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# ON THE CLASSIFICATION OF COMPLEX MULTI-GERMS OF CORANK ONE AND CODIMENSION ONE

KEVIN HOUSTON


## Abstract

Corank one multi-germs $f : (\mathbf{C}^n, S) \to (\mathbf{C}^p, 0)$, $n < p$, of $\mathscr{A}_e$-codimension one are classified. The mono-germs are given in an explicit normal form and the multi-germs are described in terms of augmentations, and concatenations of mono-germs and a special bi-germ.


## 1. Introduction

The classification of singularities of mappings up to $\mathscr{A}$-equivalence (that is, equivalence up to diffeomorphic changes in source and target) is a central problem within Singularity Theory. In his classic paper [12] Mather classified the $\mathscr{A}$-stable map-germs, which are also known as $\mathscr{A}_e$-codimension 0 germs. The next target for classification, the $\mathscr{A}_e$-codimension 1 germs, is considerably more difficult and has taken a long time to develop. One reason for this is one does not have an analogue of Mather's result that $\mathscr{K}$-equivalent $\mathscr{A}$-stable maps are $\mathscr{A}$-equivalent – a result which reduces the $\mathscr{A}$-equivalence problem to a more tractable equivalence. In the codimension 1 case for example, the two real maps, $(x, y) \to (x, y^2, y^3 \pm x^2 y)$, have $\mathscr{A}_e$-codimension 1, are $\mathscr{K}$-equivalent but not $\mathscr{A}$-equivalent, see [13]. To further complicate matters this problem does not occur in the complex situation for this example; the two maps are $\mathscr{A}$-equivalent.

Examples of $\mathscr{A}_e$-codimension 1 maps are found in low dimensional classifications, for example [13], and some general classifications of simple singularities, such as [6]. But until recently there have been few general classifications of the class itself.

In his Ph.D. thesis [1] Cooper classified corank 1 (the rank of the differential is 1 less than maximal) $\mathscr{A}_e$-codimension 1 map-germs from $\mathbf{C}^n$ to $\mathbf{C}^{n+1}$ by using explicit coordinate changes in source and target to reduce the map to a normal form. A more elementary proof of the classification is given in [2]; many of his other results are in this more accessible publication. Surprisingly, just as in

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the case of stable germs the situation involves dealing with the $\mathscr{K}$-equivalence class of the germ. This is because if the map is not an augmentation, then its $\mathscr{A}$-orbit is open in its $\mathscr{K}$-orbit, and in the complex case there is at most one open orbit.

In [2] it is also shown that one can classify the multi-germs of $\mathscr{A}_e$-codimension 1 corank 1 maps from $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n \geq p - 1$ and $(n, p)$ in the nice dimensions. (It should be noted that here and throughout the paper, corank 1 for a germ with more than 1 branch means that the individual branches can have corank at most 1, i.e. corank 0 for any or all branches is acceptable.) The classification is given in terms of the augmentation and concatenation of mono-germs and a single bi-germ.

A less explicitly stated classification for $n \geq p$ was given by Damon in [3]. The approaches from there and [2] can be combined to give a more explicit classification of $\mathscr{A}_e$-codimension 1 multi-germs, $n \geq p$, with the branches having the $\mathscr{K}$-type of simple hypersurface singularities. Since by [6] all corank 1 singularities in the nice dimensions with $n \geq p$ have branches of this type we recover the $n \geq p$ classification in [2].

In this paper we classify corank 1 $\mathscr{A}_e$-codimension 1 map-germs from $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n < p$ but with no nice dimension restrictions. First, an explicit mono-germ description is given in Theorem 3.1. Then, in Theorem 6.4 the description of multi-germs is given in terms of the augmentation and concatenation of mono-germs and a special bi-germ. The processes of augmentation and binary concatenation are in essence the same as in [2] but the monic concatenation process is slightly different.

Furthermore, the multi-germ description of Theorem 6.4 is more precise than the analogue in [2]. That is, rather than state that the multi-germs can be constructed from mono-germs by augmentation and the two concatenation processes we show precisely how this can be done. To achieve this we first prove the useful result that for codimension 1 map-germs the three processes are pairwise commmutative and that binary concatenation is associative. Applying these gives a simple and elegant statement of the classification.

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## 2. Augmentation

First we give some notation used throughout the paper. The $\mathscr{A}_e$-codimension of a map-germ $f$ will be denoted $\mathrm{cod}(f)$. If two germs $f$ and $g$ are $\mathscr{A}$-equivalent, then we use the notation $f \sim_{\mathscr{A}} g$. The set $S$ will be a finite set of points and usually these will be the origins of a collection of copies of $\mathbb{C}^n$ for some $n$.

The results of this section were originally proved in [1], a more accessible reference for them is [2]. First we define the augmentation of a map-germ.

DEFINITION 2.1. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map with a 1-parameter stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0)$, where $F(x, \lambda) = (f_\lambda(x), \lambda)$. Then the *augmentation of $f$ by $F$* is the map $A_F(f)$ given by $(x, \lambda) \mapsto (f_{\lambda^2}(x), \lambda)$.

The real picture of the example of the standard cusp $x \mapsto (x^2, x^3)$ unfolded by $(x^2, x^3 - \lambda x, \lambda)$ and augmented to $(x^2, x^3 - \lambda^2 x, \lambda)$ is shown in Figure 1.

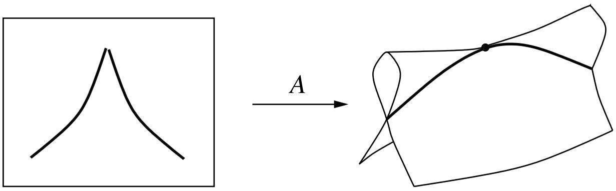


FIGURE 1. Augmentation of a cusp.

THEOREM 2.2 (See [2]). *Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a finitely $\mathscr{A}$-determined map-germ with a 1-parameter stable unfolding. Then the following are true.*

(i) *If $f$ has $\mathscr{A}_e$-codimension 1, then $A_F(f)$ has $\mathscr{A}_e$-codimension 1.*

(ii) *The $\mathscr{A}$-equivalence class of $A_F(f)$ is independent of the choice of miniversal unfolding of $f$.*

(iii) *If $f \sim_{\mathscr{A}} f'$ and both have $\mathscr{A}_e$-codimension 1, then $A(f) \sim_{\mathscr{A}} A(f')$.*

Thus we can produce new codimension 1 maps from old ones. If $f$ is not the augmentation of another germ, then $f$ is called *primitive*.

One can also generalise this definition so that the unfolding parameter is replaced by a general function and not just a quadratic one, see [7].

Since the augmentation of an $\mathscr{A}_e$-codimension one map is again codimension one we can augment repeatedly. Thus, define $A^m(f)$ to be the $m$-fold augmentation of $f$. For $m > 0$ this is the augmentation process repeated $m$ times and for the trivial case $A^0(f) = f$.

A stable map is called a *prism* if it is the trivial unfolding of another map. An important lemma is the following, taken from [2] (Theorem 2.7) but originally proved in Proposition 2.5 of [1]. Damon proved a similar theorem in [3] under the more restrictive assumptions of homogeneity and $n \geq p$. Essentially the lemma provides a process which allows us to reverse augmentation. In analogy with musical theory we shall call the process diminishing (the opposite of augmentation).

