MATH. SCAND. 96 (2005), 96–106

TOPOLOGICAL TRIVIALITY OF FAMILIES OF REAL ISOLATED SINGULARITIES AND THEIR MILNOR FIBRATIONS

RAIMUNDO NONATO ARAÚJO DOS SANTOS*

Abstract

The aim of this paper is to study the topological triviality and the topological equivalence of the Milnor fibrations for families of real analytic map germs with no coalescing of critical points.

1. Introduction

In the case of complex germs, equivalent conditions to the topological triviality of families of germs of functions with isolated singularity, had been gotten by several authors (see [12], [2], [4]). A theorem due to G. M. Greuel summarizes various of these results, presenting equivalent conditions to the topological triviality of such families. As a consequence of his result it follows that no coalescing of critical points is a necessary and sufficient condition for topological triviality of complex analytic families of function-germs. A natural question is whether or not no coalescing of singularities also is a sufficient condition for topological triviality of families of real analytic function-germs. The answer is not. H. King [5] presents an example showing that the above question is false, and gives sufficient conditions so that a family with no coalescing of critical points is topologically trivial.

The aim of this paper is to introduce sufficient conditions for the topological triviality of families of real analytic map-germs $F : \mathsf{R}^n \times \mathsf{R}, 0 \to \mathsf{R}^2, 0$, with isolated singularities at the origin. The main result is the following:

THEOREM 3.1. *Let $F(x,t) = (P(x,t), Q(x,t))$ where $P(x,t) = f(x) + t\theta(x)$, $Q(x,t) = g(x) + t\alpha(x)$ be a family of analytic map-germs. Suppose that the following conditions hold:*

(A)
$$\frac{|\langle \partial P_t(x), \partial Q_t(x)\rangle|}{\|\partial P_t(x)\|\,\|\partial Q_t(x)\|} \leq 1-\rho,$$

* Research partially supported by ICMC/USP, São Paulo, Brazil.

Received July 30, 2003.

in the neighborhood V of 0 in R^n, for all $t \in \mathsf{R}$ and $0 < \rho \leq 1$.

(B_f)

$$\overline{\langle \partial P_t(x) \rangle}_{\mathscr{A}_n} = \overline{\langle \partial P_0(x) \rangle}_{\mathscr{A}_n}$$

$$\overline{\langle \partial Q_t(x) \rangle}_{\mathscr{A}_n} = \overline{\langle \partial Q_0(x) \rangle}_{\mathscr{A}_n},$$

for all $t \in \mathsf{R}$.

Then the family F_t is topologically trivial.

The conditions (A) and (B_f) are inspired in the conditions introduced by A. Jacquemard ([7, Theorem 1]) to study the Milnor fibrations to real analytic singularities of the kind $f = (P, Q) : \mathsf{R}^n, 0 \to \mathsf{R}^2, 0$.

2. Previous results

Let $F : (\mathsf{R}^p \times \mathsf{R}^n, \mathsf{R}^p \times 0) \to (\mathsf{R}^k \times 0)$ be a one-parameter family of map germs with isolated singularity, $F_t(x) = F(x, t)$. We will usually denote a family of germs by $F_t : (\mathsf{R}^n, 0) \to (\mathsf{R}^k, 0)$, $t \in \mathsf{R}^p$, where $F_t(x) = F(t, x)$.

Definition 2.1. A family $F : (\mathsf{R}^p \times \mathsf{R}^n, \mathsf{R}^p \times 0) \to (\mathsf{R}^k, 0)$ is said to have no coalescing of critical points (or to be a good deformation of F_0) if there is a neighborhood U of $\mathsf{R}^p \times 0$ in $\mathsf{R}^p \times \mathsf{R}^n$ and a representative G of F so that G restricted to $U \cap (t \times (\mathsf{R}^n \setminus 0)$ is a submersion for each $t \in \mathsf{R}^p$. Otherwise, the family is said to coalesce.

Definition 2.2. The germs $f_i : (\mathsf{R}^n, 0) \to (\mathsf{R}^k, 0)$, $i = 0, 1$ are topologically right-equivalent ($\mathscr{C}^0 - \mathscr{R}$-equivalent) if there is a germ of a homeomorphism $h : (\mathsf{R}^n, 0) \to (\mathsf{R}^n, 0)$ so that the germs $f_0 h$ and f_1 are the same. The topological type of a germ is its topological right equivalence class.

