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C^*-ALGEBRAS ASSOCIATED WITH THE FUNDAMENTAL GROUPS OF GRAPHS OF GROUPS

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Abstract

We construct a nuclear C^*-algebra associated with the fundamental group of a graph of groups of finite type. It is well-known that every word-hyperbolic group with zero-dimensional boundary, in other words, every group acting trees with finite stabilizers is given by the fundamental group of such a graph of groups. We show that our C^*-algebra is $*$-isomorphic to the crossed product arising from the associated boundary action and is also given by a Cuntz-Pimsner algebra. We also compute the K-groups and determine the ideal structures of our C^*-algebras.

1. Introduction

There are a lot of examples of dynamical systems giving simple C^*-algebras. In these cases, we next expect the C^*-algebras to be purely infinite, because of the celebrated classification theorem of purely infinite simple separable nuclear C^*-algebras due to E. Kirchberg and N. C. Phillips [20]. For example, by the works of M. Laca and J. Spielberg [18], or C. Anantharaman-Delaroche [2], the word-hyperbolic groups, which are introduced by M. Gromov [13], acting on their boundary give such C^*-algebras, (see also [1] or [3, Appendix B] for their nuclearity). However their K-theory is unknown very well. Thus one of our purposes is to compute their K-groups and to study them through out the above mentioned Kirchberg-Phillips classification theorem.

In former work [19], we construct C^*-algebras associated with certain amalgamated free product groups by mimicking the Fock space construction (see [11], [12] for Cuntz-Krieger algebras in [9]) and give explicit formulae of their K-groups by using the method for Cuntz-Krieger algebras in [8]. The constructed C^*-algebras are $*$-isomorphic to the crossed products arising from boundary actions as word-hyperbolic groups. This is a generalization of works of M. D. Choi [7] for $\mathsf{Z}_2 * \mathsf{Z}_3$ and J. Spielberg [27] for the free products of cyclic groups.

The aim of this paper is to generalize the above mentioned results of [19] to the class of word-hyperbolic groups with zero-dimensional boundary and to

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investigate the crossed products arising from boundary actions of such word-hyperbolic groups by using the $*$-isomorphism to our constructed C^*-algebras. In other words, it means that we study the crossed products of the boundary actions arising from groups acting on trees (without inversions) with finite stabilizers. The key of our construction is the well-known result that such a group is given by the fundamental group of a graph of groups in the sense of [4] and [25]. By using graph structures (with groups) and similar techniques of Fock space construction, we will construct nuclear C^*-algebras associated with the fundamental groups of graphs of groups, which have a certain universal property. Thanks to this property, we can give other descriptions of our C^*-algebras. For instance, our C^*-algebras are $*$-isomorphic to the crossed products arising from the boundary actions. We will also show that our C^*-algebras can be realized by Cuntz-Pimsner algebras, which are defined by M. V. Pimsner in [22].

We will next give explicit formulae of the K-groups of our C^*-algebras. Since every word-hyperbolic group with zero-dimensional boundary acts on a tree, to obtain the K-groups, one can apply six-term exact sequences in [21]. As the above mentioned, our C^*-algebras are also $*$-isomorphic to Cuntz-Pimsner algebras. Thus one can also apply six-term exact sequences in [22]. However these are not trivial tasks to compute K-groups of our C^*-algebras for certain examples. We will give another formula of the K-groups of our C^*-algebras by using the method for Cuntz-Krieger algebras [8] in the similar way of [19]. The K-groups of our C^*-algebras are given by integer-valued matrices, which are determined by the representation theory of edge groups. Moreover by using the obtained matrices, we will completely determine ideal structures of our C^*-algebras by similar arguments as in the case of Cuntz-Krieger algebras with condition (II) in [8], (see also [15]). As a consequence, we also give a necessary and sufficient condition of the simplicity of our C^*-algebras in the term of the corresponding matrices. We remark that if our C^*-algebra is simple, then it is also purely infinite by [18] or [2]. Therefore, in this case, we can apply the Kirchberg-Phillips classification theorem to our C^*-algebras. Finally we consider the case of HNN-extensions and their examples.

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2. Preliminaries

2.1. Graphs of groups

We give a quick introduction on the theory of H. Bass and J-P. Serre of graphs of groups. We refer the reader to [4] and [25] for more details.

A *graph of groups* Γ consists of a connected graph (Γ^0, Γ^1), the origin and terminus $o, t : \Gamma^1 \to \Gamma^0$, a fixed point free involution $e \mapsto \overline{e}$ of Γ^1 satisfying $o(\overline{e}) = t(e)$ with vertex groups Γ_v $(v \in \Gamma^0)$, edge groups $\Gamma_e = \Gamma_{\overline{e}}$ $(e \in \Gamma^1)$ and monomorphisms $\iota_e : \Gamma_e \to \Gamma_v$ for $v = o(e)$. We write Γ^* for the set of *edge paths* $\xi = e_1 \dots e_n$ with $t(e_i) = o(e_{i+1})$ for $1 \le i < n$, and the maps o and t extend to Γ^* in an obvious way. For $\xi = e_1 \dots e_n \in \Gamma^*$, we also define *source* and *range* maps $s, r : \Gamma^* \to \Gamma^1$ by $s(\xi) = e_1$ and $r(\xi) = e_n$. For $\xi, \eta \in \Gamma^*$ with $t(\xi) = o(\eta)$, we write the concatenation $\xi\eta$. An edge path $\xi = e_1 \dots e_n \in \Gamma^*$ is called *reduced* if $e_{i+1} \ne e_i$ for $1 \le i < n$.

The *path group* π_Γ is defined by

$$\pi_\Gamma = [(\star_{v \in \Gamma^0} \Gamma_v) \star \mathsf{F}(\Gamma^1)]/\mathscr{R},$$

where $\mathsf{F}(\Gamma^1)$ denotes the free group with basis Γ^1 and $\mathscr{R}$ is the normal subgroup which imposes the relations $\overline{e} = e^{-1}$ and $e\iota_{\overline{e}}(g)e^{-1} = \iota_e(g)$ for all $e \in \Gamma^1$ and $g \in \Gamma_e = \Gamma_{\overline{e}}$. A *path* $\gamma = g_0 e_1 g_1 e_2 \dots g_{n-1} e_n g_n$ is given by an edge path $\xi = e_1 \dots e_n$ and a sequence $\mu = g_0 g_1 \dots g_n$ of elements of vertex groups, where $g_i \in \Gamma_{v_i}$ with $v_i = t(e_i) = o(e_{i+1})$ for $1 \le i < n$. We often express a path γ by a pair (ξ, μ) and we write Γ_ξ for the set of *group paths* μ *of type* ξ. The maps o, t, s and r extend to the set of paths by $o(\gamma) = o(\xi), t(\gamma) = t(\xi), s(\gamma) = s(\xi)$ and $r(\gamma) = r(\xi)$ if $\gamma = (\xi, \mu)$, respectively. We denote by 1_ξ the trivial group path $1 \dots 1$ of type ξ. An edge path ξ may be confused with a path $(\xi, 1_\xi)$. Note that any path γ gives an element in π_Γ. For $v, w \in \Gamma^0$, we write $\pi_\Gamma[v, w]$ for the set of elements in π_Γ given by paths from v to w. A path $\gamma = g_0 e_1 g_1 \dots e_n g_n$ is said to be *reduced* if either $n = 0$ and $g_0 \ne 1$, or else $n > 0$ and whenever $e_{i+1} = \overline{e_i}$, we have $g_i \notin \iota_{\overline{e_i}}(\Gamma_{\overline{e_i}})$. Note that if γ is a reduced path, then $\gamma \ne 1$ in π_Γ. Thus the canonical homomorphisms $: \Gamma_v \to \pi_\Gamma$ are injective.