LEMMA 2.3 (Diminishing Lemma). *Let $g$ be an $\mathscr{A}_e$-codimension 1 multi-germ such that the miniversal unfolding of $g$ is a prism. Then $g$ is an augmentation of an $\mathscr{A}_e$-codimension 1 map-germ.*

Using this one can often prove results by reducing to the case of a primitive map.

## 3. Classification of mono-germs

The main theorem on the classification of mono-germs is the following.

THEOREM 3.1. *Suppose that $f : (\mathsf{C}^n, 0) \to (\mathsf{C}^p, 0)$, $n < p$, is a corank 1 $\mathscr{A}_e$-codimension 1 map-germ, then the following are true.*

(i) *$f$ is $\mathscr{A}$-equivalent to a map of the form,*

$$
\begin{aligned}
&(u_1, \ldots, u_{l-1}, v_1, \ldots, v_{l-1}, w_{11}, w_{12}, \ldots, w_{rl}, x_1, \ldots, x_{n-l(r+2)+1}, y)\\
&\mapsto \Bigg(u_1, \ldots, u_{l-1}, v_1, \ldots, v_{l-1}, w_{11}, w_{12}, \ldots, w_{rl}, x_1, \ldots, x_{n-l(r+2)+1},\\
&y^{l+1} + \sum_{i=1}^{l-1} u_i y^i,\ y^{l+2} + \sum_{i=1}^{l-1} v_i y^i + y^l \sum_{i=1}^{n-l(r+2)+1} x_i^2,\ \sum_{i=1}^{l} w_{1i} y^i, \ldots, \sum_{i=1}^{l} w_{ri} y^i\Bigg),
\end{aligned}
$$

*where $r = p - n - 1$ and $l + 1$ is the multiplicity of the germ. Conversely, any such germ has $\mathscr{A}_e$-codimension 1.*

(ii) *An $\mathscr{A}_e$-versal unfolding is given by unfolding with the addition of the term $\lambda y^l$ to the $(p - rl - 1)$th component function, i.e. the term beginning $y^{l+2}$.*

(iii) *The germ is precisely $l + 2$-determined.*

The proof of Theorem 3.1 is given in the rest of this section. From the theorem one immediately deduces the following.

COROLLARY 3.2. *Corank 1 $\mathscr{A}_e$-codimension 1 map-germs from $\mathsf{C}^n$ to $\mathsf{C}^p$, $n < p$, which are $\mathscr{K}$-equivalent are $\mathscr{A}$-equivalent.*

REMARK 3.3. The squared terms in $x$ and the unfolding parameter term $y^l$ show that a map of the above form is an augmentation of the form where $n = l(r + 2) - 1$.

Mather's classification of stable singularities held for real and complex germs and obviously we would like a real version of the above theorem. Calculations and classifications in low dimensions suggest the following statement.

CONJECTURE 3.4. *Suppose $f : (\mathsf{R}^n, 0) \to (\mathsf{R}^p, 0)$ is a corank 1 $\mathscr{A}_e$-codimension 1 real map-germ with $n < p$. Then $f$ is $\mathscr{A}$-equivalent to a* *germ of the form*

$$(u, v, w, x, y) \mapsto \Bigg(u, v, w, x, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i,$$
$$y^{l+2} + \sum_{i=1}^{l-1} v_i y^i + y^l \sum_{i=1}^{n-l(r+2)+1} \pm x_i^2, \sum_{i=1}^{l} w_{1i} y^i, \ldots, \sum_{i=1}^{l} w_{ri} y^i\Bigg),$$

*That is, it has the same form as the complex version, but the sum of squares arising from the repeated use of augmentation may have negative terms.*

### *3.1. Proof of Theorem 3.1 parts (i) and (ii)*

To prove part (i) of Theorem 3.1 we use results from the classification in the $p = n + 1$ case given in [2]. Let us follow the method there and begin by defining a map $f^l : (\mathbb{C}^{2l-1}, 0) \to (\mathbb{C}^{2l}, 0)$ by

$$f^l(u, v, y) = \left(u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i\right).$$

By Lemma 4.1 of [2] the $\mathscr{A}_e$-codimension is 1 and the $\mathscr{A}_e$-tangent space is given there as

$$T\mathscr{A}_e f^l = \theta(f^l)\backslash \left\{y^l \partial/\partial Y_2, y^{l-1}\partial/\partial v_1, \ldots, y\partial/\partial v_{l-1}\right\}$$
$$+ \left\langle y^{l-1}\partial/\partial v_1 + y^l \partial/\partial Y_2, \ldots, y\partial/\partial v_{l-1} + y^l \partial/\partial Y_2\right\rangle,$$

where we label the last two coordinates of $\mathbb{C}^{2l}$ with $Y_1$ and $Y_2$. Let us now define an extension of this map, $f^{l,r} : (\mathbb{C}^{2l-1+rl}, 0) \to (\mathbb{C}^{2l+r(l+1)}, 0)$:

$$f^{l,r}(u, v, w, y)$$
$$= \left(u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i, w, \sum_{i=1}^{l} w_{1i} y^i, \ldots, \sum_{i=1}^{l} w_{ri} y^i\right).$$

By the proof of Proposition 3.7 of [8] it is known that $f^{l,r}$ is finitely determined. We can do better than this as the following shows.

THEOREM 3.5. *The map $f^{l,r}$ has $\mathscr{A}_e$-tangent space equal to*

$$T\mathscr{A}_e f^{l,r} = \theta(f^{l,r})\backslash \left\{y^l \partial/\partial Y_2, y^{l-1}\partial/\partial v_1, \ldots, y\partial/\partial v_{l-1}\right\}$$
$$+ \left\langle y^{l-1}\partial/\partial v_1 + y^l \partial/\partial Y_2, \ldots, y\partial/\partial v_{l-1} + y^l \partial/\partial Y_2\right\rangle.$$

PROOF. One can prove this via a standard calculation involving Nakayama's lemma.

Hence, $f^{l,r}$ has $\mathscr{A}_e$-codimension equal to 1 with unfolding given by $\lambda y^l$ added to the term beginning $y^{l+2}$. So augmentation of $f^{l,r}$ will produce a map of the form in Theorem 3.1.

The maps $f^{l,r}$ have the following useful property.

LEMMA 3.6. *The $\mathscr{A}$-orbit of $f^{l,r}$ is open in its $\mathscr{K}$-orbit.*

PROOF. For simplicity let the dimension of the source be $n$ and that of the target be $p$. Denote the normal space of the $\mathscr{G}$-orbit by $N\mathscr{G}$ and the extended one by $N\mathscr{G}_e$. It is easy to calculate that $\dim N\mathscr{K}_e(f^{l,r}) = p+1$. Thus we find that $\dim N\mathscr{A}_e = \dim N\mathscr{K}_e - p$. But $\dim N\mathscr{A}_e = \dim N\mathscr{A} - n$ (as $f^{l,r}$ is not $\mathscr{A}$-stable, see [14] p. 510) and $\dim N\mathscr{K}_e = \dim N\mathscr{K} + (p-n)$ ([14] p. 509). So $\dim N\mathscr{A} = \dim N\mathscr{K}$, implying that the $\mathscr{A}$-orbit is open in the $\mathscr{K}$-orbit.