Definition 2.3. We say that a family of map-germs $F_t : (\mathsf{R}^n, 0) \to (\mathsf{R}^k, 0)$, $t \in \mathsf{R}^p$ and $F_t(x) = F(t, x)$, is topologically trivial if there is a continuous family of germs of homeomorphisms $G_t : (\mathsf{R}^n, 0) \to (\mathsf{R}^n, 0)$, $t \in \mathsf{R}^p$, such that $F_0(x) = F_t \circ G_t(x)$.

H. King [6] presents an example of a family of real analytic functions which does not have coalescing of critical points and is not topolo-gically trivial; moreover, he gives sufficient conditions so that a good deformation is topologically trivial. To state King's results we need the following definition.

Definition 2.4. For a polynomial $f : (\mathsf{R}^n, 0) \to (\mathsf{R}^k, 0)$, $n \geq p$, with an isolated critical point at 0, define $r(f)$, the Milnor radius of f, to be the smallest critical value of the distance function $\|x\|^2$ restricted to $f^{-1}(0) \setminus 0$ (or ∞ if there are no critical values). In other words, $r(f)$ is the biggest $\epsilon_0 > 0$ such that there is $0 < \epsilon \leq \epsilon_0$ with $(f^{-1}(0) \setminus 0) \,\overline{\pitchfork}\, \mathsf{S}^{n-1}_\epsilon$.

In [6], H. King shows the following results:

THEOREM 2.5. *Suppose* $F_t : (\mathsf{R}^n, 0) \longrightarrow (\mathsf{R}^k, 0)$, $t \in \mathsf{R}^p$ *is a conti-nuous family of polynomial germs with no coalescing of critical points and there is a* $\delta > 0$ *so that* $r(F_t) > \delta$ *for all* $t \in \mathsf{R}^p$. *Then there is a continuous family of homeomorphism germs* $H_t : (\mathsf{R}^m, 0) \to (\mathsf{R}^m, 0)$ *so that* $F_0 = F_t \circ H_t$ *for all* $t \in \mathsf{R}^p$.

THEOREM 2.6. *Let* $F_t : (\mathsf{R}^n, 0) \longrightarrow (\mathsf{R}^k, 0)$, $t \in \mathsf{R}^p$ *be a family of germs with no coalescing of critical points and suppose there is a family of homeomorphism germs* $G_t : (\mathsf{R}^m, 0) \longrightarrow (\mathsf{R}^m, 0)$, $t \in \mathsf{R}^p$ *so that the germ at* 0 *of each set* $G_t \circ F_t^{-1}(0)$ *is the germ of* $F_0^{-1}(0)$. *Then there is a family of homeomorphism germs* $H_t : (\mathsf{R}^m, 0) \longrightarrow (\mathsf{R}^m, 0)$, $t \in \mathsf{R}^p$ *and a neighborhood* V *of* 0 *in* R^p *so that the germ at* 0 *of* $F_t \circ H_t$ *is the germ of* F_0 *for each* $t \in V$.

The $\mathscr{C}^0 - \mathscr{R}$-equivalence class of f_0 relates to the equivalence class of its Milnor fibration. More precisely, given $f_0, g_0 : (\mathsf{R}^n, 0) \to (\mathsf{R}^k, 0)$ with isolated singularity, f_0 is $\mathscr{C}^0 - \mathscr{R}$-equivalent to g_0 if and only if their Milnor fibrations are also equivalent. (See [5, Theorem 1]).

Recall that if $f : (\mathsf{C}^{n+1}, 0) \to (\mathsf{C}, 0)$ is the germ of a holomorphic function with a critical point at 0, then for every sufficiently small $\epsilon > 0$, the map $\phi := \frac{f}{\|f\|} : \mathscr{S}_\epsilon^{2n+1} \setminus \mathscr{S}^1$, is the projection map of a locally trivial fiber bundle, where $K = f^{-1}(0) \cap \mathscr{S}_\epsilon$ is the link of 0. This is the *Milnor fibration* of f. Milnor also proves in the last chapter of his book a fibration theorem for real singularities. He shows that if $f : (\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$, $n \geq p \geq 2$, is the germ of a real analytic map-germ whose derivative Df has rank p on a punctured neighborhood of $0 \in \mathsf{R}^n$, then, for every sufficiently small sphere $\mathscr{S}_\epsilon^{n-1} \subset \mathsf{R}^n$ centered at 0, one has a locally trivial fiber bundle, $f = \frac{f}{\|f\|} : \mathscr{S}_\epsilon^{n-1} \setminus N_K \to \mathscr{S}^{p-1}$, where N_K denotes a tubular neighborhood of the link K in $\mathscr{S}_\epsilon^{n-1}$.

