ldots,k}]\big\}_{k\in\mathsf{N}}\big) \in \Gamma_0 \times \Gamma^{\mathscr{F}}.$$

From this construction, we obtain a Borel measurable field of normal states

$$z \in Z \mapsto \varphi(z) = \omega_{\xi(z),\xi(z)} \in \mathscr{M}(z)_*$$

on $\mathscr{M}(z)$ and an increasing sequence of finite rank $\varphi(z)$-invariant normal conditional expectations

$$z \in Z \mapsto \mathscr{E}_k(z) = \sum_{i=1}^{k} \omega_{\xi_i^k(z), \eta_i^k(z)} \otimes x_i^k(z)$$

from $\mathcal{M}(z)$ onto finite dimensional von Neumann subalgebras $\mathcal{N}_k(z) = \mathcal{E}_k(z)(\mathcal{M}(z))$ (with $\dim \mathcal{N}_k(z) \leq k$). Then

$$\varphi = \int_Z^{\oplus} \varphi(z)\, d\mu(z)$$

is a normal state on $\mathcal{M}$ and

$$\mathcal{E}_k = \int_Z^{\oplus} \mathcal{E}_k(z)\, d\mu(z)$$

is an increasing sequence of φ-invariant normal conditional expectations from $\mathcal{M}$ onto type I (actually sub-homogeneuous) von Neumann subalgebras

$$\mathcal{N}_k = \int_Z^{\oplus} \mathcal{N}_k(z)\, d\mu(z).$$

To complete the proof of Theorem 7.1, we only need to show that φ is faithful on $\mathcal{M}$ and the union $\bigcup \mathcal{N}_k$ is weak* dense in $\mathcal{M}$. These will be proved in the following two lemmas.

LEMMA 7.2. *The normal state* $\varphi = \int_Z^{\oplus} \varphi(z)\, d\mu(z)$ *is faithful on* $\mathcal{M}$.

PROOF. Let e be a projection in $\mathcal{M}$ such that $\varphi(e) = 0$. Then there exists a measurable field of projections $z \mapsto e(z) \in \mathcal{M}(z)$ such that

$$e = \int_Z^{\oplus} e(z)\, d\mu(z) \qquad \text{and} \qquad \int_Z^{\oplus} \varphi(z)(e(z))\, d\mu(z) = \varphi(e) = 0.$$

This implies that $\varphi(z)(e(z)) = 0$ for almost all $z \in Z$. Since the $\varphi(z)$ are faithful on $\mathcal{M}(z)$, we must have $e(z) = 0$ for almost all $z \in Z$ and thus $e = \int_Z^{\oplus} e(z)\, dm(z) = 0$. Therefore, φ is faithful on $\mathcal{M}$.

LEMMA 7.3. *The union* $\bigcup \mathcal{N}_n$ *is weak* dense in* $\mathcal{M}$.

PROOF. By Lemma 6.1, for each $z \in Z$ and $m \in \mathsf{N}$

$$\| f_m(z) \circ \mathcal{E}_k(z) - f_m(z) \| \to 0 \text{ as } k \to \infty.$$

Since for every m, the function $z \in Z \mapsto \| f_m(z) \circ \mathcal{E}_k(z) - f_m(z) \| \in \mathsf{R}$ is Borel and bounded, by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{k\to\infty} \| f_m \circ \mathcal{E}_k - f_m \| &= \lim_{k\to\infty} \int_Z \| f_m(z) \circ \mathcal{E}_k(z) - f_m(z) \|\, d\mu(z) \\ &= \int_Z \lim_{k\to\infty} \| f_m(z) \circ \mathcal{E}_k(z) - f_m(z) \|\, d\mu(z) = 0.\end{aligned}$$

It follows from Lemma 6.2 that $\bigcup \mathcal{N}_k$ is weak* dense in $\mathcal{M}$.

