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STRONG PERFORATION IN INFINITELY GENERATED K_0-GROUPS OF SIMPLE C^*-ALGEBRAS

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Abstract

Let (G, G^+) be an ordered abelian group. We say that G has strong perforation if there exists $x \in G$, $x \notin G^+$, such that $nx \in G^+$, $nx \neq 0$ for some natural number n. Otherwise, the group is said to be weakly unperforated. Examples of simple C^*-algebras whose ordered K_0-groups have this property and for which the entire order structure on K_0 is known have, until now, been restricted to the case where K_0 is group isomorphic to the integers. We construct simple, separable, unital C^*-algebras with strongly perforated K_0-groups group isomorphic to an arbitrary infinitely generated subgroup of the rationals, and determine the order structure on K_0 in each case.

1. Introduction

Elliott's classification of AF C^*-algebras via the K_0-group ([2]) began a widespread effort to classify nuclear C^*-algebras. The K_0-group, which is an ordered group for stably finite C^*-algebras ([1]), has figured prominently in almost all work on this problem. (For an overview of the classification problem for nuclear C^*-algebras, see [3].) So far, every result on the classification of C^*-algebras has required the assumption that the ordered K_0-group be weakly unperforated whenever it is not zero. This assumption was shown to be nontrivial by Villadsen ([8]); the ordered abelian group $\mathsf{Z}_n := (\mathsf{Z}, \{0, n, n+1, \ldots\})$ may arise as a saturated sub-ordered group of the K_0-group of a simple nuclear C^*-algebra. In [4], Elliott and Villadsen refined the results of [8] to obtain, for each natural number n, a simple nuclear C^*-algebra A_n whose ordered K_0-group is order isomorphic to Z_n. This result was further generalised by the author in [7], where it was shown that a certain class of order structures on the integers (which might possibly comprise all such order structures giving a simple ordered group) could arise as the ordered K_0-group of a simple nuclear C^*-algebra.

The classification of a category by an invariant is not complete until one knows the range of the invariant, and any classification of simple nuclear stably finite C^*-algebras will necessarily capture the ordered K_0-group. Thus, the range of the K_0 functor bears investigation. This range is known when K_0

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is a weakly unperforated ordered group, whence our interest in instances of the ordered K_0-group which exhibit strong perforation.

2. Essential Results

In this section we review results from [4] that will be used in the sequel.

Let C, D be C^*-algebras, and let ϕ_0, ϕ_1 be $*$-homomorphisms from C to D. The generalised mapping torus of C and D with respect to ϕ_0 and ϕ_1 is

$$A := \{ (c, d) | d \in C([0, 1]; D),\ c \in C,\ d(0) = \phi_0(c),\ d(1) = \phi_1(c) \}$$

We will write $A(C, D, \phi_0, \phi_1)$ for A when clarity demands it. We now list without proof some theorems, specialised to our needs, which will be used in the sequel.

THEOREM 2.1 (Elliott and Villadsen ([4]), Sec. 2, Thm. 2). *The index map $b_* : \mathrm{K}_*C \to \mathrm{K}_{1-*}\mathrm{S}D = \mathrm{K}_*D$ in the six term periodic sequence for the extension*

$$0 \to \mathrm{S}D \to A \to C \to 0$$

is the difference

$$\mathrm{K}_*\phi_1 - \mathrm{K}_*\phi_0 : \mathrm{K}_*C \to \mathrm{K}_*D.$$

Thus, the six-term exact sequence may be written as the short exact sequence

$$0 \to \mathrm{Coker}\, b_{1-*} \to \mathrm{K}_*A \to \mathrm{Ker}\, b_* \to 0.$$

In particular, if b_{1-*} is surjective, then K_*A is isomorphic to its image, $\mathrm{Ker}\, b_*$, in K_*C.

Suppose that cancellation holds for each pair of projections in $D \otimes \mathcal{K}$ obtained as the images under the maps ϕ_0 and ϕ_1 of a single projection in $C \otimes \mathcal{K}$. Then, if b_1 is surjective,

$$(\mathrm{K}_0A)^+ \cong (\mathrm{K}_0C)^+ \cap \mathrm{K}_0(e_\infty)(\mathrm{K}_0A),$$

where e_∞ denotes the evaluation of A at the fibre at infinity.

THEOREM 2.2 (Elliott and Villadsen ([4]), Sec. 3, Thm. 3). *Let A_1 and A_2 be building block algebras as described above,*

$$A_i = A(C, D, \phi_0^i, \phi_1^i), \quad i = 1, 2.$$

Let there be given three maps between the fibres,

$$\gamma : C_1 \to C_2,$$
$$\delta, \delta' : D_1 \to D_2,$$

such that δ and δ' have mutually orthogonal images, and

$$\delta\phi_0^1 + \delta'\phi_1^1 = \phi_0^2\gamma,$$
$$\delta\phi_1^1 + \delta'\phi_0^1 = \phi_1^2\gamma.$$