Moreover, f can be extended to $\mathscr{S}_\epsilon^{n-1} \setminus K$ as the projection map of a fiber bundle, but this extension may not be given by the obvious map $\frac{f}{\|f\|}$. The problem of studying real isolated singularities at 0, for which the map $\frac{f}{\|f\|}$ extends to all of $\mathscr{S}^{n-1} \setminus K \to \mathscr{S}^{p-1}$ as the projection map of a fiber bundle (as in the case of holomorphic maps) was first studied by A. Jacquemard in [7] (see also [8]).

Let $f = (P, Q) : (\mathsf{R}^n, 0) \to (\mathsf{R}^2, 0)$ be a real analytic map-germ with an isolated singularity at 0. Let $\mathscr{A}_n$ be the ring of real analytic germs in $\mathsf{R}^n, 0$ and $I = \langle \partial P(x) \rangle$ (resp. $\langle \partial Q(x) \rangle$) the ideal in $\mathscr{A}_n$ generated by the partial derivatives of P (resp. Q), and by $\overline{I}$ the integral closure of I, that is, $\overline{I} = \{h \in \mathscr{A}_n \mid \exists\, a_i \in I^i$ with $h^n + a_1 h^{n-1} + \cdots + a_n = 0\}$.

THEOREM ([7, Theorem 1]). *Suppose that there exists a neighborhood* V

of 0 *in* R^n *such that:*

(A)
$$\frac{|\langle \partial P(x), \partial Q(x)\rangle|}{\|\partial P(x)\|\,\|\partial Q(x)\|} \leq 1-\rho,$$

in the neighborhood V *of* 0 *in* R^n, $0<\rho\leq 1$.

(B)
$$\overline{\langle \partial P(x)\rangle} = \overline{\langle \partial Q(x)\rangle}.$$

Then there exists a sufficiently small $\epsilon_0 > 0$*, such that for all* $0<\epsilon\leq\epsilon_0$*,* $\phi = \frac{f}{\|f\|} : \mathscr{S}_\epsilon^{n-1} \setminus K_\epsilon \to \mathscr{S}^1$ *is the projection map of a locally trivial fiber bundle. Furthermore, this fiber bundle is equivalent to the fibration given Milnor in* [9].

In [11] the authors replace the condition (B) in Theorem above by condition (B_{R}): $\overline{\langle \partial P(x)\rangle}_{\mathsf{R}} = \overline{\langle \partial Q(x)\rangle}_{\mathsf{R}}$, where $\overline{\langle \partial P(x)\rangle}_{\mathsf{R}}$ denotes the real integral closure as defined by T. Gaffney [3]. In the complex analytic category, both conditions are equivalent (see [3], [12]).

DEFINITION 2.7. Let I be an ideal in the ring $\mathscr{A}_n$. The real integral closure of I, denoted by $\overline{I}_{\mathsf{R}}$, is the set of $h\in\mathscr{A}_n$ such that for all real analytic curve $\gamma : (\mathsf{R},0)\to(\mathsf{R}^n,0)$, we have $h\circ\gamma\in(\gamma^*(I))\mathscr{A}_1$.

3. The Main Theorem

In this section we present the main result of this paper as well as its main consequences. Let $f=(P_0,Q_0):(\mathsf{R}^n,0)\to(\mathsf{R}^2,0)$, be the germ of a real analytic map-germ with isolated singularity at zero, and $F:(\mathsf{R}^n\times\mathsf{R},0)\to(\mathsf{R}^2,0)$ be a deformation of f given by $F(x,t)=(P(x,t),Q(x,t))$, $P(x,t)=P_0(x)+t\theta(x)$, $Q(x,t)=Q_0(x)+t\alpha(x)$.