It should be noted that this is not true for an augmentation of $f^{l,r}$ as one can easily calculate using the same method as in the proof above.

PROOF OF THEOREM 3.1. We generalise the proof of Proposition 4.3 of [2] which is essentially the $p = n+1$ case. Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a corank 1 $\mathscr{A}_e$-codimension 1 map-germ, $n < p$, with multiplicity $l+1$. The versal unfolding $G : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ is an $n - l(p-n+1)+1$-fold prism on a minimal stable map-germ of multiplicity $l+1$. Thus by Theorem 2.7 of [2], quoted above as Lemma 2.3, $f$ is the $n - l(p-n+1)+1$-fold augmentation of an $\mathscr{A}_e$-codimension 1, corank 1, multiplicity $l+1$ map-germ $f' : (\mathbb{C}^{2l+l(p-n+1)-1}, 0) \to (\mathbb{C}^{2l+(p-n-1)(l+1)}, 0)$. Such a map is obviously $\mathscr{K}$-equivalent to the map $f^{l,p-n-1}$. The $\mathscr{A}$-orbit of $f^{l,p-n-1}$ is open in its $\mathscr{K}$-orbit by Lemma 3.6 and by Lemma 3.12 of [2] there is at most one open $\mathscr{A}$-orbit in a given complex contact class, thus we conclude that $f'$ and $f^{l,p-n-1}$ are $\mathscr{A}$-equivalent.

The $n - l(p-n+1)+1$-fold augmentation of $f^{l,p-n-1}$ is $\mathscr{A}$-equivalent to $f$ by Theorem 2.2: the $\mathscr{A}$-equivalence class of the augmentation of codimension 1 map-germ $g$ depends only on the $\mathscr{A}$-equivalence class of $g$.

### *3.2. Proof of Theorem 3.1 part (iii): Order of determinacy*

To find the order of determinacy we use the techniques of [5], in particular Proposition 3.8 there, which we summarise in the next proposition. We now need some more notation. Suppose we have a finitely $\mathscr{A}$-determined map $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Let $\mathscr{O}_X$ denote the ring of function germs at 0 for the germ $(X, 0)$. The tangent space $T\mathscr{A}_e h$ is a $h^*(\mathscr{O}_{\mathbb{C}^p})$ submodule of $(\mathscr{O}_{\mathbb{C}^n})^p$. Let $e_i$ denote the standard basis vector for the $i$th copy of $\mathscr{O}_{\mathbb{C}^n}$. Denote the maximal ideal of $\mathcal{O}_{\mathbb{C}^d}$ by $\mathbf{m}_d$ and use the standard $tf$ and $wf$ notation of Singularity Theory, see [14].

PROPOSITION 3.7. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ *be a map-germ and let*

$$D \subset tf(\theta_{\mathbb{C}^n}) + wf(\theta_{\mathbb{C}^p}) + \mathbf{m}_n^s \theta_f$$

*be an* $\mathcal{O}_{\mathbb{C}^n}$*-module such that*

$$\mathbf{m}_n^s \theta_f \subset tf(\mathbf{m}_n \theta_{\mathbb{C}^n}) + f^*(\mathbf{m}_p) \cdot D + \mathbf{m}_n^{s+1} \theta_f.$$

*Then,* $f$ *is* $s$*-determined.*

Let $f$ be as in Theorem 3.1. Then by calculation one can see that $T\mathscr{A}_e f$ has the same type of structure as $T\mathscr{A}_e(f^{l,r})$: Let $m = p - rl - 1$ then $y^l e_m$, and $y^{l-i} e_{l+i-1}$, $i = 1, \ldots, l-1$ are missing from $T\mathscr{A}_e f$, but $y^l e_m + y^{l-i} e_{l+i-1}$ is included. Thus if we let $\mathbf{m}_{n-1}$ denote the ideal generated by the variables other than $y$ and

$$D = \langle \mathcal{O}_n, \ldots, \mathcal{O}_n, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^l \rangle \mathcal{O}_n, \ldots, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^2 \rangle \mathcal{O}_n, \\ \mathcal{O}_n, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^{l+1} \rangle \mathcal{O}_n, \mathcal{O}_n, \ldots, \mathcal{O}_n \rangle$$

where the $\mathbf{m}_{n-1}\mathcal{O}_n + \langle y^j \rangle \mathcal{O}_n$ terms begin at position $l$, then $D$ is an $\mathcal{O}_n$-module contained in $T\mathscr{A}_e f$.

The non-trivial problem is to show that, for all $i$, $y^{l+2} e_i$ is in the right hand side of the second inclusion in the proposition. For the positions corresponding to the functions $u_1, \ldots, u_{l-1}$, $v_1, \ldots, v_{l-1}$ and $w_{11}, \ldots, w_{rl}$ we can use elements of $tf(\mathbf{m}_n \theta_{\mathbb{C}^n})$ modulo $\mathbf{m}_n^{l+3}$. For the $r$ extension terms and position $2l - 1$ we use $y^{l+2} + \sum_{i=1}^{l-1} v_i y^i$, elements of $tf$ and $f^*(\mathbf{m}_p) \cdot D$. For the remaining position we use $y\partial f/\partial y$ and terms in $tf$ and $f^*(\mathbf{m}_p) \cdot D$.

So $f$ is at least $(l+2)$-determined. This is in fact exact. The $(l+1)$-jet is not finitely $\mathscr{A}$-determined as can be seen by using the method of [11] to show that the $(l+1)$th multiple point space has dimension greater than that arising from a finitely determined map-germ.

## 4. Concatenation

In [2] there are two methods for producing multi-germs from germs with fewer branches: monic and binary concatenation. The monic concatenation process below is a generalisation of the version in [2] for the $p = n+1$ case. The binary concatenation process is almost the same as in [2]. To avoid overcomplicated statements and proofs we ask for the codimensions of the images of the two maps used in the concatenation to be the same. Without this the resulting germ has branches with different source dimensions; this is an interesting situation but one rarely treated in Singularity Theory.

In [3] Damon uses the notion of product union to classify multi-germs. Using the two concatenation constructions is equivalent to his method. However concatenations allow us to produce a clearer classification list and also allow us to describe multiple point spaces in a simpler manner than would otherwise be the case, see [9].

### 4.1. *Monic concatenation*

Monic concatenation produces a new germ by adding an immersive branch to the stable unfolding of a map.

DEFINITION 4.1. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, be a multi-germ of finite $\mathscr{A}_e$-codimension with a stable unfolding on the single parameter $\lambda$. Let $r = p - n - 1$. Then the *monic concatenation of* $f$ with respect to $F$ is the multi-germ $C_F(f) : (\mathbb{C}^{n+r+1}, S \cup \{0\}) \to (\mathbb{C}^{p+r+1}, 0)$ given by

$$\begin{aligned}(x, x', \lambda) &\mapsto (f_\lambda(x), x', \lambda)\\ y &\mapsto (y, 0, \dots, 0),\end{aligned}$$

where $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_r)$ and $y = (y_1, \dots, y_{n+r+1})$.