Now we turn to the only remaining implication (i) $\Rightarrow$ (iv) in Theorem 1.1. Let $\mathcal{M}$ be a hyperfinite von Neumann algebra. As we have discussed at the beginning of the proof of Theorem 3.2, there exists an increasing net of projections $e_i \to 1_{\mathcal{M}}$ in $\mathcal{M}$ such that each reduced von Neumann subalgebra $e_i\mathcal{M}e_i$ is σ-finite and we can completely identify $L_p(e_i\mathcal{M}e_i)$ with a completely positively and completely contractively complemented subspace in $L_p(\mathcal{M})$. Moreover the union of these spaces is norm dense in $L_p(\mathcal{M})$. Therefore, to show that $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space, it sufficies to show that each $L_p(e_i\mathcal{M}e_i)$ is a matrix orderly rigid $\mathcal{COL}_p$ space. So we only need to prove the σ-finite case.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra equipped with a normal faithful state φ. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ a φ-invariant normal faithful conditional expectation. Then we have

$$\mathcal{E} \circ \sigma_t^{\varphi} = \sigma_t^{\varphi} \circ \mathcal{E}, \qquad \forall\, t \in \mathsf{R}.$$

It follows that the modular automorphism group of the restriction of φ to $\mathcal{N}$ is equal to the restriction of σ_t^{φ} to $\mathcal{N}$. We can thus regard $\mathcal{N} \rtimes_{\sigma^{\varphi}} \mathsf{R}$ as a von Neumann subalgebra of $\mathcal{M} \rtimes_{\sigma^{\varphi}} \mathsf{R}$. Moreover, the canonical faithful normal semifinite trace on $\mathcal{N} \rtimes_{\sigma^{\varphi}} \mathsf{R}$ coincides with the restriction of that on $\mathcal{M} \rtimes_{\sigma^{\varphi}} \mathsf{R}$ to $\mathcal{N} \rtimes_{\sigma^{\varphi}} \mathsf{R}$. Hence $L_p(\mathcal{N})$ can be naturally viewed as a subspace of $L_p(\mathcal{M})$.

Let $\mathcal{M}_a$ denote the set of all *analytic* elements in $\mathcal{M}$, i.e. $x \in \mathcal{M}$ such that $t \in \mathsf{R} \mapsto \sigma_t^{\varphi}(x)$ has an analytic extension from C into $\mathcal{M}$ and let D be the density of φ in $L_1(\mathcal{M})$. Then for $1 \le p < \infty$, $D^{\frac{1-\theta}{p}}\mathcal{M}_a D^{\frac{\theta}{p}} = \mathcal{M}_a D^{\frac{1}{p}}$ (with any $\theta \in [0, 1]$) is norm dense in $L_p(\mathcal{M})$ and we can get a well-defined map

$$\mathcal{E}_p(D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}) = D^{\frac{1-\theta}{p}} \mathcal{E}(x) D^{\frac{\theta}{p}} \qquad (x \in \mathcal{M}_a),$$

which extends to a positive and contractive projection (which is still denoted by $\mathcal{E}_p$) from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ (see [22, §2]). The map $\mathcal{E}_p$ is actually completely positive and completely contractive since for each $n \in \mathsf{N}$, $\mathrm{id}_{M_n} \otimes \mathcal{E}$ is a $\mathrm{tr}_n \otimes \varphi$-invariant normal faithful conditional expectation from $M_n \bar{\otimes} \mathcal{M}$ onto $M_n \bar{\otimes} \mathcal{N}$ and thus

$$\mathrm{id}_{S_p^n} \otimes \mathcal{E}_p = (\mathrm{id}_{M_n} \otimes \mathcal{E})_p : L_p(M_n \bar{\otimes} \mathcal{M}) \to L_p(M_n \bar{\otimes} \mathcal{N})$$

is a positive and contractive projection. Therefore, we may identify $L_p(\mathcal{N})$ with a completely positively and completely contractively complemented subspace in $L_p(\mathcal{M})$.