Then there exists a unique map

$$\theta : A_1 \to A_2,$$

respecting the canonical ideals, giving rise to the map $\gamma : C_1 \to C_2$ between the quotients (or fibres at infinity), and such that for any $0 < s < 1$, if e_s denotes evaluation at s,

$$e_s\theta = \delta e_s + \delta' e_{1-s}.$$

Let A_1 and A_2 be building block algebras as in Theorem 2.1 with $\theta : A_1 \to A_2$ as in Theorem 2.2. Let there be given a map $\beta : D_1 \to C_2$ such that the composed map $\beta\phi_1^1$ is a direct summand of the map γ, and such that the composed maps $\phi_0^2\beta$ and $\phi_1^2\beta$ are direct summands of the maps δ' and δ, respectively. Suppose that the decomposition of γ as the orthogonal sum of $\beta\phi_1^1$ and another map is such that the image of the second map is orthogonal to the image of β. (Note that this requirement is automatically satisfied if C_1, D_1, and the map $\beta\phi_1^1$ are unital.)

Let

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

be a sequence of separable building block C^*-algebras,

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \qquad i = 1, 2, \ldots$$

with each map $\theta_i : A_i \to A_{i+1}$ obtained by the construction of Theorem 2.2. For each $i = 1, 2, \ldots$ let $\beta_i : D_i \to C_{i+1}$ be a map verifying the hypotheses of the preceding paragraph.

Suppose that for every $i = 1, 2, \ldots$, the intersection of the kernels of the boundary maps ϕ_0^i and ϕ_1^i from C_i to D_i is zero.

Suppose that, for each i, the image of each of ϕ_0^{i+1} and ϕ_1^{i+1} generates D_{i+1} as a closed two-sided ideal, and that this is in fact true for the restriction of ϕ_0^{i+1} and ϕ_1^{i+1} to the smallest direct summand of C_{i+1} containing the image of β_i. Suppose that the closed two-sided ideal of C_{i+1} generated by the image of β_i is a direct summand.

Suppose that, for each i, the maps $\delta_i' - \phi_0^i\beta_i$ and $\delta_i - \phi_1^i\beta_i$ from D_i to D_{i+1} are injective.

Suppose that, for each i, the map $\gamma_i - \beta_i\phi_1^i$ takes each non-zero direct summand of C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal.

Suppose that, for each i, the map $\beta_i : D_i \to C_{i+1}$ can be deformed – inside the hereditary sub-C^*-algebra generated by its image – to a map $\alpha_i : D_i \to C_{i+1}$ with the following property: There is a direct summand of α_i, say $\bar{\alpha}_i$, such that $\bar{\alpha}_i$ is non-zero on an arbitrary given element x_i of D_i, and has image a simple sub-C^*-algebra of C_{i+1}, the closed two-sided ideal generated by which contains the image of β_i.

THEOREM 2.3 (Elliott and Villadsen ([4]), Sec. 5, Thm. 5). *If the hypotheses above are satisfied, there is a map θ_i' homotopic inside A_i to θ_i for each i such that the inductive limit of the sequence*

$$A_1 \xrightarrow{\theta_1'} A_2 \xrightarrow{\theta_2'} \cdots$$

is simple.

3. Infinitely Generated Subgroups of the Rational Numbers

A generalised integer is a symbol $\mathbf{n} = a_1^{n_1}a_2^{n_2}a_3^{n_3}\ldots$, where the a_i's are pairwise distinct prime numbers and each n_i is either a non-negative integer or ∞. The subgroup $G_\mathbf{n}$ of the rational numbers associated to the generalised integer $\mathbf{n}$ is the group of all rationals whose denominators (when in lowest terms) are products of powers of the a_i's not exceeding a^{n_i}. If $n_i = \infty$, then an arbitrarily large power of a_i may appear in the denominator.

THEOREM 3.1. *For each pair $(\mathbf{n}, k)$ consisting of a generalised integer $\mathbf{n}$ and a positive rational $k < 1$, there exists a simple, separable, unital, nuclear C^*-algebra $A_{(\mathbf{n},k)}$ such that*

$$(\mathrm{K}_0(A_{(\mathbf{n},k)}), \mathrm{K}_0(A_{(\mathbf{n},k)})^+, [1_{A_{(\mathbf{n},k)}}]) = (G_\mathbf{n}, G_\mathbf{n} \cap (k, \infty), 1).$$

PROOF. Given a 2-tuple $(\mathbf{n}, k)$ we will construct a sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

where $A_j = A(C_j, D_j, \phi_0^j, \phi_1^j)$, and the θ_j constructed as in Theorem 2.2 from maps

$$\gamma_j : C_j \to C_{j+1}, \qquad \delta_j, \delta_j' : D_j \to D_{j+1}.$$

In order to obtain a simple inductive limit, we will require a map

$$\beta_j : D_j \to C_{j+1}$$

having the properties listed in Section 2.

For each j let
$$C_j = p_j(\mathrm{C}(X_j) \otimes \mathscr{K})p_j$$
where p_j is a projection in $\mathrm{C}(X_j) \otimes \mathscr{K}$ and $\mathscr{K}$ denotes the compact operators. Express k in lowest terms, say $\frac{a}{b}$, and set $X_1 = \mathrm{S}^{2\times(a+1)}$. Let $X_{j+1} = X_j{}^{\times n_j}$, where n_j is a natural number to be specified.