For each $e \in \Gamma^1$ with $o(e) = v$, we choose a set $\Delta_e \subseteq \Gamma_v$ of coset representatives for $\Gamma_v/\iota_e(\Gamma_e)$ with $1 \in \Delta_e$. Relative to these choices, a path γ is called $(\Delta$-)*normalized* if it has the form $\gamma = g_1 e_1 \dots g_n e_n g$, where $g_i \in \Delta_{e_i}$ $(1 \le i \le n)$, $g \in \Gamma_{v_n}$ and γ is reduced, i.e., $e_{i-1} = \overline{e_i}$ implies $g_i \ne 1$. For $v, w \in \Gamma^0$, every element of $\pi_\Gamma[v, w]$ can be represented by the unique normalized path from v to w.

DEFINITION 2.1.

(i) The *fundamental group of* Γ *at a base* $v_0 \in \Gamma^0$ is given by

$$\Gamma_0 := \pi_\Gamma[v_0, v_0].$$

(ii) The *fundamental group of* Γ *relative to a maximal subtree* (T^0, T^1) of

(Γ^0, Γ^1) is defined by

$$\Gamma_T := \pi_\Gamma/(\text{relations} : e = 1 \ \text{ for all } \ e \in T^1).$$

The above two definitions give the same group essentially because the natural projection $q : \pi_\Gamma \to \Gamma_T$ restricts to an isomorphism $q_T : \Gamma_0 \to \Gamma_T$. The inverse of q_T is given as follows. For $v, w \in \Gamma^0$, let $\xi_{v \mapsto w}$ denote the reduced edge path in T from v to w. Then we have

$$\begin{aligned} q_T^{-1}(q(g)) &= \xi_{v_0 \mapsto v} \cdot g \cdot \xi_{v \mapsto v_0} && \text{for} \quad g \in \Gamma_v, \\ q_T^{-1}(q(e)) &= \xi_{v_0 \mapsto o(e)} \cdot e \cdot \xi_{t(e) \mapsto v_0} && \text{for} \quad e \in \Gamma^1. \end{aligned}$$

Hence we often denote by $\pi_1(\Gamma)$ or Γ for simplicity, the fundamental group of Γ.

Finally we introduce the tree $\widetilde{X}$ called *universal cover* of Γ. The vertices $\widetilde{X}^0$ are given by

$$\widetilde{X}^0 = \coprod_{v \in \Gamma^0} \pi_\Gamma[v_0, v]/\Gamma_v.$$

and the edges $\widetilde{X}^1$ are certain ordered pairs $\mathscr{E} = (\gamma\Gamma_v, \gamma'\Gamma_w)$ of distinct vertices, where $\gamma_1 \in \pi_\Gamma[v_0, v_1]$ and $\gamma_2 \in \pi_\Gamma[v_0, v_2]$ such that $\gamma_1^{-1}\gamma_2 = g_1 e g_2$ with $e \in \Gamma^1$, $o(e) = v_1$, $t(e) = v_2$, $g_1 \in \Gamma_{v_1}$ and $g_2 \in \Gamma_{v_2}$. Then we can define the origin and terminus and the orientation on $(\widetilde{X}^0, \widetilde{X}^1)$ in an obvious way. Moreover there is a natural left action of Γ_0 on $\widetilde{X}$, which is oriented preserved. Then the graph $\widetilde{X}$ constructed above is, in fact, a tree.

EXAMPLE 2.2.

(i) If $\Gamma_v = \{1\}$ for all $v \in \Gamma^0$, then

$$\Gamma_T = \pi_1(\Gamma^1, T^1) := \mathsf{F}(\Gamma^1)/(\overline{e} = e^{-1} \ \text{ and } \ e = 1 \ \text{ if } \ e \in T^1),$$

which is a free group based on half the edges of $\Gamma^1 \setminus T^1$.

(ii) If $\Gamma_e = \{1\}$ for all $e \in \Gamma^1$, then $\Gamma_T = (\star_{v \in \Gamma^0} \Gamma_v) \star \pi_1(\Gamma^1, T^1)$.

(iii) If Γ is the tree of Figure 1, then the fundamental group is the amalgamated free product group $\Gamma_{v_1} \star_{\Gamma_e} \Gamma_{v_2}$.

(iv) If Γ is the loop of Figure 2, then the fundamental group is the HNN-extension

$$G \star_H \theta = \langle G, x \mid hx = x\theta(h) \ \text{ for } \ h \in H \rangle,$$

where $G = \Gamma_{v_0}$, $H = \iota_x(\Gamma_x)$, $\overline{H} = \iota_{\overline{x}}(\Gamma_{\overline{x}})$ and $\theta : H \to \overline{H} \subseteq G$ given by $\theta = \iota_{\overline{x}} \circ \iota_x^{-1}$.

(v) A group G acts on a tree Y without inversion. Let $(\Gamma^0, \Gamma^1) = G\backslash Y$ and Γ_v (resp. Γ_e) denote the stabilizer of a vertex $v \in Y^0$ (resp. an edge $e \in Y^1$). Then Γ becomes a graph of groups in the natural way, (see [25] for more details). Moreover the fundamental group of this Γ is isomorphic to G and the universal cover $\widetilde{X}$ of Γ coincides with Y.

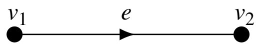

FIGURE 1

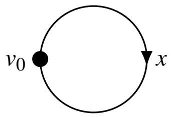

FIGURE 2

2.2. Hyperbolic groups with zero-dimensional boundary

For the definition and basic properties of word-hyperbolic groups and hyperbolic boundaries, we refer the reader to [13], [14] and [16]. Let G be a word-hyperbolic group with the hyperbolic boundary ∂G. If ∂G has infinitely many points, G contains a subgroup isomorphic to the free group F_2 and ∂G is a infinite perfect compact metrizable space. In this paper, we focus on word-hyperbolic groups with zero-dimensional boundary, which are equivalent to *virtually free*, i.e., groups containing a free subgroup of finite index. Moreover, a virtually free group can be also given by the fundamental group of a graph of groups of *finite type*, i.e., (Γ^0, Γ^1) is a finite graph and Γ_v is a finite group for every $v \in \Gamma^0$.

THEOREM 2.3 (cf. Theorem 7.3 in [24]). *A finitely generated group G has a free subgroup F_n of finite index if and only if G is the fundamental group of a graph of groups Γ of finite type, where $\mathsf{F}_0 = \{1\}$ and F_1 is the integer group* Z.

Let Γ be a graph of groups of finite type with a base v_0. Notice that each vertex in $\widetilde{X}$ can be represented by the unique normalized path. It is well-known that the hyperbolic boundary $\partial\widetilde{X}$ can be identified with the *end space*. Namely if $\partial\widetilde{X} \neq \emptyset$, then $\partial\widetilde{X}$ can be expressed by the *infinite normalized path space*

$$\left\{ g_1 e_1 g_2 e_2 \dots \;\middle|\; \begin{array}{l} o(e_1) = v_0, t(e_i) = o(e_{i+1}), \mu_i \in \Delta_{e_i}, i = 1, 2, \dots \\ e_{i-1} = \overline{e_i} \implies g_i \neq 1 \end{array} \right\}$$

with the natural left action of Γ_0. Moreover the following fact is easily checked, (see [14], [16]) and we use this later.

PROPOSITION 2.4. *Let Γ be a graph of groups of finite type. Let $\widetilde{X}$ be the universal cover of Γ. If $\partial\widetilde{X} \neq \emptyset$, then the hyperbolic boundary $\partial\Gamma$ is homeomorphic to the end space $\partial\widetilde{X}$ of the tree $\widetilde{X}$, and two actions of Γ on $\partial\Gamma$ and $\partial\widetilde{X}$ are conjugate.*

This is a reason why we can construct C^*-algebras, such as Cuntz-Krieger algebras, which is $*$-isomorphic to the crossed products $C(\partial\Gamma) \rtimes \Gamma$ arising from the boundary actions of word-hyperbolic groups with zero-dimensional boundary, because the end space structure of trees is similar to the Fock space.