Note that $p = n + r + 1$ and so the zeroes in the lower branch above correspond to the coordinates $x'$ and $\lambda$ in the upper branch. The definition is different to that given in [2] and generalises their $C_0$ operation for the case $p = n + 1$.

The real picture of the example of the monic concatenation of the standard cusp $x \mapsto (x^2, x^3)$ unfolded by $(x^2, x^3 - \lambda x, \lambda)$ is shown in Figure 2.

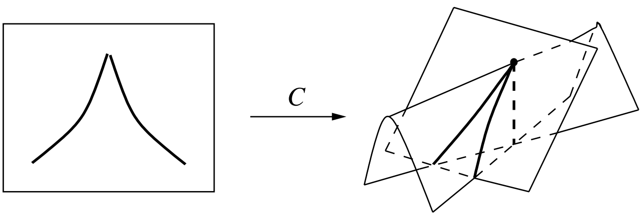


FIGURE 2. Monic concatenation of a cusp.

The next proposition is analogous to statements in Theorem 2.2 on augmentation.

PROPOSITION 4.2. *Suppose that* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, *is a finitely* $\mathscr{A}$*-determined multi-germ with a 1-parameter stable unfolding. Then the following are true.*

(i) *The $\mathscr{A}_e$-codimension of $C_F(f)$ is equal to the $\mathscr{A}_e$-codimension of $f$.*

(ii) *If the $\mathscr{A}_e$-codimension of $f$ is one, then under $\mathscr{A}$-equivalence $C_F(f)$ is independent of the choice of the unfolding $F$. Hence we use the notation $C(f)$.*

(iii) *If $f$ has $\mathscr{A}_e$-codimension one and $f \sim_{\mathscr{A}} f'$, then $C(f) \sim_{\mathscr{A}} C(f')$.*

PROOF. The proofs of the first two statements are just trivially modified versions of the proofs of Theorems 3.1 and 3.3 in [2]. The proof of the third statement follows from the fact that $f$ and $f'$ can be induced from the same unfolding.

For $\mathscr{A}_e$-codimension one germs we can apply this concatenation operation repeatedly without being concerned about the precise unfolding used and so we define $C^m(f)$ to be the $m$-fold concatenation of $f$, where $C^0(f) = f$.

*4.2. Binary Concatenation*

Suppose that $f : (\mathbf{C}^n, S) \to (\mathbf{C}^p, 0)$, $n < p$, and $g : (\mathbf{C}^{n'}, T) \to (\mathbf{C}^{p'}, 0)$, $n' < p'$, are finitely $\mathscr{A}$-determined map-germs with 1-parameter stable unfoldings and $p - n = p' - n'$. Let $F(x, \lambda) = (f_\lambda(x), \lambda)$ and $G(y, \mu) = (g_\mu(y), \mu)$ be the unfoldings of $f$ and $g$ respectively. Purely for reasons of exposition the following definition is slightly less general than in [2].

DEFINITION 4.3. The *binary concatenation of $f$ and $g$* with respect to $F$ and $G$, denoted $B_{F,G}(f, g)$, is the multi-germ $B_{F,G}(f, g) : (\mathbf{C}^{n+p'+1}, S \cup T) \to (\mathbf{C}^{p+p'+1}, 0)$ given by

$$(X, x, u) \mapsto (X, f_u(x), u)$$
$$(y, Y, u) \mapsto (g_u(y), Y, u).$$

It is difficult to draw real (non-schematic) pictures of binary concatenation. The only relevant example possible is the concatenation of two copies of the bi-germ which maps two distinct isolated points to the origin in $\mathbf{C}$. In Figure 3 it is shown that this bi-germ can be unfolded to give a map with the axes in $\mathbf{C}^2$ as its image. The two copies then combine to give a quadruple point.
We now state some important results from [2] about the binary concatenation process.

PROPOSITION 4.4. *The following are true.*

(i) *([2] Proposition 3.8) We have* $\mathrm{cod}(B_{F,G}(f, g)) \geq \mathrm{cod}(f) \times \mathrm{cod}(g)$, *with equality if $\lambda \in d\lambda(Derlog(D(F)))$ or $\mu \in d\mu(Derlog(D(G)))$, and where $Derlog(D(F))$ is the module of liftable vector fields over $F$ (see [4] for the properties of this module).*

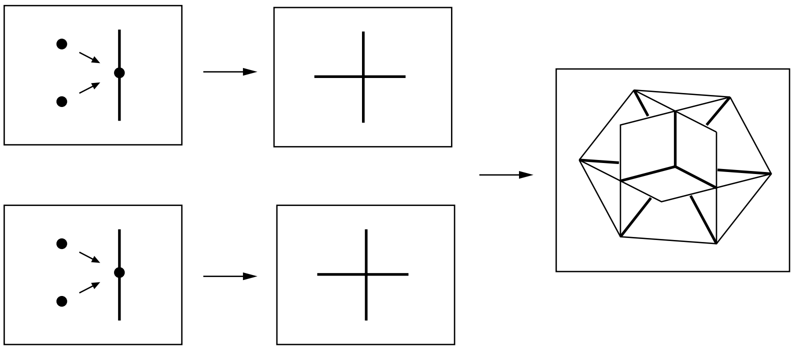

FIGURE 3. Binary concatenation of two simple bi-germs.

(ii) *([2] Proposition 3.11) Suppose that $B_{F,G}(f, g)$, $f$ and $g$ all have $\mathscr{A}_e$-codimension 1. Then, up to $\mathscr{A}$-equivalence, $B_{F,G}(f, g)$ is independent of the choice of unfoldings $F$ and $G$. Hence we use the notation $B(f, g)$.*

(iii) *Suppose that $B(f, g)$, $f$ and $g$ all have $\mathscr{A}_e$-codimension 1. If $f \sim_{\mathscr{A}} f'$ and $g \sim_{\mathscr{A}} g'$, then $B(f', g') \sim_{\mathscr{A}} B(f, g)$.*

REMARK 4.5. Though it is not stated there Proposition 3.11 of [2] needs the extra condition that the concatenation has $\mathscr{A}_e$-codimension 1.

REMARK 4.6. Asking for $B(f, g)$, to have codimension 1 means asking for $B_{F,G}(f, g)$ to have $\mathscr{A}_e$-codimension 1 for some (and hence any) unfoldings $F$ and $G$.

In view of (i) above and unlike the other two methods of creating new germs, we do not yet have a guarantee that the $\mathscr{A}_e$-codimension of $B(f, g)$ is equal to 1 when $f$ and $g$ are $\mathscr{A}_e$-codimension 1. For corank 1 mono-germs we have the following.

THEOREM 4.7. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, has both corank and $\mathscr{A}_e$-codimension equal to 1, that $g : (\mathbb{C}^{n'}, S) \to (\mathbb{C}^{p'}, 0)$ is a multi-germ with a 1-parameter stable unfolding $G$ and $p' - n' = p - n$. Then* $\operatorname{cod}(B_{F,G}(f, g)) = \operatorname{cod}(g)$.