Let $D_j = C_j \otimes \mathrm{M}_{\dim(p_j)k_j}$, where k_j is a natural number to be specified. Let μ_j and ν_j be maps from C_j to $C_j \otimes \mathrm{M}_{\dim(p_j)}$ given by
$$\mu_j(a) = p_j \otimes a(x_j) \cdot 1_{\dim(p_j)}$$
(where x_j is a point to be specified in X_j and $1_{\dim(p_j)}$ is the unit of $\mathrm{M}_{\dim(p_j)}$) and
$$\nu_j(a) = a \otimes 1_{\dim(p_j)}.$$

For $t \in \{0, 1\}$, let $\phi_j^t : C_j \to D_j$ be the direct sum of l_j^t and $k_j - l_j^t$ copies of μ_j and ν_j, respectively, where the l_j^t are non-negative integers such that $l_j^0 \neq l_j^1$ for all $j \geq 1$.

Note that both C_j and D_j are unital, as are the maps ϕ_j^t. The ϕ_j^t are also injective and as such satisfy the hypotheses of Section 2 concerning them alone.

By Theorem 2.1, for each $e \in \mathrm{K}_0(C_j)$,
$$\begin{aligned} b_0(e) &= (l_j^1 - l_j^0)(\mathrm{K}_0(\mu_j) - \mathrm{K}_0(\nu_j))(e) \\ &= (l_j^1 - l_j^0)(\dim(p_j) \cdot \mathrm{K}_0(p_j) - \dim(p_j) \cdot e). \end{aligned}$$

Since $l_j^1 - l_j^0$ is non-zero for every j and $\mathrm{K}_0(X_j)$ is torsion free, $b_0(e) = 0$ implies that e belongs to the maximal free cyclic subgroup of $\mathrm{K}_0(C_j)$ containing $\mathrm{K}_0(p_j)$. As $\mathrm{K}_1(C_j) = 0$, b_1 is surjective. $\mathrm{K}_0(A_j)$ is thus group isomorphic (by Theorem 2.1) to its image, in $\mathrm{K}_0(C_j)$ – which is isomorphic as a group to $\mathbf{Z}$.

In order for $\mathrm{K}_0(A_j)$ to be isomorphic as an ordered group to its image in $\mathrm{K}_0(C_j)$, with the relative order, it is sufficient (by Theorem 2.1) that for any projection q in $C_j \otimes \mathscr{K}$ such that the images of q under $\phi_j^0 \otimes \mathrm{id}$ and $\phi_j^1 \otimes \mathrm{id}$ have the same K_0 class, these images be in fact equivalent. For any such q, the image of $\mathrm{K}_0(q)$ under $b_0 = \mathrm{K}_0(\phi_j^1) - \mathrm{K}_0(\phi_j^0)$ is zero, so that $\mathrm{K}_0(q)$ belongs to $\operatorname{Ker} b_0$. It will be clear from the construction below that the dimension of both $\phi_j^1(q)$ and $\phi_j^0(q)$ is at least half the dimension of X_j. Thus, by Theorem 8.1.5 of [5], $\phi_j^1(q)$ and $\phi_j^0(q)$ are equivalent, as they have the same K_0 class.

Let us now specify the projection p_1. Let ξ be the Hopf line bundle over S^2. Set $g_1 = [\xi^{\times a+1}] - [\theta_a] \in \mathrm{K}^0(X_1)$, where $[\,\cdot\,]$ denotes the stable isomorphism class of a vector bundle and θ_l denotes the trivial vector bundle of fiber dimension l. By Theorem 8.1.5 of [5], we have that $(a+1) \cdot g_1$ and hence $b \cdot g_1$

are positive. Let p_1 be a projection in $C(X_1) \otimes \mathcal{K}$ corresponding to the K^0 class $b \cdot g_1$. By [8] we know that the ordered, saturated, free cyclic subgroup of $K_0(C_1)$ generated by g_1 is equal to

$$(\mathbb{Z}, \{0, a+1, a+2, \ldots\}),$$

where the class of the unit is the integer $b \geq a+1$.

Decompose b into powers of primes, $b = a_{i_1}^{m_1} a_{i_2}^{m_2} \ldots a_{i_n}^{m_n}$. Set $\mathbf{n}' = \frac{\mathbf{n}}{b}$, with the convention that $\infty - l = \infty$ for all natural numbers l. Let L_j be an enumeration of the primes appearing in $\mathbf{n}'$ for $j \geq 2$, $j \in \mathbb{N}$, and set $L_1 = b$.