3. Construction

Let Γ be a graph of groups of finite type. We fix a base v_0 and a maximal subtree T of Γ. From now, we identify $\iota_e(\Gamma_e)$ with Γ_e, and so Γ_e may not be equal to $\Gamma_{\overline{e}}$, but $\Gamma_e \simeq \Gamma_{\overline{e}}$ via $\iota_{\overline{e}} \circ \iota_e^{-1}$. For each $e \in \Gamma^1$ with $o(e) = v$, we choose a set $\Delta_e \subseteq \Gamma_v$ of coset representatives for Γ_v/Γ_e with $1 \in \Delta_e$.

Now we assume that $\partial\widetilde{X} \neq \emptyset$ and (Γ^0, Γ^1) is a finite graph. Let $\omega \in \partial\widetilde{X}$, which is expressed by the infinite normalized path $g_1e_1g_2e_2\ldots$. Then there is n_0 such that either $g_{n_0} \neq 1$ or $e_{n_0} \notin T^1$ holds with $g_k = 1$, $e_k \in T^1$ for all $k \leq n_0 - 1$. We put $\gamma_0 = g_1e_1\ldots g_{n_0-1}e_{n_0-1} = (\xi_0, 1_{\xi_0})$ where $\xi_0 = e_1\ldots e_{n_0-1}$. Since (Γ^0, Γ^1) is finite, there is $n_1 > n_0$ such that either $\overline{e_{n_1-1}} = e_{n_1}$ or $e_{n_1} \notin T^1$ holds with $e_{n_0+1}, \ldots, e_{n_1-1} \in T^1$. Then we set $\gamma_1 = (\xi_1, \mu_i)$, where $\xi_1 = e_{n_0}\ldots e_{n_1-1}$ and $\mu_1 = g_{n_0}\ldots g_{n_1-1}$. By repeating this argument, we obtain infinite sequence $\{\gamma_i\}_{i=0}^{\infty}$ of (finite) normalized paths such that

(i) ω is represented by the normalized path $\gamma_0\gamma_1\ldots$,

(ii) $\gamma_0 = (\xi_0, 1_{\xi_0})$ is a reduced edge path in T with $o(\xi_0) = v_0$,

(iii) γ_i is given by a reduced edge path $\xi_i = e_1^{(i)}e_2^{(i)}\ldots e_{n_i}^{(i)}$ and $\mu_i = g_1^{(i)}g_2^{(i)}\ldots g_{n_i}^{(i)}$,

(iv) $e_2^{(i)}, \ldots, e_{n_i}^{(i)} \in T^1$ for $i \geq 1$,

(v) either $s(\xi_i) = \overline{r(\xi_{i-1})}$ with $g_1^{(i)} \neq 1$ or $s(\xi_i) \notin T^1$ for $i \geq 1$,

(vi) either $g_1^{(1)} \neq 1$ or $s(\xi_1) \notin T^1$.

We denote by N_Γ^n the set of all normalized paths of the form $\gamma_k\ldots\gamma_{k+n-1}$ and by W_Γ^0 the set of all pair (e, f) of edges with $e = r(\gamma_k)$, $f = s(\gamma_{k+1})$ for some $\omega = \{\gamma_i\}_{i=0}^{\infty} \in \partial\widetilde{X}$ and $k \geq 1$. For $n \geq 1$, we write W_Γ^n for the set of reduced paths $\gamma = (\xi, \mu)$ such that $\gamma^{-1}\gamma' \in \Gamma_{\overline{r(\gamma')}}$ for some $\gamma' = (\xi, \mu') \in N_\Gamma^n$. We put $E_\Gamma^n = \{\xi \in \Gamma^* \mid (\xi, \mu) \in N_\Gamma^n \text{ for some } \mu\}$. We set

$$N_\Gamma^* = \bigcup_{n\geq 1} N_\Gamma^n, \qquad W_\Gamma^* = \bigcup_{n\geq 1} W_\Gamma^n, \qquad \text{and} \qquad E_\Gamma^* = \bigcup_{n\geq 1} E_\Gamma^n.$$

For $\alpha, \beta \in W_\Gamma^*$, we denote $\alpha \rightsquigarrow \beta$ if $t(\alpha) = o(\beta)$ and $\alpha\beta$ is reduced. For $e \in \Gamma^1$, we denote $\Gamma(e) = \Gamma_e$ if $e \in T^1$ and $\Gamma(e) = \Gamma_{o(e)}$ if $e \notin T^1$.

Definition 3.1. A *Cuntz-Krieger* Γ*-family* (*with respect to* T) in a C^*-algebra consists of a family $\{S(\gamma) \mid \gamma \in W_\Gamma^1\}$ of partial isometries and a family

$\{U(g) \mid g \in \Gamma(e), e \in s(W_\Gamma^1)\}$ of partial unitaries satisfying the *Cuntz-Krieger* Γ*-relations* (*with respect to* T): We denote by $P(\gamma)$ the range projection of $S(\gamma)$. Let $\alpha, \beta \in W_\Gamma^1$ and $g \in \Gamma(e)$ with $e \in s(W_\Gamma^1)$.

$$P(\alpha)P(\beta) = \begin{cases} P(\alpha) = P(\beta) & \text{if } \alpha\Gamma_{r(\alpha)} = \beta\Gamma_{r(\beta)}, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

$$S(\alpha)^* S(\alpha) = \sum_{\substack{\gamma \in N_\Gamma^1 \\ \alpha \rightsquigarrow \gamma}} P(\gamma), \tag{2}$$

$$I = \sum_{\gamma \in N_\Gamma^1} P(\gamma), \tag{3}$$

$$U(g)^* U(g) = U(g)U(g)^* = \sum_{\substack{\gamma \in N_\Gamma^1 \\ s(\gamma) = e}} P(\gamma) (= P(e)), \tag{4}$$

$$U(g)S(\alpha) = S(g\alpha) \quad \text{if } e = s(\alpha), \tag{5}$$

$$S(\alpha)U(g) = S(\alpha g)P(e) \quad \text{if } t(\alpha) = o(e) \text{ and } g \in \Gamma_{\overline{r(\alpha)}}, \tag{6}$$

where $g\alpha = (gg_0)e_1g_1 \dots e_ng_n$ and $\alpha g = g_0e_1g_1 \dots e_n(g_ng)$ if $\alpha = g_0e_1g_1 \dots e_ng_n \in W_\Gamma^1$.

For $\gamma = \gamma_1 \dots \gamma_n \in W_\Gamma^n$ with $\gamma_i \in W_\Gamma^1$, we define

$$S(\gamma) = S(\gamma_1) \dots S(\gamma_n).$$

REMARK 3.2. The above relations (2), (3) and (4) may be infinite sums even if (Γ^0, Γ^1) is finite. For simplicity, we always assume that Γ is of *finite type*. In this case, the above sums are finite. This assumption corresponds to finiteness of associated 0-1 matrices of Cuntz-Krieger algebras. As many generalizations of Cuntz-Krieger algebras, for example, Exel-Laca algebras [10], graph algebras [5], [17], Cuntz-Pimsner algebras [22] and so on, we might define C^*-algebras associated with graphs of groups without the assumption of finiteness. However, for investigations of crossed products arising from boundary actions, which we focus on in this paper, it is not necessary.

DEFINITION 3.3. We define the *universal* C^*-algebra $\mathcal{O}_\Gamma$ generated by a universal Cuntz-Krieger Γ-family $\{S_\gamma, U_g\}$. In other words, the C^*-algebra $\mathcal{O}_\Gamma$ is generated by a Cuntz-Krieger Γ-family $\{S_\gamma, U_g\}$ such that for every Cuntz-Krieger Γ-family $\{S(\gamma), U(g)\}$ on a Hilbert space $\mathcal{H}$ there is a canonical $*$-representation $\pi : \mathcal{O}_\Gamma \to \mathcal{B}(\mathcal{H})$ with $\pi(S_\gamma) = S(\gamma)$ and $\pi(U_g) = U(g)$.