THEOREM 3.1. *Let* F *be as above and* $\partial_x P(x,t) = \mathrm{Grad}_x\, P_t(x)$*,* $\partial P = (\partial_x P(x,t), \frac{\partial P}{\partial t})$*. Suppose that the following conditions hold:*

(A)
$$\frac{|\langle \partial_x P(x,t), \partial_x Q(x,t)\rangle|}{\|\partial_x P(x,t)\|\,\|\partial_x Q(x,t)\|} \leq 1-\rho,$$

$x\neq 0$ *near zero, for all* $t\in\mathsf{R}$ *and* $0<\rho\leq 1$.

(B_f)
$$\overline{\langle \partial_x P(x,t)\rangle}_{\mathsf{R}} = \overline{\langle \partial_x P(x,0)\rangle}_{\mathsf{R}}$$
$$\overline{\langle \partial_x Q(x,t)\rangle}_{\mathsf{R}} = \overline{\langle \partial_x Q(x,0)\rangle}_{\mathsf{R}},$$

for all $t\in\mathsf{R}$.

Then the family F_t is topologically trivial.

PROOF. The idea of the proof is to construct a vector field $V(x,t) = a(x,t)\partial_x P(x,t) + b(x,t)\partial_x Q(x,t) + \frac{\partial}{\partial t}$ (where $\frac{\partial}{\partial t}$ is the unit vector in t-direction) such that:

$$\begin{cases} \langle \partial P(x,t), V(x,t)\rangle = 0 \\ \langle \partial Q(x,t), V(x,t)\rangle = 0 \end{cases} \tag{1}$$

The vector field $V(x,t)$ is tangent to the levels $X = F^{-1}(c) = P^{-1}(c_1) \cap Q^{-1}(c_2)$, where $c = (c_1, c_2) \in \mathsf{R}^2$. In particular for $c = 0$, V is tangent to the variety $F^{-1}(0) = P^{-1}(0) \cap Q^{-1}(0)$.

The flow $\phi(x,t)$, $\phi(x,0) = x$, satisfies the following:

$$\begin{cases} \dfrac{\partial P(\phi(x,t))}{\partial t} = \langle \partial P, V\rangle = 0 \\ \dfrac{\partial Q(\phi(x,t))}{\partial t} = \langle \partial Q, V\rangle = 0. \end{cases} \tag{2}$$

Hence, $F(\phi(x,t)) = F(\phi(x,0)) = (f(x), g(x))$.

From (1) it follows that:

$$\begin{cases} \langle \partial P, V\rangle = a\|\partial_x P\|^2 + b\langle \partial_x P, \partial_x Q\rangle + \dfrac{\partial P}{\partial t} = 0 \\ \langle \partial Q, V\rangle = a\langle \partial_x P, \partial_x Q\rangle + b\|\partial_x Q\|^2 + \dfrac{\partial Q}{\partial t} = 0 \end{cases} \tag{3}$$

The matrix of the system is

$$\begin{pmatrix} \|\partial_x P\|^2 & \langle \partial_x P, \partial_x Q\rangle \\ \langle \partial_x P, \partial_x Q\rangle & \|\partial_x Q\|^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} \theta(x) \\ \alpha(x) \end{pmatrix}$$

and it follows from condition (A) that its determinant:

$$\Delta(x,t) := \|\partial_x P(x,t)\|^2 \|\partial_x Q(x,t)\|^2 - \langle \partial_x P(x,t), \partial_x Q(x,t)\rangle^2 \tag{4}$$

satisfies the condition $\Delta(x,t) \neq 0$ for all $x \in V$, in a neighborhood of $0 \in \mathsf{R}^n$, $\forall t$. Then we can write:

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{\Delta} \begin{pmatrix} \|\partial_x Q\|^2 & -\langle \partial_x P, \partial_x Q\rangle \\ -\langle \partial_x P, \partial_x Q\rangle & \|\partial_x P\|^2 \end{pmatrix} \begin{pmatrix} \theta(x) \\ \alpha(x) \end{pmatrix}$$

Hence,

$$\text{(5)} \qquad \begin{cases} a = -\dfrac{1}{\Delta}(\|\partial_x Q\|^2\theta(x) - \langle \partial_x P, \partial_x Q\rangle \alpha(x)) \\ b = -\dfrac{1}{\Delta}(\|\partial_x P\|^2\alpha(x) - \langle \partial_x P, \partial_x Q\rangle \theta(x)) \end{cases}$$

This vector field is well defined and smooth in U, where $U \subset \mathsf{R}^n \times \mathsf{R} \setminus (\{0\} \times \mathsf{R})$. To show that V is integrable at the points $(0, t)$, it is sufficient to guarantee that there exists a constant $C > 0$ such that:

$$\text{(6)} \qquad \left\| V(x, t) - \frac{\partial}{\partial t} \right\| \leq C\|x\|,$$

for x sufficiently close to zero. This will follow from Lemma 3.2 and Proposition 3.3 below.