We now define a family of continuous maps from S^2 to S^2, indexed by the integers, to be used in the construction of the maps γ_j from C_j to C_{j+1}. Consider S^2 as being embedded in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ as the unit sphere with center the origin, with the identification $(x, y, z) = (x + yi, z)$. For each $\eta \in \mathbb{N}$, let $\omega'_\eta : \mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{C} \times \mathbb{R}$ be defined by $\omega'_\eta(w, z) = (w^\eta / |w^{\eta-1}|, z)$ when $w \neq 0$ and otherwise by $\omega'_\eta(0, z) = (0, z)$. This defines a map from S^2 to itself by restriction. Let ω_η be the composition of ω'_η with the antipodal map. Note that ω'_η is the suspension of the η^{th} power map on S^1, and thus has the same degree, namely $-\eta$, as this map ([6]). As the antipodal map has degree -1, the composed map ω_η has degree η. In the language of vector bundles, $K^0(\omega_\eta)([\xi]) = [\xi^{\otimes \eta}]$.

Define a map γ'_j from $C(X_j)$ to $M_{n_j} \otimes C(X_{j+1}) = M_{n_j}(C({X_j}^{\otimes n_j})$ as follows:

$$\gamma'_j(f(x)) = (f(\omega_{L_{j+1}}(x)) \otimes 1 \otimes \cdots \otimes 1) \oplus (1 \otimes f(\omega_{L_{j+1}}(x)) \otimes \cdots \otimes 1) \oplus \cdots \oplus \ (1 \otimes 1 \otimes \cdots \otimes f(\omega_{L_{j+1}}(x))).$$

Let

$$\beta'_j = 1 \cdot e_{x_j}$$

be a map from $C(X_j)$ to $C(X_{j+1})$, where e_{x_j} denotes the evaluation of an element of $C(X_j)$ at a point $x_j \in X_j$ and 1 is the unit of $C(X_{j+1})$. Fix $x_1 \in S^2$ and define $x_{j+1} := (\omega_{L_{j+1}}(x_j), \ldots, \omega_{L_{j+1}}(x_j)) \in {X_j}^{\times n_j} = X_{j+1}$.

Let us define $\gamma_j : C(X_j) \to M_{n_j}(C(X_{j+1})) \otimes M_2(\mathcal{K})$ inductively as the direct sum of two maps. For the first map, take the restriction to $C_j \subseteq C(X_j) \otimes \mathcal{K}$ of the tensor product of γ'_j with the identity map from $\mathcal{K}$ to $\mathcal{K}$. The second map is obtained as follows: compose the map ϕ^1_j with the direct sum of q_j copies of the tensor product of β'_j with the identity map from $\mathcal{K}$ to $\mathcal{K}$ (restricted to $D_j \subseteq C(X_j) \otimes \mathcal{K}$), where q_j is to be specified. The induction consists of first considering the case $j = 1$ (since p_1 has already been chosen), then setting $p_2 = \gamma_j(p_1)$, so that C_2 is specified as the cut-down of $C(X_j) \otimes M_2(\mathcal{K})$, and continuing in this way.

With β_j taken to be the restriction to $D_j \subseteq C(X_j) \otimes \mathcal{K}$ of $\beta'_j \otimes \mathrm{id}$ we have, by construction, that $\beta_j \phi_j^1$ is a direct summand of γ_j and, furthermore, the second direct summand and β_j map into orthogonal blocks (and hence orthogonal subalgebras) as desired.

We will now need to verify that p_j has the following property: the set of all rational multiples of $K_0(p_j)$ in the ordered group $K_0(C_j) = K^0(X_j)$ is isomorphic (as a sub ordered group) to

$$(\mathsf{Z}, \{0, l_j + 1, l_j + 2, \ldots\}),$$

where

$$l_j := L_j l_{j-1}, \qquad l_1 := a$$

and the class of the unit (i.e., of p_j) is $\prod_{k=1}^{j} L_k$.

Our verification will proceed by induction. The case $j = 1$ has been established by construction. Suppose that the assertion of the preceding paragraph holds for all p_k, $k \leq j$. Suppose further that the group of rational multiples of $K_0(p_k)$ (being isomorphic as a group to Z) is generated by a K_0 class of the form $[\xi^{\times n}] - [\theta_m]$, where $m < n$ and (this is again true by construction for $k = 1$). We will show that $K_0(p_j)$ has both the property of the preceding paragraph and the property just mentioned.

Let $g_k \in K^0(X_k)$ be the generator of the group of rational multiples of p_k. Note that, as is the case for all maps on $K^0(S^2)$ induced by a continuous map from S^2 to itself, $K_0(\omega_\eta)([\theta_1]) = [\theta_1]$. Write $g_k = [\xi^{\times d_k}] - [\theta_{m_k}]$. Then

$$K_0(\gamma_j)(g_j) = [(\xi^{\otimes L_{j+1}})^{\times d_j n_j}] - [\theta_{m'_{j+1}}]$$

for some integers $d_j > 0$ and m'_{j+1}. We may assume that the multiplicity of the map $K_0(\gamma_j)$ is divisible by L_{j+1}, as we have yet to specify n_j. We recall that for any integer l, the K_0 class $[\xi^{\otimes l}]$ corresponds to the element $(1, l)$ in $K^0(S^2) = \langle[\theta_1]\rangle \oplus \langle e(\xi)\rangle$, which is also the difference of K_0 classes $l[\xi] - [\theta_{l-1}]$. Thus we have

$$K_0(\gamma_j)(g_j) = L_{j+1}([\xi^{\times(a+1)n_1 n_2 \ldots n_j}] - [\theta_{m_{j+1}}]).$$

for some integer m_{j+1}. Setting $g_j := [\xi^{\times(a+1)n_1 n_2 \ldots n_j}] - [\theta_{m_{j+1}}]$, we have established that $K_0(\gamma_j)(g_j) = L_{j+1} g_{j+1}$ for all natural numbers j.