PROOF. From Theorem 3.1 we know that the map $f$ and its unfolding $F$ are quasihomogeneous. The image of $F$ is quasihomogeneous and hence the usual Euler vector field is in $Derlog(D(F))$. Since the unfolding parameter, call it $\lambda$, has non-zero weight we deduce that $\lambda \in d\lambda(Derlog(D(F)))$ and so by Proposition 4.4 we have the required equality.

## 5. The three operations commute and the binary one is associative

We have 3 operations, $A$, $B$ and $C$. It is natural to look for relations between these and in this section we show that for codimension 1 maps they are pairwise commutative and binary concatenation is associative. We also show that a binary concatenation of a particularly special bi-germ and a germ $f$ is equivalent to the 2-fold monic concatenation of $f$.

Throughout this section $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ and $g : (\mathbb{C}^{n'}, T) \to (\mathbb{C}^{p'}, 0)$ are $\mathscr{A}_e$-codimension 1 multi-germs with $n < p$ and $p' - n' = p - n$.

Figure 4 shows an example of the commutation of the processes of augmentation and monic concatenation.

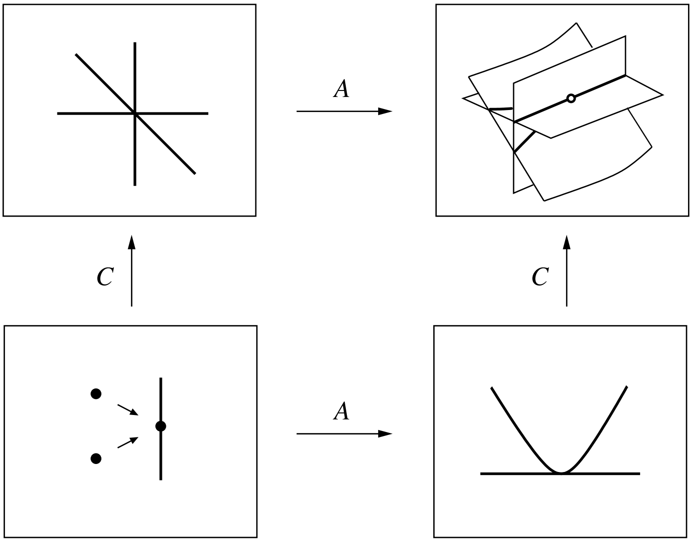


FIGURE 4. Commutation of $A$ and $C$.

THEOREM 5.1. *The A and C operations commute:* $A(C(f)) \sim_{\mathscr{A}} C(A(f))$.

PROOF. The assumption that $f$ is codimension 1 implies, by Proposition 4.2, that the $\mathscr{A}$-class of a monic concatenation is independent of the unfolding used.

Suppose that $F(x, \lambda) = (f_\lambda(x), \lambda)$ is a versal unfolding of $f$ with $f_0(x) = f(x)$. Then $C(f)$ is given by

$$
\begin{aligned}
(x, x', \lambda) &\mapsto (f_\lambda(x), x', \lambda),\\
y &\mapsto (y, 0, \dots, 0),
\end{aligned}
$$

where $x' = (x'_1, \dots, x'_r)$, ($r = p - n - 1$ as usual) and $y = (y_1, \dots, y_p)$. We can unfold with the map

$$\begin{aligned}(x, x', \lambda, \mu) &\mapsto (f_\lambda(x), x', \lambda, \mu),\\ (y, \mu) &\mapsto (y, 0, \dots, 0, \mu, \mu).\end{aligned}$$

By Mather's criterion for multi-germs ([12]) this is stable and so $A(C(f))$ can be given by

$$\begin{aligned}(x, x', \lambda, \mu) &\mapsto (f_\lambda(x), x', \lambda, \mu),\\ (y, \mu) &\mapsto (y, 0, \dots, 0, \mu^2, \mu).\end{aligned}$$

Now we turn to $CA(f)$. The map

$$(x, \lambda, \mu) \mapsto (f_{\lambda^2+\mu}(x), \lambda, \mu)$$

is a stable unfolding of $A(f)$ and so we can construct $CA(f)$:

$$\begin{aligned}(x, \lambda, x', \mu) &\mapsto (f_{\lambda^2+\mu}(x), \lambda, x', \mu),\\ (y, \widetilde{y}) &\mapsto (y, \widetilde{y}, 0, \dots, 0).\end{aligned}$$

The right change of coordinates given by $\mu' = \lambda^2 + \mu$ produces

$$\begin{aligned}(x, \lambda, x', \mu') &\mapsto (f_{\mu'}(x), \lambda, x', \mu' - \lambda^2),\\ (y, \widetilde{y}) &\mapsto (y, \widetilde{y}, 0, \dots, 0).\end{aligned}$$

An equally simple left change of coordinates produces

$$\begin{aligned}(x, \lambda, x', \mu') &\mapsto (f_{\mu'}(x), \lambda, x', \mu'),\\ (y, \widetilde{y}) &\mapsto (y, \widetilde{y}, 0, \dots, 0, \widetilde{y}^2).\end{aligned}$$

By exchanging coordinates in the target and relabelling we obtain $A(C(f))$ as described earlier.

Augmentation commutes with binary concatenation.

THEOREM 5.2. *Suppose that $B(f, g)$ or $B(A(f), g)$ is $\mathscr{A}_e$-codimension 1. Then the A and B operations commute: $A(B(f, g)) \sim_{\mathscr{A}} B(A(f), g)$.*

PROOF. Take as an unfolding of $B(f, g)$ the map

$$\begin{aligned}(X, y, u, v) &\mapsto (X, f_u(y), u - v, v)\\ (x, Y, u, v) &\mapsto (g_u(x), Y, u, v).\end{aligned}$$

Then $A(B(f, g))$ is given by

$$
\begin{aligned}
(X, y, u, v) &\mapsto (X, f_u(y), u - v^2, v)\\
(x, Y, u, v) &\mapsto (g_u(x), Y, u, v).
\end{aligned}
$$

Turning our attention to $B(A(f), g)$ the components of $A(f)$ are $(f_{\lambda^2}(y), \lambda)$ and $(f_{\lambda^2+\mu}(y), \lambda, \mu)$ is a versal unfolding. So $B(A(f), g)$ can be given by

$$
\begin{aligned}
(X, y, \lambda, \mu) &\mapsto (X, f_{\lambda^2+\mu}(y), \lambda, \mu)\\
(x, Z, u) &\mapsto (g_u(x), Z, u).
\end{aligned}
$$

We make the change of coordinates $\mu' = \lambda^2 + \mu$ to get

$$
\begin{aligned}
(X, y, \lambda, \mu') &\mapsto (X, f_{\mu'}(y), \lambda, \mu' - \lambda^2)\\
(x, Z, u) &\mapsto (g_u(x), Z, u).
\end{aligned}
$$

This is the same as the description of $A(B(f, g))$ given above (the coordinates $Z$ correspond to the coordinates $Y$ and $v$), and shows that $A(B(f, g))$ is $\mathscr{A}$-equivalent to $B(A(f), g)$.

Thus for these particular unfoldings $A(B(f, g)) \sim_{\mathscr{A}} B(A(f), g)$. However, by assumption one of these is codimension 1 (and hence so is the other). Therefore we conclude from Proposition 4.4 that the unfoldings used to produce the binary concatentions are not significant.