LEMMA 3.2. *There exist constants $c_1 > 0$ and $c_2 > 0$ such that:*

(a) $|\theta(x)| \leq c_1\|x\|\,\|\partial_x P\|$ *and*

(b) $|\alpha(x)| \leq c_2\|x\|\,\|\partial_x Q\|$.

PROOF. (a) The function $\theta(x)$ is analytic and $\theta(0) = 0$, then it follows from the Bochnak-Lojasiewicz inequality [1] that,

$$\text{(7)} \qquad |\theta(x)| \leq c_0\|x\|\,\|\partial_x\theta(x)\|$$

On the other hand,

$$|t|\,\|\partial_x\theta\| - \|\partial_x f(x)\| \leq \|\partial_x f(x) + t\partial_x\theta(x)\| = \|\partial_x P(x, t)\| \leq c_1\|\partial_x f(x)\|,$$

where in the last inequality we use condition (B).

Thus, $|t|\,\|\partial_x\theta\| \leq (1 + c_1)\|\partial_x f(x)\|\ \forall t$.

In particular taking $t = 1$, we have

$$\|\partial_x\theta\| \leq (1 + c_1)\|\partial_x f(x)\|.$$

Now using again condition (B), we have:

$$\|\partial_x\theta\| \leq (1 + c_1)\|\partial_x f(x)\| \leq c_2\|\partial_x P(x, t)\|$$

for all t. Therefore,

$$\text{(8)} \qquad \|\partial_x\theta(x)\| \leq c_2\|\partial_x P(x, t)\|, \quad \forall t$$

From (7) and (8) it follows that:

$$\|\theta(x)\| \leq c_0\|x\|\,\|\partial_x\theta(x)\| \leq c\|x\|\,\|\partial_x P(x, t)\|.$$

Part (b) can be proved in the same way.

PROPOSITION 3.3. $\left\| V(x,t) - \frac{\partial}{\partial t} \right\| \leq c\|x\|$.

PROOF.

$$\begin{aligned}\left\| V(x,t) - \frac{\partial}{\partial t} \right\| &= \|a(x,t)\partial_x P(x,t) + b(x,t)\partial_x Q(x,t)\| \\ &\leq \|a(x,t)\partial_x P(x,t)\| + \|b(x,t)\partial_x Q(x,t)\|\end{aligned}$$

From (5) we have

$$\begin{aligned}&\|a(x,t)\partial_x P(x,t)\| \\ &= \left| \frac{1}{\Delta}(\|\partial_x Q\|^2 \theta(x) - \langle \partial_x P, \partial_x Q\rangle \alpha(x)) \right| \|\partial_x P(x,t)\| \\ &\leq \frac{1}{|\Delta|}\|\partial_x Q\|^2 |\theta(x)|\, \|\partial_x P\| + \frac{1}{|\Delta|}|\langle \partial_x P, \partial_x Q\rangle|\, |\alpha(x)|\, \|\partial_x P\|.\end{aligned} \tag{9}$$

Furthermore,

$$\begin{aligned}&\frac{1}{|\Delta|}\|\partial_x Q\|^2 |\theta(x)|\, \|\partial_x P\| \left(\frac{\|\partial_x P\|^2 \|\partial_x Q\|^2}{\|\partial_x P\|^2 \|\partial_x Q\|^2} \right) \\ &= \frac{|\theta(x)|}{\|\partial_x P\|} \left(\frac{\|\partial_x P(x,t)\|^2 \|\partial_x Q(x,t)\|^2 - \langle \partial_x P(x,t), \partial_x Q(x,t)\rangle^2}{\|\partial_x P\|^2 \|\partial_x Q\|^2} \right)^{-1} \\ &= \frac{|\theta(x)|}{\|\partial_x P\|} \left(1 - \frac{\langle \partial_x P(x,t), \partial_x Q(x,t)\rangle^2}{\|\partial_x P\|^2 \|\partial_x Q\|^2} \right)^{-1}.\end{aligned} \tag{10}$$