We now show that n_j may be chosen so as to ensure that the maximal, free, cyclic subgroup of $K_0 C_{j+1}$ generated by g_{j+1} is indeed isomorphic as an ordered group to the integers with positive cone $\{0, l_{j+1} + 1, l_{j+1} + 2, \ldots\}$. That $\prod_{k=1}^{j} L_k$ is the class of the unit follows directly from the fact that $L_1 = b$ (the class of the unit in $K_0 C_1$) and that $K_0(\gamma_j)(g_j) = L_{j+1} g_{j+1}$.

As the Euler class of the Hopf line bundle on S^2 is non-zero we have, by [8], that for $q, m, h \in \mathbb{N}$ such that $0 < h(q-m) < q$,

$$h([\xi^{\times q}] - [\theta_m]) \notin (K^0 S^{2\times q})^+.$$

To apply this we note that

$$g_{j+1} = [\xi^{\times(a+1)n_1 n_2 \dots n_j}] - [\theta_{m_j}].$$

With $q = (a+1)n_1 n_2 \dots n_j$ and $m = m_j$ we wish to have

$$0 < l_j(q-m) < q$$

as then $0 < h(q-m) < q$ for all $0 < h < l_j + 1$.

Since

$$q - m = \dim g_{j+1} = \frac{n_j + k_j q_j \dim p_j}{L_{j+1}} \dim g_j$$

we want

$$\dim g_{j+1} < \frac{(a+1)n_1 n_2 \dots n_j}{l_{j+1}}.$$

Assume inductively that $n_1, n_2, \dots, n_{j-1}$ have been chosen so that

$$\dim g_j < \frac{(a+1)n_1 n_2 \dots n_{j-1}}{l_j}.$$

Choose n_j large enough so that

$$\frac{n_j + k_j q_j \dim p_j}{n_j} \dim g_j < \frac{(a+1)n_1 n_2 \dots n_{j-1}}{l_j}.$$

Then we have that

$$\frac{n_j + k_j q_j \dim p_j}{L_{j+1}} \dim g_j < \frac{(a+1)n_1 n_2 \dots n_j}{L_{j+1} l_j}.$$

Recalling that $l_{j+1} = L_{j+1} l_j$ we conclude that

$$\dim g_{j+1} = \frac{n_j + k_j q_j \dim p_j}{L_{j+1}} \dim g_j < \frac{(a+1)n_1 n_2 \dots n_j}{l_{j+1}},$$

as desired.

Note that $\gamma_j - \beta_j \phi_j^1$ is non-zero and so, as required in the hypotheses of Theorem 2.4, takes C_j into a subalgebra of C_{j+1} not contained in any proper closed two-sided ideal.

It remains to construct maps δ_j and δ'_j from D_j to D_{j+1} with orthogonal images such that

$$\delta_j\phi_j^0 + \delta'_j\phi_j^1 = \phi_{j+1}^0\gamma_j,$$
$$\delta_j\phi_j^1 + \delta'_j\phi_j^0 = \phi_{j+1}^1\gamma_j,$$

and $\phi_{j+1}^0\beta_j$ and $\phi_{j+1}^1\beta_j$ are direct summands of δ'_j and δ_j respectively. To do this we shall have to modify ϕ_{j+1}^0 and ϕ_{j+1}^1 by inner automorphisms; this is permissible since it has no effect on K-groups. The definition of δ_j and δ'_j along with the proof that they satisfy the hypotheses of section 2 is taken from [4].

In order to carry out this step we define $x_{j+1} := \omega_{L_{j+1}}(x_j)$, so that

$$e_{x_{j+1}}\gamma_j = \mathrm{mult}(\gamma_j)e_{x_j},$$

where $\mathrm{mult}(\gamma_j)$ denotes the factor by which γ_j multiplies dimension. It follows that

$$\begin{aligned}\mu_{j+1}\gamma_j &= p_{j+1} \otimes e_{x_{j+1}}\gamma_j\\ &= \gamma_j(p_j) \otimes \mathrm{mult}(\gamma_j)e_{x_j}\\ &= \mathrm{mult}(\gamma_j)\gamma_j(p_j \otimes e_{x_j})\\ &= \mathrm{mult}(\gamma_j)\gamma_j\mu_j,\end{aligned}$$

and

$$\begin{aligned}\nu_{j+1}\gamma_j &= \gamma_j \otimes 1_{\dim(p_{j+1})}\\ &= \mathrm{mult}(\gamma_j)\gamma_j \otimes 1_{\dim(p_j)}\\ &= \mathrm{mult}(\gamma_j)\gamma_j\nu_j.\end{aligned}$$