Next we will introduce the "Fock space construction", which gives the existence of non-zero Cuntz-Krieger Γ-family and hence it allows us to define the above universal C^*-algebra $\mathcal{O}_\Gamma$.

DEFINITION 3.4. We define a Hilbert space $\mathscr{H}(\Gamma, T)$ by

$$\mathscr{H}(\Gamma, T) = \bigoplus_{n\geq 0} \mathscr{H}_n(\Gamma, T),$$

where

$$\mathscr{H}_0(\Gamma, T) = \overline{\mathrm{span}}\{\delta_{\Gamma_v} \mid v \in t(N_\Gamma^1)\},$$
$$\mathscr{H}_n(\Gamma, T) = \overline{\mathrm{span}}\{\delta_{\gamma\Gamma_{t(\gamma)}} \mid \gamma \in N_\Gamma^n\}.$$

Then we define the partial isometries $T_{\mathrm{Fock}}(\alpha)$ for $\alpha \in W_\Gamma^1$ and partial unitaries $V_{\mathrm{Fock}}(g)$ for $g \in \Gamma(e)$ with $e \in s(W_\Gamma^1)$ by

$$T_{\mathrm{Fock}}(\alpha) \cdot \delta_{\Gamma_v} = \begin{cases} \delta_{\alpha\Gamma_v} & \text{if } t(\alpha) = v, \\ 0 & \text{otherwise,} \end{cases}$$

$$T_{\mathrm{Fock}}(\alpha) \cdot \delta_{\gamma\Gamma_{t(\gamma)}} = \begin{cases} \delta_{\alpha\gamma\Gamma_{t(\gamma)}} & \text{if } \alpha \rightsquigarrow \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_{\mathrm{Fock}}(g) \cdot \delta_{\Gamma_v} = \begin{cases} \delta_{\Gamma_v} & \text{if } g \in \Gamma_v, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_{\mathrm{Fock}}(g) \cdot \delta_{\gamma\Gamma_{t(\gamma)}} = \begin{cases} \delta_{g\gamma\Gamma_{t(\gamma)}} & \text{if } g \in \Gamma(s(\gamma)), \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\pi_{\mathrm{Fock}} : \mathscr{B}(\mathscr{H}(\Gamma, T)) \to \mathscr{B}(\mathscr{H}(\Gamma, T))/\mathscr{K}(\mathscr{H}(\Gamma, T))$ be a quotient and we set $S_{\mathrm{Fock}}(\gamma) = \pi_{\mathrm{Fock}}(T_{\mathrm{Fock}}(\gamma))$ and $U_{\mathrm{Fock}}(g) = \pi_{\mathrm{Fock}}(V_{\mathrm{Fock}}(g))$.

One can easily check the following and hence we can obtain the universal C^*-algebra $\mathscr{O}_\Gamma$, (the reader may refer to the relevant proof in [15] or [17]).

PROPOSITION 3.5. *The above family* $\{S_{\mathrm{Fock}}(\gamma), U_{\mathrm{Fock}}(g)\}$ *is a non-zero Cuntz-Krieger* Γ*-family.*

For a group G, we denote by $C^*(G)$ the full group C^*-algebra. We get another property of the family $\{S_{\mathrm{Fock}}(\gamma), U_{\mathrm{Fock}}(g)\}$.

Let $\{S(\gamma), U(g)\}$ be a Cuntz-Krieger Γ-family. For $(e, f) \in W_\Gamma^0$, we define

$$P(e, f) = \sum_{\substack{\gamma \in N_\Gamma^1, s(\gamma) = f \\ e\gamma\,:\,\text{reduced}}} P(\gamma).$$

PROPOSITION 3.6. *For* $(e, f) \in W_\Gamma^0$*, the canonical* $*$*-isomorphism*

$$C^*(\Gamma_{\overline{e}}) \simeq C^*(U_{\mathrm{Fock}}(g) P_{\mathrm{Fock}}(e, f) \mid g \in \Gamma_{\overline{e}})$$

holds.

PROOF. For $(e, f) \in W_\Gamma^0$, we define the projection $Q_{\text{Fock}}(e, f)$ on $\mathscr{H}(\Gamma, T)$ by

$$Q_{\text{Fock}}(e, f) = \sum_{\substack{\gamma \in N_\Gamma^1, s(\gamma)=f \\ e\gamma\,:\,\text{reduced}}} T_{\text{Fock}}(\gamma) T_{\text{Fock}}(\gamma)^*.$$

Note that for $g \in \Gamma_{\overline{e}}$, every $V_{\text{Fock}}(g)$ commutes with $Q_{\text{Fock}}(e, f)$ and $\pi_{\text{Fock}}(Q_{\text{Fock}}(e, f)) = P_{\text{Fock}}(e, f)$. Since the unitary representation $V_{\text{Fock}}(f, \cdot)$ $Q_{\text{Fock}}(e, f)$ of $\Gamma_{\overline{e}}$ contains the left regular representation of $\Gamma_{\overline{e}}$ with infinite multiplicity, we obtain the required result.

COROLLARY 3.7. *For $(e, f) \in W_\Gamma^0$, then the canonical $*$-isomorphism*

$$C^*(\Gamma_{\overline{e}}) \simeq C^*(U_g P_{e,f} \mid g \in \Gamma_{\overline{e}})$$

holds.

The following proposition can be easily proved by using the Cuntz-Krieger Γ-relations.

PROPOSITION 3.8.

$$\mathscr{O}_\Gamma = \overline{\text{span}} \left\{ S_\alpha U_g P_{e,f} S_\beta^* \;\middle|\; \begin{array}{l} \alpha, \beta \in W_\Gamma^*, e = r(\alpha) = r(\beta), \\ (e, f) \in W_\Gamma^0, g \in \Gamma_{\overline{e}} \end{array} \right\}.$$

DEFINITION 3.9. We define the action Θ of $\mathsf{T} = \{z \in \mathsf{C} \mid |z| = 1\}$ on $\mathscr{O}_\Gamma$, which is called the *gauge action*, by

$$\Theta_z(S_\gamma) = z S_\gamma,$$
$$\Theta_z(U_g) = U_g,$$

for $z \in \mathsf{T}$. We define the conditional expectation on $\mathscr{O}_\Gamma$ by

$$\Phi(\cdot) = \int_{\mathsf{T}} \Theta(\cdot)\, dz,$$

where dz is the Haar measure on T.

LEMMA 3.10. *The fixed-point subalgebra $\mathscr{F}_\Gamma$ of $\mathscr{O}_\Gamma$ under the gauge action is an* AF-*algebra.*

PROOF. For $n \in \mathsf{N}$, $(e, f) \in W_\Gamma^0$, we define the C^*-algebra

$$\mathscr{F}_n(e, f) = \overline{\text{span}} \left\{ S_\alpha U_g P_{e,f} S_\beta^* \mid \alpha, \beta \in W_\Gamma^n, e = r(\alpha) = r(\beta), g \in \Gamma_{\overline{e}} \right\}.$$

Since the family $\{S_\alpha P_{e,f} S_\beta^*\}_{\alpha,\beta\in N_\Gamma^n}$ gives the matrix units, we have

$$\mathscr{F}_n(e, f) \simeq \mathsf{M}_{K_n(e,f)} \otimes C^*(\Gamma_{\overline{e}})$$

for some $K_n(e, f) \in \mathsf{N}$. We set

$$\mathscr{F}_n = \bigoplus_{(e,f)\in W_\Gamma^0} \mathscr{F}_n(e, f).$$

Since the Cuntz-Krieger Γ-relation gives the embedding $\mathscr{F}_n \hookrightarrow \mathscr{F}_{n+1}$, we have an AF-algebra

$$\overline{\bigcup_{n\geq 1} \mathscr{F}_n}.$$

By using the conditional expectation Φ, one can show that it is, in fact, the fixed-point subalgebra $\mathscr{F}_\Gamma$ under the gauge action Θ.