The binary and monic concatenation processes also commute.

THEOREM 5.3. *Suppose that $B(f, g)$ or $B(C(f), g)$ has $\mathscr{A}_e$-codimension 1. Then the $B$ and $C$ operations commute: $C(B(f, g)) \sim_{\mathscr{A}} B(C(f), g)$.*

PROOF. As in the previous theorem we show the equivalence for certain unfoldings and then by the assumption that the maps are codimension 1 we can assume that the operations are independent of the unfoldings chosen.

Suppose that we have versal unfoldings of $f$ and $g$ given by $(f_\lambda(x), \lambda)$ and $(g_\mu(y), \mu)$ respectively. Then $C(f)$ is given by

$$
\begin{aligned}
(x, x', \lambda) &\mapsto (f_\lambda(x), x', \lambda)\\
z &\mapsto (z, 0, \dots, 0),
\end{aligned}
$$

which can be versally unfolded with

$$
\begin{aligned}
(x, x', \lambda, \nu) &\mapsto (f_\lambda(x), x', \lambda, \nu)\\
(z, \nu) &\mapsto (z, 0, \dots, 0, \nu, \nu),
\end{aligned}
$$

to yield $B(C(f), g)$ as

$$\begin{aligned}(X, x, x', \lambda, \nu) &\mapsto (X, f_\lambda(x), x', \lambda, \nu)\\ (X, z, \nu) &\mapsto (X, z, 0, \dots, 0, \nu, \nu)\\ (y, Y, \nu) &\mapsto (g_\nu(y), Y_1, Y_2, Y_3, \nu),\end{aligned}$$

where $Y_1$ is a set of $p$ coordinates, $Y_2$ a set of $r$ coordinates and $Y_3$ is a single coordinate.

By the left change of coordinates given by subtracting the last coordinate from the second to last we produce

$$\begin{aligned}(X, x, x', \lambda, \nu) &\mapsto (X, f_\lambda(x), x', \lambda - \nu, \nu)\\ (X, z, \nu) &\mapsto (X, z, 0, \dots, 0, 0, \nu)\\ (y, Y, \nu) &\mapsto (g_\nu(y), Y_1, Y_2, Y_3 - \nu, \nu).\end{aligned}$$

Using the right change of coordinates $\lambda = \Lambda + \nu$ and $Y_3 = \widetilde{Y}_3 + \nu$ we find that $B(C(f), g)$ is equivalent to

$$\begin{aligned}(X, x, x', \Lambda, \nu) &\mapsto (X, f_{\Lambda+\nu}(x), x', \Lambda, \nu)\\ (X, z, \nu) &\mapsto (X, z, 0, \dots, 0, 0, \nu)\\ (y, Y, \nu) &\mapsto (g_\nu(y), Y_1, Y_2, \widetilde{Y}_3, \nu).\end{aligned}$$

This is equivalent to $C(B(f, g))$ since $B(f, g)$ can be induced from the stable unfolding given by

$$\begin{aligned}(X, x, \Lambda, \nu) &\mapsto (X, f_{\Lambda+\nu}(x), \nu, \Lambda)\\ (y, Y_1, \widetilde{Y}_3, \nu) &\mapsto (g_\nu(y), Y_1, \nu, \Lambda),\end{aligned}$$

where $\Lambda$ is the unfolding parameter. Then $C(B(f, g))$ is $\mathscr{A}$-equivalent to the earlier description by a trivial permutation of coordinates in the target.

We now show that the $B$ operation is associative. Assume that $h$ is an $\mathscr{A}_e$-codimension 1 multi-germ with the codimension of its image the same as for $f$ and $g$.

THEOREM 5.4. *Suppose that $B(f, g)$ and $B(B(f, g), h)$ have $\mathscr{A}_e$-codimension 1. Then $B$ is associative: $B(f, B(g, h)) \sim_{\mathscr{A}} B(B(f, g), h)$.*

PROOF. Again as the maps are codimension 1 we can assume that the operations are independent of the unfoldings chosen. Suppose that we have versal unfoldings of $f$, $g$ and $h$ given by $(f_\lambda(x), \lambda)$, $(g_\mu(y), \mu)$ and $(h_\eta(z), \eta)$ respectively.

Then, $B(B(f, g), h)$ can be given by

$$
\begin{aligned}
(X_1, X_2, x, u, w) &\mapsto (X_2, X_1, f_u(x), u, w)\\
(Y_1, Y_2, y, u, w) &\mapsto (Y_2, g_u(y), Y_1, u + w, w)\\
(z, Z_1, Z_2, Z_3, w) &\mapsto (h_w(z), Z_1, Z_2, Z_3, w).
\end{aligned}
$$

By letting $u = v - w$ in the source coordinates for the second row and by exchanging the last two coordinates in the target we produce

$$
\begin{aligned}
(X_1, X_2, x, u, w) &\mapsto (X_2, X_1, f_u(x), w, u)\\
(Y_1, Y_2, y, w, v) &\mapsto (Y_2, g_{v-w}(y), Y_1, v, w)\\
(z, Z_1, Z_2, Z_3, w) &\mapsto (h_w(z), Z_1, Z_2, w, Z_3).
\end{aligned}
$$

Now we describe $B(f, B(g, h))$. The unfolding $(g_{-w}(y), w)$ is also a stable unfolding of $g$ and so we can construct $B(g, h)$ by

$$
\begin{aligned}
(Y_2, y, w) &\mapsto (Y_2, g_{-w}(y), w)\\
(z, Z_1, w) &\mapsto (h_w(z), Z_1, w).
\end{aligned}
$$

A stable unfolding of this is

$$
\begin{aligned}
(Y_2, y, w, v) &\mapsto (Y_2, g_{v-w}(y), w, v)\\
(z, Z_1, w, v) &\mapsto (h_w(z), Z_1, w, v).
\end{aligned}
$$

From this we can produce $B(f, B(g, h))$. By relabelling this is the form for the concatenation $B(B(f, g), h)$ given above.

We finish this section with a relation between $B$ and $C$ when one of the germs is a special bi-germ that will be of interest later.

THEOREM 5.5. *Let $g : (\mathbb{C}^n, \{0, 0\}) \to (\mathbb{C}^{2n+1}, 0)$, $n \geq 0$, be the $\mathscr{A}_e$-codimension 1 bi-germ*

$$
\begin{aligned}
(w_1, \dots, w_n) &\mapsto (w_1, \dots, w_n, 0, \dots, 0, 0)\\
(z_1, \dots, z_n) &\mapsto (0, \dots, 0, z_1, \dots, z_n, 0),
\end{aligned}
$$

*(this gives two n-planes intersecting in a single point), and let $f : (\mathbb{C}^{n'}, S) \to (\mathbb{C}^{n'+n+1}, 0)$, $n' \geq 0$, be an $\mathscr{A}_e$-codimension 1 multi-germ.*
*Then $B(f, g) \sim_{\mathscr{A}} C^2(f)$, and has $\mathscr{A}_e$-codimension 1.*

PROOF. By Proposition 4.4(i) and an elementary calculation for $g$ we can show that $B(f, g)$ is $\mathscr{A}_e$-codimension 1 and so by (ii) of the same proposition we can assume that this concatenation is independent of the unfoldings chosen.