Take δ_j and δ'_j to be the direct sum of r_j and s_j copies of γ_j, where r_j and s_j are to be specified. The condition, for $t = 0, 1$, that

$$\delta_j\phi_j^t + \delta'_j\phi_j^{1-t} = \phi_{j+1}^t\gamma_j,$$

understood up to unitary equivalence, then becomes the condition

$$\begin{aligned}r_j\gamma_j(l_j^t\mu_j + (k_j - l_j^t)\nu_j) + s_j\gamma_j(l_j^{1-t}\mu_j + (k_j - l_j^{1-t})\nu_j)&\\ = (l_{j+1}^t\mu_{j+1} + (k_{j+1} - l_{j+1}^t)\nu_{j+1})\gamma_j,&\end{aligned}$$

also up to unitary equivalence. As $\mathrm{K}_0(\mu_j)$ and $\mathrm{K}_0(\nu_j)$ are independent this is equivalent to the two equations

$$\begin{aligned}r_jl_j^t + s_jl_j^{1-t} &= \mathrm{mult}(\gamma_j)l_{j+1}^t,\\ (r_j + s_j)k_j &= \mathrm{mult}(\gamma_j)k_{j+1}.\end{aligned}$$

Choose $r_j = 2\,\mathrm{mult}(\gamma_j)$ and $s_j = \mathrm{mult}(\gamma_j)$, so that

$$k_{j+1} = 3k_j$$

and

$$l_{j+1}^t = 2l_j^t + l_j^{1-t}$$

Taking $k_1 = 1, l_1^0 = 0$, and $l_1^1 = 1$ we have $k_j = 3^{j-1}$ and $l_j^1 - l_j^0 = 1$ for all j and, in particular, these quantities are non-zero, as required above.

Next let us show that, up to unitary equivalence preserving the equations

$$\delta_j \phi_j^t + \delta_j' \phi_j^{1-t} = \phi_{j+1}^t \gamma_j,$$

$\phi_{j+1}^0 \beta_j$ is a direct summand of $\delta_j' = \mathrm{mult}(\gamma_j)\gamma_j$, and $\phi_{j+1}^1 \beta_j$ is a direct summand of $\delta_j = 2\,\mathrm{mult}(\gamma_j)\gamma_j$.

Note that $\phi_{j+1}^t \beta_j$ is the direct sum of l_{j+1}^t copies of $p_{j+1} \otimes \beta_j$ and $(k_{j+1} - l_{j+1}^t)\dim(p_{j+1})$ copies of β_j, whereas δ_j' and δ_j contain, respectively, $q_j\,\mathrm{mult}(\gamma_j)$ and $2q_j\,\mathrm{mult}(\gamma_j)$ copies of β_j. Note also that by Theorem 8.1.5 of [Hu] that a trivial projection of dimension at least $\dim(p_{j+1}) + \dim X_{j+1}$ in $\mathrm{C}(X_{j+1}) \otimes K$ contains a copy of p_{j+1}. Therefore, $\dim(p_{j+1}) + \dim X_{j+1}$ copies of β_j contain a copy of $p_{j+1} \otimes \beta_j$. It follows that $k_{j+1}(2\dim(p_{j+1}) + \dim X_{j+1})$ copies of β_j contain a copy of ϕ_{j+1}^t when t is either 1 or 0. Here, by a copy of a given map from D_j to D_{j+1} we mean another map obtained from it by conjugating by a partial isometry in D_{j+1} with initial projection the image of the unit.

Note that

$$\begin{aligned} k_{j+1}(2\dim(p_{j+1}) + \dim X_{j+1}) &= 3k_j(2\,\mathrm{mult}(\gamma_j)\dim(p_j) + n_j \dim X_j) \\ &\le 3k_j(2\dim(p_j) + \dim X_j)\,\mathrm{mult}(\gamma_j), \end{aligned}$$

and that k_j, $\dim(p_j)$ and $\dim X_j$ have already been specified and do not depend on n_j. It follows that, with

$$q_j = 3k_j(2\dim(p_j) + \dim X_j),$$

$q_j\,\mathrm{mult}(\gamma_j)$ copies of β_j contain a copy of $\phi_{j+1}^t \beta_j$ for $t = 0, 1$. In particular δ_j' and δ_j contain copies, respectively, of $\phi_{j+1}^0 \beta_j$ and $\phi_{j+1}^1 \beta_j$.

With this choice of q_j, let us show that for each $t = 0, 1$ there exists a unitary $u_t \in D_{j+1}$ commuting with the image of $\phi_{j+1}^t \gamma_j$, such that $(\mathrm{Ad}\, u_0)\phi_{j+1}^0 \beta_j$ is a direct summand of δ_j' and $(\mathrm{Ad}\, u_1)\phi_{j+1}^1 \beta_j$ is a direct summand of δ_j. In other words, for each $t = 0, 1$ we must show that the partial isometry constructed in the preceding paragraph, producing a copy of $\phi_{j+1}^t \beta_j$ inside either δ_j' or δ_j, may be chosen in such a way that it extends to a unitary element of D_{j+1} – which in addition commutes with the image of $\phi_{j+1}^t \gamma_j$.