From condition (A), there exist δ, $0 < \delta < 1$ such that $\frac{|\langle \partial_x P_t(x), \partial_x Q_t(x)\rangle|^2}{|\partial_x P_t(x)|^2 |\partial_x Q_t(x)|^2} \leq \delta$, hence

$$1 - \frac{|\langle \partial P_t(x), \partial Q_t(x)\rangle|^2}{\|\partial P_t(x)\|^2 \|\partial Q_t(x)\|^2} \geq 1 - \delta \tag{11}$$

From Lemma 3.2(a), it follows that

$$\frac{|\theta(x)|}{\|\partial_x P\|} \leq c_1\|x\|. \tag{12}$$

Replacing (11) and (12) in (10), we obtain

$$\frac{1}{|\Delta|}\|\partial_x Q\|^2 |\theta(x)|\, \|\partial_x P\| \leq c_2\|x\| \tag{13}$$

Similarly, we have

$$\frac{1}{|\Delta|}|\langle \partial_x P, \partial_x Q\rangle|\,|\alpha(x)|\,\|\partial_x P\| \leq c_3\|x\|. \tag{14}$$

Replacing (13) and (14) in (9), we get

$$\|a(x,t)\partial_x P(x,t)\| < c_4\|x\|.$$

In the same way, we can prove that

$$\|b(x,t)\partial_x Q(x,t)\| < c_5\|x\|.$$

It is clear that condition (A) in Theorem 3.1 is a sufficient condition for the no coalescing of critical points for the family $F(x,t)$. It is not clear whether (A) implies the constance of the Milnor radius of the family. However, we can replace condition (B_f) by a weaker condition namely, condition(B_f) along the zero sets of P and Q, and use King's result (see Theorem 2.6) to get the following:

PROPOSITION 3.4. *Let F be as above and $\partial_x P(x,t) = \mathrm{Grad}_x\, P_t(x)$, $\partial P = \left(\partial_x P(x,t), \frac{\partial P}{\partial t}\right)$. Suppose that the following conditions hold:*

(A)

$$\frac{|\langle \partial_x P(x,t), \partial_x Q(x,t)\rangle|}{\|\partial_x P(x,t)\|\,\|\partial_x Q(x,t)\|} \leq 1-\rho,$$

$x \neq 0$ *near zero, for all* $t \in \mathsf{R}$ *and* $0 < \rho \leq 1$.

(B_f^*)

$$\overline{\langle \partial_x P(x,t)\rangle}_{\mathcal{O}_{X_t}} = \overline{\langle \partial_x P(x,0)\rangle}_{\mathcal{O}_{X_0}}$$
$$\overline{\langle \partial_x Q(x,t)\rangle}_{\mathcal{O}_{X_t}} = \overline{\langle \partial_x Q(x,0)\rangle}_{\mathcal{O}_{X_0}},$$

where $\mathcal{O}_{X_t}$ is the local ring of the real analytic variety for $X_t = F_t^{-1}(0)$.

Then the family F_t is topologically trivial.

PROOF. The proof is analogous to the proof of Theorem 3.1. We use condition (A) and (B_f^*) to construct a vector field V which leaves $X = F^{-1}(0)$ invariant.

The next result is an application of the *Theorem 3.1* to deformations of Newton non-degenerate map-germs $F_0 = (P_0, Q_0) : \mathsf{R}^n, 0 \to \mathsf{R}^2, 0$. We first recall the basic notions of Newton diagram of an ideal.

For this, we fix a coordinate system $x = (x_1, \dots, x_n)$ in R^n, so that $\mathscr{A}_n$ is identified with the ring $\mathsf{R}[[x]]$ of convergent power series. For each germ $g(x) = \sum a_k x^k$, we define $\operatorname{supp}(g) = \{k \in \mathsf{Z}^n : a_k \neq 0\}$.

DEFINITION 3.5.

(i) Let I be an ideal in $\mathscr{A}_n$, define

$$\operatorname{supp} I = \cup\{\operatorname{supp} g : g \in I\}.$$

(ii) The Newton polyhedron of I, denoted by $\Gamma_+(I)$, is the convex hull in R^n_+ of the set

$$\cup\{r + v : r \in \operatorname{supp} I, v \in \mathsf{R}^n_+\}.$$

(iii) $\Gamma(I)$ is the union of all compact faces of $\Gamma_+(I)$.