THEOREM 3.11. *Let Γ be a graph of groups of finite type such that the end space $\partial\widetilde{X}$ has infinitely many ends, which is equivalent to that the fundamental group $\pi_1(\Gamma)$ is virtually free. Let $\{S(\alpha), U(g)\}$ be a non-zero Cuntz-Krieger Γ-family and π be the canonical surjective $*$-homomorphism from $\mathscr{O}_\Gamma$ onto the C^*-algebra generated by the family $\{S(\alpha), U(g)\}$. If the canonical $*$-isomorphism*

$$C^*(\Gamma_{\overline{e}}) \simeq C^*(U(g)P(e, f) \mid g \in \Gamma_{\overline{e}})$$

holds for any $(e, f) \in W_\Gamma^0$, then π is $*$-isomorphic.

PROOF. One can easily show that π is faithful on the fixed-point subalgebra $\mathscr{F}_\Gamma$ under the gauge action Θ by the similar arguments as in the proof of Lemma 3.10.

We will show that $\|\pi(\Phi(X))\| \leq \|\pi(X)\|$ for any $X \in \mathscr{O}_\Gamma$. By Proposition 3.8, it suffices to check it for a finite sum

$$X = \sum_{e,f,g} \sum_{\alpha,\beta} C_{e,f,g}^{\alpha,\beta} S_\alpha U_g P_{e,f} S_\beta^*, \qquad (\spadesuit)$$

where $C_{e,f,g}^{\alpha,\beta} \in \mathsf{C}$. Let $n \in \mathsf{N}$ be sufficiently large. We may assume that if $\alpha \in W_\Gamma^k$ and $\beta \in W_\Gamma^l$ in the above sum ($\spadesuit$), then $\min\{k, l\} = n$ holds. Note that there are e_0, f_0 such that

$$\|\pi(\Phi(X))\| = \left\| \sum_g \sum_{\alpha,\beta} C_{e_0,f_0,g}^{\alpha,\beta} S(\alpha)U(g)P(e_0, f_0)S(\beta)^* \right\|.$$

By the assumptions, the fundamental group $\pi_1(\Gamma)$ contains a free group F_r ($r \geq 2$). Hence we can take a sufficiently long aperiodic normalized path γ_0 with $s(\gamma_0) = f_0$ and $r(\gamma_0) = e_1$. Now we put a non-zero projection

$$Q = \sum_{\gamma \in N_\Gamma^n} S(\gamma)S(\gamma_0)S(\gamma_0)^* S(\gamma)^*.$$

If $\alpha, \beta \in N_\Gamma^n$, then

$$Q(S(\alpha)P(e_0, f_0)S(\beta)^*)Q = S(\alpha)S(\gamma_0)S(\gamma_0)^* S(\beta).$$

Note that $\{S(\alpha)S(\gamma_0)S(\gamma_0)^* S(\beta)^*\}_{\alpha,\beta \in N_\Gamma^n}$ is a family of matrix units. Hence the argument as in the proof of Lemma 3.10 gives the faithfulness of $A \mapsto Q\pi(A)Q$ on $\mathscr{F}_n$. In particular we get $\|\pi(\Phi(X))\| = \|Q\pi(\Phi(X))Q\|$.

We next claim that $Q\pi(\Phi(X))Q = Q\pi(X)Q$. Let $\alpha \in W_\Gamma^k$, $\beta \in W_\Gamma^l$ be in the sum (♠) with $k \neq l$. We may assume that $k = n$ and $l > n$ without loss of generality. Then

$$\begin{aligned} &QS(\alpha)U(g)P(e, f)S(\beta)^* Q \\ &\qquad = QS(\alpha)U(g)P(e, f)S(\beta_2)^* S(\gamma_0)S(\gamma_0)^* S(\beta_1)^*, \end{aligned}$$

where $\beta = \beta_1\beta_2$. The above element is non-zero if

$$S(\gamma_0)^* U(g)P(e, f)S(\beta_2)^* S(\gamma_0) \neq 0.$$

However it is impossible by the choice of γ_0. Thus $QS(\alpha)U(g)P(e, f)S(\beta)^* Q = 0$ if $k \neq l$, namely we have shown our claim. Hence we can obtain

$$\|\pi(\Phi(X))\| = \|Q\pi(\Phi(X))Q\| = \|Q\pi(X)Q\| \leq \|\pi(X)\|.$$

Therefore the proof is complete, thanks to [6].

COROLLARY 3.12. *The C^*-algebra generated by $\{S_{\mathrm{Fock}}(\gamma), U_{\mathrm{Fock}}(g)\}$ is $*$-isomorphic to the C^*-algebra $\mathcal{O}_\Gamma$ via $S_{\mathrm{Fock}}(\gamma) \mapsto S_\gamma$ and $U_{\mathrm{Fock}}(g) \mapsto U_g$.*

We can also prove the following by the same arguments as in the proof of Theorem 3.11. We will use this to get the ideal structure theorem for $\mathcal{O}_\Gamma$ in Section 6.

THEOREM 3.13. *Let $\{S_1(\gamma), U_1(g)\}$ and $\{S_2(\gamma), U_2(g)\}$ be two non-zero Cuntz-Krieger Γ families. If the canonical $*$-isomorphism*

$$C^*(U_1(g)P_1(e, f) \mid g \in \Gamma_{\overline{e}}) \simeq C^*(U_2(g)P_2(e, f) \mid g \in \Gamma_{\overline{e}})$$

holds for any $(e, f) \in W_\Gamma^0$, then the C^*-algebras $C^*(S_1(\gamma), U_1(g))$ and $C^*(S_2(\gamma), U_2(g))$ are $*$-isomorphic via $S_1(\gamma) \mapsto S_2(\gamma)$ and $U_1(g) \mapsto U_2(g)$.

REMARK 3.14. We consider the C^*-algebra $\mathcal{O}_\Gamma$ associated with Γ in Example 2.2. In the case of Example 2.2 (i), Γ is the free group and $\mathcal{O}_\Gamma$ is some Cuntz-Krieger algebra, which is given in [27]. In the case of Example 2.2 (ii) or (iii), Γ is an amalgamated free product groups and $\mathcal{O}_\Gamma$ is the same one defined in [19]. The case of Example 2.2 (iv) is discussed in Section 7.

4. Other descriptions

4.1. Cuntz-Pimsner algebras

DEFINITION 4.1. Let $e \in s(W_\Gamma^0)$. If $e \in T^1$, then we define the C^*-algebra

$$B_e = C^*(\Gamma_e) = \text{span}\{g \in \Gamma_e\}.$$

If $e \notin T^1$ with $v = o(e)$, then we consider the right action of Γ_v on Γ_v / Γ_e and define the C^*-algebra

$$B_e = C(\Gamma_v / \Gamma_e) \rtimes \Gamma_v = \text{span}\{p_e(x)g \mid x \in \Delta_e, g \in \Gamma_v\},$$

where $p_e(x) \in C(\Gamma_v / \Gamma_e)$ is defined by

$$p_e(x)(y\Gamma_e) = \begin{cases} 1 & \text{if } x^{-1}y \in \Gamma_e, \\ 0 & \text{otherwise,} \end{cases}$$

and $gp_e(x)g^{-1} = p_e(y)$ for $y \in \Gamma_e$ with $gx\Gamma_e = y\Gamma_e$.