We will consider the case $t = 0$. The case $t = 1$ is similar. Let us first show that the partial isometry in D_{j+1}, transforming $\phi^0_{j+1}\beta_j$ into a direct summand of δ'_j, may be chosen to lie in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Note first that the unit of the image of $\phi^0_{j+1}\beta_j$ – the initial projection of the partial isometry – lies in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Indeed, this projection is the image by ϕ^1_j of the unit of C_j. The property that $\beta_j\phi^1_j$ is a direct summand of γ_j implies in particular that the image by $\beta_j\phi^1_j$ of the unit of C_j commutes with the image of γ_j. The image by $\phi^0_{j+1}\beta_j\phi^1_j$ of the unit of C_j (i.e. the unit of the image of $\phi^0_{j+1}\beta_j$) therefore commutes with the image of $\phi^0_{j+1}\gamma_j$, as asserted.

Note also that the final projection of the partial isometry commutes with the image of $\phi^0_{j+1}\gamma_j$. Indeed, it is the unit of the image of a direct summand of δ'_j, and since D_j is unital it is the image of the unit of D_j by this direct summand; since C_j is unital and $\phi^1_j : C_j \longrightarrow D_j$ is unital, the projection in question is the image of the unit of C_j by a direct summand of $\delta'_j\phi^1_j$. But $\delta'_j\phi^1_j$ is itself a direct summand of $\phi^0_{j+1}\gamma_j$ (as $\phi^0_{j+1}\gamma_j = \delta_j\phi^0_j + \delta'_j\phi^1_j$), and so the projection in question is the image of the unit of C_j by a direct summand of $\phi^0_{j+1}\gamma_j$, and in particular commutes with the image of $\phi^0_{j+1}\gamma_j$.

Note that both direct summands of $\phi^0_{j+1}\gamma_j$ under consideration ($\phi^0_{j+1}\beta_j\phi^1_j$ and a copy of it) factor through the evaluation of C_j at the point x_j, and so are contained in the largest such direct summand of $\phi^0_{j+1}\gamma_j$; this largest direct summand, say π_j, is seen to exist by inspection of the construction of $\phi^0_{j+1}\gamma_j$. Since both projections under consideration (the images of the unit of C_j by the two copies of $\phi^0_{j+1}\beta_j\phi^1_j$) are less than $\pi_j(1)$, to show that they are unitarily equivalent in the commutant of the image of $\phi^0_{j+1}\gamma_j$ (in D_{j+1}) it is sufficient to show that they are unitarily equivalent in the commutant of the image of π_j in $\pi_j(1)D_{j+1}\pi_j(1)$. Note that this image is isomorphic to $\mathrm{M}_{\dim p_j}(C)$. By construction, the two projections in question are Murray-von Neumann equivalent – in D_{j+1} and therefore in $\pi_j(1)D_{j+1}\pi_j(1)$ – but all we shall use from this is that they have the same class in K^0X_{j+1}. Note that the dimension of these projections is $(k_{j+1}\dim(p_{j+1}))(k_j\dim(p_j))$, and that the dimension of $\pi_j(1)$ is $k_{j+1}\dim(p_{j+1}) + l^0_{j+1}(\dim(p_{j+1}))^2$. Since the two projections under consideration commute with $\pi_j(C_j)$, and this is isomorphic to $\mathrm{M}_{\dim(p_j)}(C)$, to prove unitary equivalence in the commutant of $\pi_j(C_j)$ in $\pi_j(1)D_{j+1}\pi_j(1)$ it is sufficient to prove unitary equivalence of the product of these projections with a fixed minimal projection of $\pi_j(C_j)$, say e. Since K^0X_{j+1} is torsion free, the products of the two projections under consideration with e still have the same class in K^0X_{j+1}. To prove that they are unitarily equivalent in $eD_{j+1}e$, it is sufficient (and necessary) to prove that both they and their complements inside e are Murray von-Neumann equivalent. Since both the cut-down projections and their complements inside e have the same class in $K^0 X_{j+1}$, to prove that the two pairs are equivalent it is sufficient, by Theorem 8.1.5 of [Hu], to show that all four projections have dimension at least $\frac{1}{2}\dim X_{j+1}$. Dividing the dimensions above by $\dim(p_j)$ (the order of the matrix algebra), we see that the dimension of the first pair of projections is $k_{j+1}k_j \dim(p_{j+1}) = k_{j+1}k_j \operatorname{mult}(\gamma_j)\dim(p_j)$. The dimension of e is $k_{j+1}\operatorname{mult}(\gamma_j) + l^0_{j+1}\operatorname{mult}(\gamma_j)\dim(p_{j+1})$, so that the dimension of the second pair of projections is $\operatorname{mult}(\gamma_j)(k_{j+1}+l^0_{j+1}\dim(p_{j+1})-k_{j+1}k_j\dim(p_j))$. Since $\dim(p_1) \geq \frac{1}{2}\dim X_1$, $\dim(p_{j+1}) = \operatorname{mult}(\gamma_j)\dim(p_j)$, $\dim X_{j+1} = n_j \dim X_j$, and $\operatorname{mult}(\gamma_j) \geq n_j$ (for all j), we have $\dim(p_{j+1}) \geq \frac{1}{2}\dim X_{j+1}$ (for all j). Since $k_{j+1}k_j$ is non-zero for all j, the first inequality holds. Since l^0_{j+1} is non-zero for all j, the second inequality holds if $\operatorname{mult}(\gamma_j)$ is strictly greater than $k_{j+1}k_j$. (One then has, using $\dim(p_{j+1}) = \operatorname{mult}(\gamma_j)\dim(p_j)$ twice, that the dimension of the second pair of projections is at least $\dim(p_{j+1})$.) Since $k_{j+1}k_j = 3{k_j}^2$, and k_j was specified before n_j, we may modify the choice of n_j so that $\operatorname{mult}(\gamma_j)$ – which is greater than n_j – is sufficiently large.