(iv) $I = \langle g_1, \dots, g_s\rangle$ is Newton non-degenerate if for each compact face $\triangle \subset \Gamma(I)$, the equations $g_{1\triangle}(x) = g_{2\triangle}(x) = \cdots = g_{s\triangle}(x) = 0$ have no common solution in $(\mathsf{R} \setminus \{0\})^n$, where $g_{i\triangle}$ is the restriction of g_i to the face $\triangle$, that is, if $g_i(x) = \sum a_r x^r$ then $g_{i\triangle}(x) = \sum_{r \in \triangle} a_r x^r$.

COROLLARY 3.6. *Let* $F_0 = (P_0, Q_0) : \mathsf{R}^n, 0 \to \mathsf{R}^2, 0$ *and* $F(x,t) = (P_0(x) + t\theta(x), Q_0(x) + t\alpha(x))$. *Suppose* $\Gamma_+(\partial P_0(x))$ *and* $\Gamma_+(\partial Q_0(x))$ *are Newton non-degenerate and* $\Gamma_+(\partial\theta(x)) \subset \Gamma_+(\partial P_0(x))$, $\Gamma_+(\partial\alpha(x)) \subset \Gamma_+(\partial Q_0(x))$.

(1) *If* F_0 *satisfies condition* (A), *then* F *is topologically trivial.*

(2) *If* F_0 *satisfies conditions* (A) *and* (B_{R}), *then there exists* ϵ_0 *such that for all* $0 < \epsilon \leq \epsilon_0$, $\frac{F_t}{\|F_t\|} : \mathscr{S}^{n-1}_\epsilon \setminus K_t \to \mathscr{S}^1$ *is the projection of a locally trivial fiber bundle, where* $K_t = F_t^{-1}(0) \cap \mathscr{S}^{n-1}_\epsilon$. *Moreover,* $\forall t, t' \in \mathsf{R}$, *the Milnor fibrations associated to* F_t *and* $F_{t'}$ *are equivalent.*

PROOF. To verify condition (A) for the family F_t, let $r : \mathsf{R}, 0 \to \mathsf{R}^n, 0$ be a non constant real analytic curve, with $r(0) = 0$.

Therefore, for each t, fixed, we have:

$$\partial_x P_t(r(s)) = \partial_x P_0(r(s)) + \partial_x R(r(s))$$
$$\partial_x Q_t(r(s)) = \partial_x Q_0(r(s)) + \partial_x S(r(s))$$

Taking the Taylor developments we get,

$$\partial_x P_t(r(s)) = \alpha_1 s^{a_1} + \cdots + \alpha_2 s^{n_1} + \cdots$$
$$\partial_x Q_t(r(s)) = \beta_1 s^{a_1} + \cdots + \beta_2 s^{n_2} + \cdots.$$

Since $\Gamma_+(\partial P_0(x))$ and $\Gamma_+(\partial Q_0(x))$ are Newton non degenerate, then $a_1 \leq n_1$ and $a_1 \leq n_2$.

Then,

$$\frac{|\langle \partial P_t(r(s)), \partial Q_t(r(s))\rangle|}{\|\partial P_t(r(s))\| \, \|\partial Q_t(r(s))\|} = \frac{|\langle \alpha_1 s^{a_1} + \cdots, \beta_1 s^{a_1} + \cdots\rangle|}{\|\alpha_1 s^{a_1} + \cdots \| \, \|\beta_1 s^{a_1} + \cdots \|}$$

$$= \frac{|\langle \alpha_1, \beta_1\rangle| s^{2a_1} + \cdots}{s^{2a_1}\|\alpha_1\|(1+\cdots)\|\beta_1\|(1+\cdots)} = \frac{|\langle \alpha_1, \beta_1\rangle| s^{2a_1}(1+\cdots)}{\|\alpha_1\| \, \|\beta_1\| s^{2a_1}(1+\cdots)}$$

$$= \frac{|\langle \alpha_1, \beta_1\rangle|}{\|\alpha_1\|\|\beta_1\|} + s(\cdots) = \frac{|\langle \alpha_1, \beta_1\rangle|}{\|\alpha_1\| \, \|\beta_1\|} + u(s)$$