Let $\xi \in E_\Gamma^1$ be fixed. We define the right Hilbert $C^*(\Gamma_{\overline{r(\xi)}})$-module

$$H_\xi = \text{span}\{\mu \in \Gamma_\xi \mid (\xi, \mu) \in W_\Gamma^1\}$$

with the natural right action of $C^*(\Gamma_{\overline{r(\xi)}})$ and the inner product

$$\langle \mu_1, \mu_2 \rangle_{H_\xi} = \begin{cases} \gamma_1^{-1}\gamma_2 & \text{if } \gamma_1^{-1}\gamma_2 \in \Gamma_{\overline{r(\xi)}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma_i = (\xi, \mu_i)$ for $i = 1, 2$. Moreover we define the left action of $B_{s(\xi)}$ on H_ξ in the following way. If $s(\xi) \in T^1$, then $g \cdot \mu = (gg_0) \dots g_n$, and if $s(\xi) \notin T^1$, then

$$p_e(s)g \cdot g_0 \dots g_n = \begin{cases} (gg_0) \dots g_n & \text{if } s^{-1}gg_0 \in \Gamma_{s(\xi)}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu = g_0 g_1 \dots g_n \in H_\xi$. Hence we obtain the Hilbert $B_{s(\xi)}$-$C^*(\Gamma_{\overline{r(\xi)}})$ bimodule H_ξ.

Next let $(r(\xi), e) \in W_\Gamma^0$. If $e \neq \overline{r(\xi)}$, then $e \notin T^1$ and $C^*(\Gamma_{\overline{r(\xi)}}) \subset B_e$. Thus we put

$$E_{(\xi,e)} = H_\xi \otimes_{C^*(\Gamma_{\overline{r(\xi)}})} B_e = \operatorname{span}\{\mu\} \otimes \operatorname{span}\{p_e(x)g\}.$$

If $e = \overline{r(\xi)}$ and $e \in T^1$, then $B_e = C^*(\Gamma_{r(\xi)})$ and we set

$$E_{(\xi,e)} = H_\xi \otimes_{C^*(\Gamma_{r(\xi)})} B_e = \operatorname{span}\{\mu\} \otimes \operatorname{span}\{g\}(= H_\xi).$$

If $e = \overline{r(\xi)}$ and $e \notin T^1$, then we define

$$E_{(\xi,e)} = H_\xi \otimes_{\phi_e} B_e = \operatorname{span}\{\mu\} \otimes \operatorname{span}\{p_e(x)g \mid 1 \neq x \in \Delta_e\},$$

where $\phi_e(g)b = g \sum_{1\neq y \in \Delta_e} p_e(y)b$ for $g \in \Gamma_e$ and $b \in B_e$. Note that $E_{(\xi,e)}$ is a Hilbert $B_{s(\xi)}$-B_e bimodule. Then we define

$$B = \bigoplus_e B_e,$$

$$E = \bigoplus_\xi \bigoplus_e E_{(\xi,e)}.$$

Note that E is a Hilbert B-bimodule.

REMARK 4.2. The above constructed E may be not full, in general. So to identify $\mathcal{O}_\Gamma$ with the Cuntz-Pimsner algebra associated with E, we have to consider $\widetilde{\mathcal{O}}_E$. (See [22, Remark 1.2 (3)].)

THEOREM 4.3. *Let Γ be a graph of groups of finite type with infinitely many ends. Let $\widetilde{\mathcal{O}}_E$ be the Cuntz-Pimsner algebra associated with the above B-bimodule E. Then $\mathcal{O}_\Gamma$ is canonically $*$-isomorphic to $\widetilde{\mathcal{O}}_E$.*

PROOF. Note that $\widetilde{\mathcal{O}}_E$ is generated by $\{S_{\mu\otimes I_e}\}$ and the C^*-algebra B, where I_e is the unit of the C^*-algebra B_e. One can easily show that the universality gives an $*$-isomorphism between $\mathcal{O}_\Gamma$ and $\widetilde{\mathcal{O}}_E$.

$$S_\gamma \longleftrightarrow \sum_e S_{\mu\otimes I_e} \qquad \text{if } \gamma = (\xi, \mu) \in W_\Gamma^1,$$

$$U_{e,g} \longleftrightarrow g \qquad \text{if } g \in \Gamma(e).$$

4.2. Crossed products of the boundary actions

THEOREM 4.4. *Let Γ be a graph of groups of finite type with infinitely many ends. Then*

$$\mathcal{O}_\Gamma \simeq C(\partial\Gamma) \rtimes \Gamma.$$

PROOF. In this proof, we will confuse the fundamental groups $\Gamma_0 = \pi_1(\Gamma, v_0)$ and $\Gamma_T = \pi_1(\Gamma, T)$ via the isomorphism q_T and the boundaries $\partial\Gamma$ and $\partial\widetilde{X}$. (See Definition 2.1 below and Proposition 2.4.) Let us denote by λ the implementing unitary in $C(\partial\Gamma) \rtimes \Gamma$. We write $p(\gamma) \in C(\partial\widetilde{X})$ for the characteristic function of the set of all infinite normalized paths with beginning of the form γ.

Let $\gamma = (\xi, \mu) \in W_\Gamma^1$. We define the partial isometry $S(\gamma)$ in $C(\partial\widetilde{X}) \rtimes \Gamma_0$ by

$$S(\gamma) = \lambda(\xi_{v_0 \mapsto o(\xi)} \cdot \gamma \cdot \xi_{t(\xi) \mapsto v_0}) \sum_{(r(\xi),e) \in W_\Gamma^0} \sum_g p(\xi_{v_0 \mapsto t(\xi)} g e),$$

where g runs over all elements in $\Delta_e \setminus \{1\}$ if $e = \overline{r(\xi)}$ and in Δ_e if $e \neq \overline{r(\xi)}$. Let $e \in s(W_\Gamma^*)$ and $g \in \Gamma(e)$. We define the partial unitary $U(g)$ in $C(\partial\widetilde{X}) \rtimes \Gamma_0$ by

$$U(g) = \lambda(\xi_{v_0 \mapsto o(e)} \cdot g \cdot \xi_{t(e) \mapsto v_0}) \sum_\alpha S(\alpha) S(\alpha)^*,$$

where α runs over all elements $\alpha = (\eta, \nu) \in N_\Gamma^1$ with $s(\alpha) = e$. Then one can show that $\{S(\gamma), U(g)\}$ is a Cuntz-Krieger Γ-family. Thus there is the canonical $*$-homomorphism from $\mathcal{O}_\Gamma$ to $C(\partial\widetilde{X}) \rtimes \Gamma_0$.

Conversely, to define a unitary $u(g) \in \mathcal{O}_\Gamma$ for $g \in \Gamma_v$ with $v \in \Gamma^0$, we introduce some notations and definitions. Let $g \in \Gamma_v$ for $v \in \Gamma^0$ and $\gamma = (\xi, \mu) \in N_\Gamma^*$ with $\mu = g_0 g_1 \ldots g_n$. Consider the path $g \cdot \gamma$, which is given by the edge path $\xi_{v \mapsto o(\xi)}\xi$ and the sequence $(g, 1, \ldots, 1, g_0, g_1, \ldots, g_n)$ of vertex groups. Note that $g \cdot \gamma$ is reduced, but it may be not in W_Γ^*. If $v \neq o(\gamma)$, then $g \cdot \gamma = \gamma_0 \gamma_1$, where γ_0 is a path $(\xi_0, 1_{\xi_0})$ in T and $\gamma_1 \in W_\Gamma^*$. In this case we define $S_{g \cdot \gamma} = S_{\gamma_1}$. If $v = o(\gamma)$, then $g \cdot \gamma$ may be in $\Gamma_{\overline{r(\gamma)}}$. For any $\alpha \in N_\Gamma^1$ with $\gamma \rightsquigarrow \alpha$, $g \cdot \gamma\alpha = \alpha_0 \alpha_1$, where α_0 is a path $(\eta_0, 1_{\eta_0})$ in T and $\alpha_1 \in W_\Gamma^*$. In this case we define $S_{g \cdot \gamma\alpha} = S_{\alpha_1}$.