LEMMA 7.4. *Let φ be a normal faithful state on $\mathcal{M}$ and let $\{\mathcal{E}_\alpha\}$ be an increasing family of φ-invariant normal conditional expectations from $\mathcal{M}$ onto* *von Neumann subalgebras $\{\mathcal{N}_\alpha\}$. Assume that the $*$-subalgebra $\bigcup_\alpha \mathcal{N}_\alpha$ is weak* dense in $\mathcal{M}$ and each $L_p(\mathcal{N}_\alpha)$ is a matrix orderly rigid $\mathcal{COL}_p$ space $(1 \le p < \infty)$. Then $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space.*

PROOF. Let $\{\mathcal{E}_\alpha\}$ be an increasing family of φ-invariant normal faithful conditional expectations from $\mathcal{M}$ onto von Neumann subalgebras $\{\mathcal{N}_\alpha\}$. According to the above discussion, we may identify $\{L_p(\mathcal{N}_\alpha)\}$ with an increasing family of completely positively and completely contractively complemented subspaces in $L_p(\mathcal{M})$. Since the union $\bigcup_\alpha \mathcal{N}_\alpha$ is weak* dense in $\mathcal{M}$, we can conclude from [17, Lemma 2.2] that for $1 \le p < \infty$, the union $\bigcup_\alpha L_p(\mathcal{N}_\alpha)$ is norm dense in $L_p(\mathcal{M})$. In fact, with this weak* density assumption, we have $\mathcal{E}_\alpha(x) \to x$ in $L_p(\mathcal{M})$ for every $x \in L_p(\mathcal{M})$. If we assume that each $L_p(\mathcal{N}_\alpha)$ is a matrix orderly rigid $\mathcal{COL}_p$ space, then it is easy to see that $L_p(\mathcal{M})$ is also a matrix orderly rigid $\mathcal{COL}_p$ space.

We are finally ready to accomplish the proof of Theorem 1.1.

END OF THE PROOF OF THEOREM 1.1. (i) $\Rightarrow$ (iv). Let us first assume that $\mathcal{M}$ is a hyperfinite von Neumann algebra with a separable predual. Then we can apply Theorem 7.1 to get a normal faithful state φ on $\mathcal{M}$ and an increasing sequence of φ-invariant normal faithful conditional expectations $\{\mathcal{E}_k\}$ from $\mathcal{M}$ onto type I von Neumann subalgebras $\mathcal{N}_k$ of $\mathcal{M}$ such that $\bigcup \mathcal{N}_k$ is weak* dense in $\mathcal{M}$. It is clear that each $L_p(\mathcal{N}_k)$ is a matrix orderly rigid $\mathcal{COL}_p$ and thus we can conclude from Lemma 7.4 that $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$.

The σ-finite case can be deduced from separable case by applying a result of Haagerup (see [13, Appendix]). Indeed, we may let $\mathcal{I}$ denote the family of all subsets of $\mathcal{M}$ which are at most countable and we partially order $\mathcal{I}$ by inclusion. Given any $\alpha \in \mathcal{I}$, let $\mathcal{N}_\alpha$ denote the von Neumann subalgebra generated by $\{\sigma_t^\varphi(x) : x \in \alpha, t \in \mathsf{Q}\}$, where φ is a fixed normal faithful state on $\mathcal{M}$ and Q is the set of rational numbers. Then each $\mathcal{N}_\alpha$ has a separable predual and is σ^φ-invariant. By Takesaki [41], there exists a unique φ-invariant normal faithful conditional expectation $\mathcal{E}_\alpha : \mathcal{M} \to \mathcal{N}_\alpha$. The uniqueness of $\mathcal{E}_\alpha$ implies that the family $\{\mathcal{E}_\alpha\}$ is increasing. It is clear that each $\mathcal{N}_\alpha$ is hyperfinite and the union of $\{\mathcal{N}_\alpha\}$ is weak* dense in $\mathcal{M}$. Then each $L_p(\mathcal{N}_\alpha)$ is a matrix orderly rigid $\mathcal{COL}_p$ space and we can conclude from Lemma 7.4 that $L_p(\mathcal{M})$ is a matrix orderly rigid $\mathcal{COL}_p$ space. This completes the proof of Theorem 1.1.

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