From the hypothesis it follows that $\frac{|\langle \alpha_1, \beta_1\rangle|}{\|\alpha_1\|\|\beta_1\|} < 1$ and $\lim_{s\to 0} u(s) = 0$. Then for sufficiently small s, we have:

$$\frac{|\langle \partial P_t(r(s)), \partial Q_t(r(s))\rangle|}{\|\partial P_t(r(s))\| \, \|\partial Q_t(r(s))\|} \leq 1 - \rho$$

where $0 < \rho \leq 1$ for $t \in \mathsf{R}$ fixed. Then, there exist a neighborhood $V \subsetneq \mathsf{R}^n$, $0 \in V$ such that

$$\forall x \in V \setminus \{0\} : \frac{|\langle \partial P_t(x)), \partial Q_t(x)\rangle|}{\|\partial P_t(x)\| \, \|\partial Q_t(x)\|} \leq 1 - \rho, \quad 0 < \rho \leq 1.$$

Since the Newton polyhedrons $\Gamma_+(\partial P_0(x))$ and $\Gamma_+(\partial Q_0(x))$ are non degenerate and $\Gamma_+(\partial P_t(x)) \subset \Gamma_+(\partial P_0(x)), \Gamma_+(\partial Q_t(x)) \subset \Gamma_+(\partial Q_0(x))$, it follows from Theorem 3.4 [10] that $\overline{\langle \partial P_t(x)\rangle}_{\mathscr{A}_n} = \overline{\langle \partial P_0(x)\rangle}_{\mathscr{A}_n}$ and $\overline{\langle \partial Q_t(x)\rangle}_{\mathscr{A}_n} = \overline{\langle \partial Q_0(x)\rangle}_{\mathscr{A}_n}$. Then (1) follows from Theorem 3.1.

Now, if condition (B_{R}) holds for F_0, it follows that $\overline{\langle \partial P_t(x)\rangle}_{\mathscr{A}_n} = \overline{\langle \partial Q_t(x)\rangle}_{\mathscr{A}_n}$. Thus, we can apply Jacquemard's result to prove that for each t, there exists ϵ_0, such that for all $0 < \epsilon \leq \epsilon_0$, $\frac{F_t}{\|F_t\|} : \mathscr{S}^{n-1}_\epsilon \setminus K_t \to \mathscr{S}^1$ is the projection of a locally trivial fiber bundle. Moreover, it follows from [5] Theorem 1 that these fibrations are equivalent.

Acknowledgments. I would like to thank my thesis advisor Maria Aparecida Ruas for discussions and numerous remarks on mathematics and English usage of the paper.

REFERENCES

1. Parusiński, Adam, *Topological Triviality of μ-constant deformations of type $f(x) + tg(x)$*, Canad. J. Math. 53 (2001), 73–97.
2. Damon, J., Gaffney, T., *Topological triviality of deformations of functions and Newton filtrations*, Invent. Math. 72 (1983), 335–358.
3. Gaffney, T., *The integral closure of Modules and Whitney equisingularity*, Invent. Math. 102 (1992), 301–322.

4. Martin Greuel, G., *Constant Milnor number implies constant multipllicity for quasihomogeneous singularities*, Manuscripta Math. 56 (1986), 159–166.
5. King, H. C., *Topological type of isolated critical points*, Ann. of Math. 107 (1978), 385–397.
6. King, H. C., *Topological Type in Families of Germs*, Invent. Math. 62 (1980), 1–13.
7. Jaquemard, A., *Fibrations de Milnor pour des applications réelles*, Boll. Un. Mat. Ital. B (7) 37, 1 (1989), 45–62.
8. Jacquemard, A., *Thèse 3ème cycle Université de Dijon*, 1982.
9. Milnor, J., *Singular Points of Complex Hypersurfaces*, Ann. of Math. Stud. 61 (1968).
10. Saia, M. J., *The integral closure of ideals and the Newton filtration*, J. Algebraic Geom. 6 (1996), 1–11.
11. Ruas, M. A. S., Seade, J., and Verjovsky, *On real singularities with a Milnor fibration*, Trends Math.: trends in Singularities (2002), 191–213.
12. Teissier, B., *Introduction to equisingularity problems,* Proc. Sympos. Pure Math. 29 (1975), 539–632.

INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DA BAHIA
AV. ADEMAR DE BARROS, S/N
ONDINA SALVADOR
BAHIA – BRASIL
E-mail: rnonato@ufba.br