This shows that the two projections in D_{j+1} under consideration are unitarily equivalent by a unitary in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Replacing ϕ^0_{j+1} by its composition with the corresponding inner automorphism, we may suppose that the two projections in question are equal. In other words $\phi^0_{j+1}\beta_j$ is unitarily equivalent to the cut-down of δ'_j by the projection $\phi^0_{j+1}\beta_j(1)$.

Now consider the compositions of these two maps with ϕ^1_j, namely $\phi^0_{j+1}\beta_j\phi^1_j$ and the cut-down of $\delta'_j\phi^1_j$ by the projection $\phi^0_{j+1}\beta_j(1)$. Since both of these maps can be viewed as the cut-down of $\phi_{j+1}\gamma_j$ by the same projection, they are in fact the same map. Thus any unitary inside the cut-down of D_{j+1} by $\phi^0_{j+1}\beta_j(1)$ taking $\phi^0_{j+1}\beta_j$ into the cut-down of δ'_j by this projection (such a unitary is known to exist) must commute with the image of $\phi^0_{j+1}\beta_j\phi^1_j$ and hence with the image of $\phi^0_{j+1}\gamma_j$, since this commutes with the projection $\phi^0_{j+1}\beta_j(1) = \phi^0_{j+1}(\beta_j\phi^1_j(1))$. The extension of such a partial unitary to a unitary u_0 in D_{j+1} equal to one inside the complement of this projection then belongs to the commutant of the image of $\phi^0_{j+1}\gamma_j$, and transforms $\phi^0_{j+1}\beta_j$ into the cut-down of δ'_j by this projection, as desired.

As stated above, the proof for the case $t = 1$ is similar.

Inspection of the construction of the maps $\delta'_j - \phi^0_j\beta_j$ and $\delta_j - \phi^1_j\beta_j$ shows that they are injective, as required by the hypotheses of section 2.

Replacing ϕ^t_{j+1} with $(\operatorname{Ad} u_t)\phi^t_{j+1}$, we have an inductive sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

satisfying the hypotheses of section 2. (The existence of α_j homotopic to β_j and non-zero on a given element of D_j, defined by another point evaluation, is clear.)

By Theorem 2.3 there exists a sequence

$$A_1 \xrightarrow{\theta_1'} A_2 \xrightarrow{\theta_2'} \cdots,$$

with θ_j' homotopic to θ_j (and so agreeing with θ_j on K_0), the inductive limit of which is simple.

Since the map $K_0(\theta_j')$ (considered as a map between single copies of the integers) takes the canonical generator $1 \in \mathbb{Z}$ to L_{j+1}, we may conclude that the simple inductive limit in question has the desired K_0-group. That the positive elements are all those greater than k follows from the fact that at each stage, $l_j + 1$ is the smallest positive element in $K_0 A_j = \mathbb{Z}$ and

$$\lim \frac{l_j + 1}{\prod_{k=1}^{j} L_k} = \lim \frac{a \prod_{k=2}^{j} L_j + 1}{b \prod_{k=2}^{j} L_j} = k + \lim \frac{1}{\prod_{k=1}^{j} L_k} = k.$$

Theorem 3.1 follows.

Finally, one might reasonably ask whether $K_0(A_{(\mathbf{n},k)})^+$ can be made to contain k. There is no reason *a priori* why this should not be possible, but the construction above does not seem amenable to modifications which would achieve this result. Roughly speaking, the K_0-group in Theorem 3.1 can be thought of as an inductive limit of sub-ordered groups of ordered K_0-groups of homogeneous C^*-algebras. In order that the inductive limit of Theorem 3.1 be simple, one must introduce point evaluations via the maps β_j. In the absence of these point evaluations, one could have maps $\Psi : \mathbb{Z}_{mk} \to \mathbb{Z}_{mnk}$ with $\Psi(nk) = mnk$ at the level of K_0 between the building blocks A_i and A_{i+1}. With these point evaluations, however, one is forced into a situation where $\Psi(nk)$ is necessarily strictly less than mnk.

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