We can versally unfold the bi-germ with

$$\begin{aligned}(w_1, \dots, w_n, \lambda) &\mapsto (w_1, \dots, w_n, 0, \dots, 0, \lambda, \lambda)\\(z_1, \dots, z_n, \lambda) &\mapsto (0, \dots, 0, z_1, \dots, z_n, 0, \lambda).\end{aligned}$$

Then $B(f, g)$ can be given by

$$\begin{aligned}(X, x, \lambda) &\mapsto (X_1, X_2, X_3, f_\lambda(x), \lambda)\\(w_1, \dots, w_n, Y, \lambda) &\mapsto (w_1, \dots, w_n, 0, \dots, 0, \lambda, Y, \lambda)\\(z_1, \dots, z_n, Y, \lambda) &\mapsto (0, \dots, 0, z_1, \dots, z_n, 0, Y, \lambda),\end{aligned}$$

where $X_1$, $X_2$ and $Y$ are each a collection of $n$ coordinates and $X_3$ is a single coordinate.

Now consider $C(f)$ which is given by

$$\begin{aligned}(x, X_2, u) &\mapsto (f_u(x), X_2, u)\\Y &\mapsto (Y, 0, \dots, 0, 0),\end{aligned}$$

and can be unfolded with

$$\begin{aligned}(x, X_2, u, \lambda) &\mapsto (f_u(x), X_2, u, \lambda)\\(Y, \lambda) &\mapsto (Y, 0, \dots, 0, \lambda, \lambda).\end{aligned}$$

This is stable by Mather's criteria for stability. So $C(C(f))$ is given by

$$\begin{aligned}(X_1, x, X_2, u, \lambda) &\mapsto (f_u(x), X_2, u, X_1, \lambda)\\(X_1, Y, \lambda) &\mapsto (Y, 0, \dots, 0, \lambda, X_1, \lambda)\\(Z_1, Z_2, Z_3) &\mapsto (Z_1, Z_2, Z_3, 0, \dots, 0, 0),\end{aligned}$$

which by exchanging left coordinates and relabelling gives $B(f, g)$ above.

## 6. Classification of multi-germs

In this section we classify corank 1 $\mathscr{A}_e$-codimension 1 multi-germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, by constructing them through augmentation and concatenation of a bi-germ and primitive $\mathscr{A}_e$-codimension 1 mono-germs. The initial part of the proof is similar to that given in [2] for corank 1 codimension 1 map-germs with $n \geq p - 1$ and so we only highlight the differences.

First we need to make some definitions and state some results from [2]. Suppose that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a multi-germ with branches $f^1, \dots, f^s$. Then the *analytic stratum* of $f$ is the submanifold in the target along which $f$ is trivial. If $f$ is stable, then define $\tau(f)$ to be the tangent space at 0 of the analytic stratum of $f$.

THEOREM 6.1.

(i) *(Corollary 5.7 of [2]) If $f$ has $\mathscr{A}_e$-codimension 1, then for any proper subset $S'$ of $S$ the restriction of $f$ to $(\mathbb{C}^n, S') \to (\mathbb{C}^p, 0)$ is stable.*

(ii) *(Corollary 5.13 of [2]) Suppose that $h = \{f, g\}$ is a primitive $\mathscr{A}_e$-codimension 1 multi-germ. Then there is a decomposition $T_0(\mathbb{C}^p) = \tau(f) \oplus \tau(g) \oplus \mathbb{C}$.*

The following proposition is a generalisation of the $p = n + 1$ case in Proposition 5.16 of [2].

PROPOSITION 6.2. *Suppose that $h = \{f, g\}$ is a primitive $\mathscr{A}_e$-codimension 1 multi-germ such that $g$ is not transverse to $\tau(f)$. Then the following two cases can occur.*

(i) *If $f$ and $g$ are transverse, then $g$ has exactly one immersive branch and $h$ is $\mathscr{A}$-equivalent to $C(f_0)$, where $f$ is a versal unfolding of the $\mathscr{A}_e$-codimension 1 germ $f_0$.*

(ii) *If $f$ and $g$ are not transverse, then $h$ is $\mathscr{A}$-equivalent to the bi-germ given by two $n$-planes intersecting in a point in $(2n + 1)$-space.*

PROOF. (i) In the same way as in the proof of Proposition 5.16 in [2] we prove that $g$ has one branch and that it is a prism on a germ of rank 0. Hence, as $n < p$ it is an immersion.

Using right and left changes of coordinates we put $g$ into the form $y \mapsto (y, 0, \dots, 0)$. Then using $df^i(T_{x_i}\mathbb{C}^n) + dg(T_{x_i}\mathbb{C}^n) = T_0(\mathbb{C}^p)$, where $x_i \in S$, we can put $f$ into the form $(x, Y, u) \mapsto (f(x, Y, u), Y, u)$. As, by Theorem 6.1(ii), $T_0(\mathbb{C}^p) = \tau(g) \oplus \tau(f) \oplus \mathbb{C}$, we can assume that $f$ is trivial in the $Y$ coordinates and so by a change of coordinates $h$ is $\mathscr{A}$-equivalent to

$$\begin{aligned}(x, Y, u) &\mapsto (f'(x, u), Y, u)\\ y &\mapsto (\varphi(y), 0, \dots, 0),\end{aligned}$$

for some $f'$ and $\varphi$. Through a change of coordinates we can produce the form

$$\begin{aligned}(x, Y, u) &\mapsto (f''(x, u), Y, u)\\ y &\mapsto (y, 0, \dots, 0).\end{aligned}$$

That is, $\{f, g\}$ is $\mathscr{A}$-equivalent to $C(f_0)$ where $f_0(x) = f''(x, 0)$. By Theorem 4.2(i) the $\mathscr{A}_e$-codimension of $f_0$ is 1. Since $(x, Y, u) \mapsto (f''(x, u), Y, u)$ is stable by Theorem 6.1(i) it must be an $\mathscr{A}$-versal unfolding of $f_0$.

(ii) If $f$ and $g$ are not transverse, then the argument above reverses and $f$ has exactly one immersive branch as well. Now as $T_0(\mathbb{C}^p) = \tau(g) \oplus \tau(f) \oplus \mathbb{C}$

we deduce that $h$ is $\mathscr{A}$-equivalent to the bi-germ

$$(w_1, \ldots, w_n) \mapsto (w_1, \ldots, w_n, 0, \ldots, 0, 0)$$
$$(z_1, \ldots, z_n) \mapsto (0, \ldots, 0, z_1, \ldots, z_n, 0).$$

In particular, $p = 2n + 1$.

If $f$ is transverse to $\tau(g)$ and $g$ is transverse to $\tau(f)$, then, subject to extra conditions on our map, we can construct $h$ by binary concatenation. Using the proof of Theorem 5.21 of [2] and the observation that its statement can be modified so that we require the existence of an unfolding parameter of positive weight for $g_0$ we have the following.