Now we define the unitary $u(g) \in \mathcal{O}_\Gamma$ for $g \in \Gamma_v$ with $v \in \Gamma^0$ by

$$u(g) = \sum_{\substack{\gamma \in N_\Gamma^1 \\ v \neq o(\gamma)}} S_{g \cdot \gamma} S_\gamma^* + \sum_{\substack{\gamma \in N_\Gamma^1 \\ v = o(\gamma)}} \sum_{\substack{\alpha \in N_\Gamma^1 \\ \gamma \rightsquigarrow \alpha}} S_{g \cdot \gamma\alpha} S_{\gamma\alpha}^* = \sum_{\gamma \in N_\Gamma^2} S_{g\gamma} S_\gamma^*.$$

Let $e \in \Gamma^1 \setminus T^1$ and $\gamma = (\xi, \mu) \in N_\Gamma^1$ with $\xi = e_1 \ldots e_n$ and $\mu = g_0 g_1 \ldots g_n$. Consider the path $e \cdot \gamma$, which is given by the edge path $e\xi_{t(e) \mapsto o(\xi)}\xi$ and the sequence $(1, \dots, 1, g_0, g_1, \dots, g_n)$ of vertex groups. If $s(\xi) = \overline{e}$ and $g_0 = 1$, then $e \cdot \gamma$ has the form either $\gamma_0\gamma_1$ or ξ', where $\gamma_0 = (\xi_0, 1_{\xi_0})$, $\gamma_1 \in N^1_\Gamma$ and $\xi' = e_2 \dots e_n$ is the reduced edge path in T. In this case we define either $S_{e\cdot\gamma} = S_{\gamma_1}$ or $S_{e\cdot\gamma} = I$, respectively. In other cases, $e \cdot \gamma \in N^*_\gamma$. Then we define $u(e) \in \mathcal{O}_\Gamma$ by

$$u(e) = \sum_{\gamma \in N^1_\Gamma} S_{e\cdot\gamma} S^*_\gamma.$$

One can show that the conjugates of the family $\{P_\gamma\}$ by Γ_0 generates a commutative C^*-algebra. Hence they give a covariant representation of the C^*-dynamical system $(C(\partial\widetilde{X}), \Gamma_0)$. Thus there is the canonical $*$-homomorphism from $C(\partial\widetilde{X}) \rtimes \Gamma_0$ to $\mathcal{O}_\Gamma$. One can check that the above two $*$-homomorphism are mutual inverse $*$-isomorphisms.

REMARK 4.5. We do not need to care about the deference between the full and the reduced crossed product on the above, because of the amenability of the action of Γ on $\partial\Gamma$ (see [1] or Appendix B by E. Germain in [3]). The amenability also follows from next lemma.

The following can be easily proved by using the gauge-invariant uniqueness theorem. (See [15] and [26].)

LEMMA 4.6. *If Γ is a graph of groups of finite type and $\partial\widetilde{X}$ has infinitely many ends, then*

$$\mathcal{O}_\Gamma \simeq \mathcal{F}_\Gamma \rtimes \mathsf{N}.$$

5. K-theory

Let Γ be a graph of groups of finite type with infinitely many ends. In this section, we give a formula of the K-groups of $\mathcal{O}_\Gamma$.

Let $\{\chi^e_a\}_{a\in A_e}$ be the set of characters corresponding with all irreducible unitary representations of the finite groups Γ_e and $\Gamma_{\overline{e}}$ with degrees $\{d_e(a)\}_{a\in A_e}$. For $(e, f) \in W^0_\Gamma$ and $a \in A_e$, we write

$$P_{e,f}(a) = \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_{g\in\Gamma_{\overline{e}}} \overline{\chi^e_a(g)} U_g P_{e,f}.$$

For $\gamma = (\xi, \mu) \in N^1_\Gamma$ with $t(e) = o(\gamma)$ let us denote

$$\Gamma_{\overline{e}}(\gamma) = \{g \in \Gamma_{\overline{e}} \mid g\gamma\Gamma_{t(\gamma)} = \gamma\Gamma_{t(\gamma)}\}.$$

For $g \in \Gamma_{\overline{e}}(\gamma)$, there is $h \in \Gamma_{\overline{r(\gamma)}}$ such that $g\gamma = \gamma h$. We write $F_\gamma(g) = h$. For $\xi \in E^1_\Gamma$ and $(e, s(\xi)) \in W^0_\Gamma$, we choose a set $N(\overline{e}, \xi)$ of double coset representatives for

$$\Gamma_{\overline{e}}\backslash\{\gamma = (\xi, \mu) \mid \mu \in \Gamma_\xi, e\gamma \text{ is reduced}\}/\Gamma_{t(\xi)}$$

such that $N(\overline{e}, \xi) \subset N_\Gamma^1$.

DEFINITION 5.1. We set an index set

$$\Sigma = \left\{(e, f, a) \mid (e, f) \in W_\Gamma^0, a \in A_e\right\}$$

and $m = \sharp\Sigma$. We define the $m \times m$-matrix M with Z-valued entries by

$$M((e, e', a), (f, f', b)) = \begin{cases} \sum_{\gamma \in N(\overline{e},\xi)} \langle \chi_a^e \mid \chi_b^f \circ F_\gamma \rangle_{\Gamma_{\overline{e}}(\gamma)} & \text{if there is } \alpha = (\xi, \mu) \in N_\Gamma^1 \\ & \text{with } s(\xi) = e', r(\xi) = f, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\langle \chi_a^e \mid \chi_b^f \circ F_\gamma \rangle_{\Gamma_{\overline{e}}(\gamma)} = \frac{1}{\sharp\Gamma_{\overline{e}}(\gamma)} \sum_{g \in \Gamma_{\overline{e}}(\gamma)} \overline{\chi_a^e(g)} \chi_b^f(F_\gamma(g)).$$

PROPOSITION 5.2.

$$K_0(\mathscr{F}_\Gamma) = \varinjlim\{M^t : \mathsf{Z}^m \to \mathsf{Z}^m\}.$$

PROOF. Note that

$$\mathscr{F}_n(e, e') \simeq \mathsf{M}_{K_n(e,e')} \otimes C^*(\Gamma_{\overline{e}}) \simeq \bigoplus_{a \in A_e} \mathsf{M}_{K_n(e,e')} \otimes \mathsf{M}_{d_e(a)}.$$

We can express the projection $e_{11} \otimes I$ in $\mathsf{M}_{K_n(e,e')} \otimes \mathsf{M}_{d_e(a)}$ by

$$P = S_\alpha P_{e,e'}(a) S_\alpha^*$$

for some $\alpha \in N_\Gamma^n$, where e_{11} is a minimal projection in the matrix algebras. The unit of $\mathsf{M}_{K_{n+1}(f,f')} \otimes \mathsf{M}_{d_f(b)}$ in $\mathscr{F}_{n+1}(f, f')$ is given by

$$Q = \sum_{\beta \in N_\Gamma^{n+1}} S_\beta P_{f,f'}(b) S_\beta^*.$$

It therefore suffices to compute $\mathrm{Tr}(PQ)/d_e(a)$.

$$\begin{aligned}\frac{\mathrm{Tr}(PQ)}{d_e(a)} &= \frac{1}{\sharp\Gamma_{\overline{e}}}\sum_{g\in\Gamma_{\overline{e}}}\overline{\chi_a^e(g)}\,\mathrm{Tr}\left(S_\alpha U_g P_{e,e'}S_\alpha^* Q\right)\\ &= \frac{1}{\sharp\Gamma_{\overline{e}}}\sum_{\gamma}\sum_{g\in\Gamma_{\overline{e}}(\gamma)}\overline{\chi_a^e(g)}\,\mathrm{Tr}\left(S_{\alpha\gamma}U_{F_\gamma(g)}P_{f,f'}(b)S_{\alpha\gamma}^*\right)\\ &= \frac{1}{\sharp\Gamma_{\overline{e}}}\sum_{\gamma}\sum_{g\in\Gamma_{\overline{e}}(\gamma)}\overline{\chi_a^e(g)}\chi_b^f(F_\gamma(g)), \qquad \ldots(\clubsuit)\end{aligned}$$

where γ runs over all elements in N_Γ^1 such that $s(\gamma) = e', r(\gamma) = f$ and $e\gamma$ is reduced. Note that $F_\gamma(g) = \gamma^{-1}g\gamma$ in the above. By the same argument as in [19, Section 7], we have

$$(\clubsuit) = \sum_{\gamma\in N(\overline{e},\xi)}\langle\chi_a^e \mid \chi_b^f\circ F_\gamma\rangle_{\Gamma_{\overline{e}}(\gamma)}.$$

Therefore we obtain the K-groups of the C^*-algebra $\mathcal{O}_\Gamma$ by the Pimsner-Voiculescu exact sequence [23].