THEOREM 6.3. *If $h = \{f, g\}$ is a multi-germ of $\mathscr{A}_e$-codimension 1, in which $f$ is transverse to $\tau(g)$ and $g$ is transverse to $\tau(f)$, and if either the pullback of $f$ by $\tau(g)$ or the pullback of $g$ by $\tau(f)$ is quasihomogeneous with a unfolding parameter of positive weight, then $\{f, g\}$ is equivalent to a binary concatenation $B(f_0, g_0)$; that is, to a germ of the form*

$$(X, y, u) \mapsto (X, f_u(y), u)$$
$$(x, Y, v) \mapsto (g_v(x), Y, v).$$

We are now in a position to classify corank 1 $\mathscr{A}_e$-codimension 1 maps germs in terms of augmentations and concatenations of primitive $\mathscr{A}_e$-codimension 1 mono-germs and a special $\mathscr{A}_e$-codimension 1 bi-germ.

The statement, included in the following, that $h$ is quasihomogeneous and can be constructed from the 3 operations, was proved in [2] in the case $p = n+1$ with $(n, p)$ in the nice dimensions. Here we are more precise about how the construction is made, we assume more generally that $n < p$ and also drop the nice dimensions condition.

THEOREM 6.4. *Suppose that $h : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, $n < p$, is a corank 1 multi-germ of $\mathscr{A}_e$-codimension 1. Then, the following are true.*

(i) *The map-germ $h$ is $\mathscr{A}$-equivalent to one of the following distinct mappings:*

(a) *$A^m C^s g$, (all branches non-singular),*
(b) *$A^m C^s f_1$, (exactly one singular branch),*
(c) *$A^m C^s B(f_1, B(f_2, B(f_3, \ldots, B(f_{t-1}, f_t))))$, (multiple singular branches),*

*where $f_i : (\mathbb{C}^{n'}, 0) \to (\mathbb{C}^{n'+p-n}, 0)$ is a primitive corank 1 $\mathscr{A}_e$-codimension 1 mono-germ; $g$ is the bi-germ given by two $(p - n - 1)$-planes intersecting at a single point in $2(p - n) - 1$ space; $m, s, n' \in \mathbb{N} \cup \{0\}$; and $t \geq 2$.*

(ii) *The map h is quasihomogeneous.*

(iii) *Conversely, any map of the form above has $\mathscr{A}_e$-codimension 1.*

PROOF. That the mappings are distinct is obvious. As in [2] we use induction on the number of branches but in contrast we work by removing one branch at a time until we reduce to a mono-germ or the special bi-germ. The statements are true for $|S| = 1$ by Theorem 3.1. Thus, suppose the statements (i) and (ii) are true for $|S| \leq k$ and that $h$ is a multi-germ with $k + 1$ branches. Then $h = \{f, g\}$ where $g$ is a mono-germ and $f$ has $k$ branches. By Lemma 2.3 we can assume that $h$ is primitive. We have two cases.

(a) $g$ not transverse to $\tau(f)$: If $f$ and $g$ are transverse, then by Proposition 6.2 $g$ is an immersive branch and $h$ is equivalent to $C(f')$ where $f'$ is a codimension 1 map germ arising from $f$.

If $f$ and $g$ are not tranverse, then $h$ is the augmentation of the bi-germ above, again by Proposition 6.2.

(b) $g$ is tranverse to $\tau(f)$ and $f$ is transverse to $\tau(g)$: The germ $g$ is quasihomogeneous with positive unfolding weight by Theorem 3.1 and so by Theorem 6.3 $h$ is equivalent to the binary concatenation of two codimension 1 germs. By the inductive hypothesis these two germs are quasihomogeneous. The concatenation of two such germs must also have this property.

Using the above we see that any corank 1 codimension 1 multi-germ can be constructed from primitive mono-germs or the special bi-germ by the $A$, $B$ and $C$ processes. Using Theorems 5.1, 5.2 and 5.3 on the commutativity of the operations we can move any $A$ and $C$ operations to the front of our form for $h$. We can then use the associativity of $B$ as described in Theorem 5.4 to produce our desired form. If the special bi-germ occurs in a binary concatenation, then we use Theorem 5.5 to replace it with two monic concatenations.

Part (ii) follows from (i) as clearly quasihomogeneity is preserved under the augmentation and concatenation operations and the mono-germs and bi-germ are both obviously quasihomogeneous.

Part (iii) follows from Theorem 2.2(i), Proposition 4.2(i), and Theorem 4.7.

REMARKS 6.5. (i) This formulation reveals more about the structure of corank 1 $\mathscr{A}_e$-codimension 1 singularities with $p = n+1$ than in [2]. For example, it is not obvious from there that the bi-germ need only be used in creating maps without singular branches, nor that the order of monic and binary concatenations is unimportant.

(ii) Using this theorem it can be seen that one can encode the information needed to construct a codimension 1 multi-germ in a finite set of non-negative integers: the number of augmentations, the number of monic concatenations, the number of binary concatenations, the number of bi-germs and the multiplicities of the mono-germs.

(iii) It follows from the theorem that for complex corank 1 $\mathscr{A}_e$-codimension 1 multi-germs with $n < p$ we have a situation similar to the stable case: $\mathscr{A}_e$-codimension one maps that are $\mathscr{K}$-equivalent are $\mathscr{A}$-equivalent. As noted in the introduction this does not hold in the real case.

(iv) It would be nice to conjecture that the real corank 1 $\mathscr{A}_e$-codimension 1 map-germs have a form similar to that in the theorem. However, for real maps the $\mathscr{A}$-equivalence class of $B_{F,G}(f, g)$ depends on the unfoldings used, see Example 3.10 of [2]. The precise dependence is not entirely clear at the time of writing, and I hope to return to this problem in a subsequent paper.

From the classification it is possible to calculate the number of $\mathscr{A}_e$-codimension 1 multi-germs for each pair of dimensions $(n, p)$. It would be interesting to give a formula for this number.

In one of the founding papers of modern Singularity Theory, [15], Whitney classified the stable singularities from $n$-space to $2n$-space. We can do the same for $\mathscr{A}_e$-codimension 1 maps in the complex case.

COROLLARY 6.6. *Suppose that* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{2n}, 0)$, $n > 0$, *is an* $\mathscr{A}_e$-*codimension 1 map-germ. Then* $f$ *is* $\mathscr{A}$-*equivalent to one of the following distinct maps:*

(i) $C(g)$, *if* $n = 1$,

(ii) $A(g)$, $n \geq 1$,

(iii) $(x_1, \dots, x_{n-1}, y) \mapsto (x_1, \dots, x_{n-1}, y^2, y^3, x_1 y, \dots x_{n-1} y)$, $n \geq 1$,

*where* $g$ *is the special bi-germ produced by two* $n-1$-*planes intersecting in* $(2n-1)$-*space.*

PROOF. A miniversal unfolding of $f$ is a stable map from $m$-space to $(2m-1)$-space and so is corank 1. By Whitney's well known classification (also in [15]) we have two $m$-planes intersecting transversely, a generalised Whitney umbrella, and if $m = 2$ we have three planes intersecting transversely in 3-space. As corank and multiplicity are preserved by unfolding we can deduce the required codimension 1 germs from the theorem.

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SCHOOL OF MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS, LS2 9JT
U.K.
*E-mail:* k.houston@leeds.ac.uk