THEOREM 5.3. *Let Γ be a graph of groups of finite type with infinitely many ends and M be as above. Then the K-groups of the C^*-algebra $\mathcal{O}_\Gamma$ are given by*

$$K_0(\mathcal{O}_\Gamma) = \mathsf{Z}^m/(I - M^t)\mathsf{Z}^m,$$
$$K_1(\mathcal{O}_\Gamma) = \mathrm{Ker}\{I - M^t : \mathsf{Z}^m \to \mathsf{Z}^m\}.$$

6. Ideal structure

By the same argument as the Cuntz-Krieger algebras with condition (II) in [8], we can give the ideal structure theorem for $\mathcal{O}_\Gamma$ by using the matrix M, which is obtained in Section 5. Here, we do not need the condition (II) of the matrix M, thanks to our uniqueness theorem (Theorem 3.11).

DEFINITION 6.1 (cf. [8]). We denote by $\mathcal{W}_M = \mathcal{W}_M(\Sigma)$ the set of finite admissible words $(\sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in \Sigma$ and $M(\sigma_i, \sigma_{i+1}) = 1$. For $\sigma, \tau \in \Sigma$, we write $\sigma \geq \tau$ if there is a admissible word $(\sigma_1, \ldots, \sigma_n) \in \mathcal{W}_M$ with $\sigma_1 = \sigma$ and $\sigma_n = \tau$. For $\sigma \in \Sigma$, let us denote $[\sigma] = \{\tau \in \Sigma \mid \sigma \geq \tau \geq \sigma\}$. Notice that the relation $\geq$ is a well-defined partial order on $\widetilde{\Sigma} = \{[\sigma] \mid \sigma \in \Sigma\}$.

A subset $\Lambda \subseteq \widetilde{\Sigma}$ is *hereditary* if $\lambda_1 \in \Lambda$ and $\lambda_1 \geq \lambda_2$ implies $\lambda_2 \in \Lambda$. We put

$$\Sigma(\Lambda) = \left\{\sigma \in \Sigma \;\middle|\; \tau_1 \geq \sigma \geq \tau_2 \text{ for some } \tau_1, \tau_2 \in \bigcup_{\lambda\in\Lambda}\lambda\right\}.$$

The *saturation* $\overline{\Sigma(\Lambda)}$ is the smallest subset Σ' of Σ which contains $\Sigma(\Lambda)$ and is *saturated* in the sense that it contains every element $\sigma \in \Sigma$ for which $M(\sigma, \tau) \neq 0$ implies $\tau \in \Sigma'$.

For any hereditary subset Λ of $\widetilde{\Sigma}$, we denote by $\mathscr{I}_\Lambda$ the two-sided closed ideal of $\mathscr{O}_\Gamma$ generated by $P_{e,f}(a)$ with $(e, f, a) \in \Sigma(\Lambda)$. We write $P_\sigma = P_{e,f}(a)$ when $\sigma = (e, f, a) \in \Sigma$.

LEMMA 6.2 (cf. Lemma 3.1 in[15]). *Let Λ be a hereditary subset of $\widetilde{\Sigma}$. Then*

$$\mathscr{I}_\Lambda = \overline{\text{span}} \left\{ S_{\sigma,\alpha} U_g P_{e,f}(a) S^*_{\tau,\beta} \;\middle|\; \begin{array}{l} \alpha \in W^k_\Gamma, \beta \in W^l_\Gamma, e = s(\alpha) = s(\beta) \\ g \in \Gamma_{\overline{e}}, (e, f, a) \in \overline{\Sigma(\Lambda)} \\ \sigma = (\sigma_1, \dots, \sigma_k), \tau = (\tau_1, \dots, \tau_l) \in \mathscr{W}_M, \\ \quad k, l \in \mathbf{N} \\ \sigma_i = (e_i, f_i, a_i) \in \Sigma, e_i = r(\alpha_{i-1}), f_i = s(\alpha_i) \end{array} \right\},$$

where $S_{\sigma,\alpha} = P_{\sigma_1} S_{\alpha_1} \dots P_{\sigma_k} S_{\alpha_k}$ for $\alpha = \alpha_1 \dots \alpha_n$ with $\alpha_i \in W^1_\Gamma$ for $1 \leq i \leq k$.

PROOF. We first show that if $\sigma \in \overline{\Sigma(\Lambda)}$, then $P_\sigma \in \mathscr{I}_\Lambda$. It suffices to see that the set $\{\sigma \in \Sigma \mid P_\sigma \in \mathscr{I}_\Lambda\}$ is saturated. We take $\sigma = (e, e', a) \in \Sigma$ such that $P_\tau \in \mathscr{I}_\Lambda$ whenever $M(\sigma, \tau) \neq 0$. Then

$$\begin{aligned} P_\sigma &= P_{e,e'}(a) P_{e,e'} \\ &= P_{e,e'}(a) \sum_{\substack{\gamma \in N^1_\Gamma, s(\gamma) = e' \\ e\gamma\,:\,\text{reduced}}} S_\gamma S^*_\gamma \\ &= P_{e,e'}(a) \sum_{\substack{\gamma \in N^1_\Gamma, s(\gamma) = e' \\ e\gamma\,:\,\text{reduced}}} \left(\sum_{\substack{(f,f') \in W^0_\Gamma \\ f = r(\gamma)}} \sum_{b \in A_f} S_\gamma P_{f,f'}(b) S^*_\gamma \right). \end{aligned}$$

Thus it suffices to show that

$$\sum_\alpha P_{e,e'}(a) S_\alpha P_{f,f'}(b) S^*_\alpha \in \mathscr{I}_\Lambda,$$

where α runs over all elements $(\xi, \mu) \in N^1_\Gamma$ such that $e\alpha$ is reduced for each $(f, f') \in W^0_\Gamma, b \in A_f$ and $\xi \in E^1_\Gamma$ with $s(\xi) = e'$ and $r(\xi) = f$. If

$X = \sum_\alpha P_{e,e'}(a) S_\alpha P_{f,f'}(b) S_\alpha^* \neq 0$, then

$$\begin{aligned}\mathrm{Tr}(XX^*) &= \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_\alpha \sum_{g\in\Gamma_{\overline{e}}} \overline{\chi_k^e(g)}\, \mathrm{Tr}(U_g S_\alpha P_{f,f'}(b) S_\alpha^*) \\ &= \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_\alpha \sum_{g\in\Gamma_{\overline{e}}(\alpha)} \overline{\chi_k^e(g)}\, \mathrm{Tr}(S_\alpha U_{F_\alpha(g)} P_{f,f'}(b) S_\alpha^*) \\ &= d_e(a) M(\sigma,\tau) \neq 0,\end{aligned}$$

where $\tau = (f, f', b)$. Hence we have $X \in \mathscr{I}_\Lambda$ and thus $P_\sigma \in \mathscr{I}_\Lambda$.

One can easily check that the right hand on the equation of the statement is a two-sided closed ideal. The proof is complete.

LEMMA 6.3 (cf. Lemma 3.4 in [15]). *The elements*

$$E_n = P_\Lambda + \sum_{\sigma=(\sigma_1,\ldots,\sigma_n)\in\mathscr{W}_\Lambda(\Sigma\setminus\overline{\Sigma(\Lambda)})} \sum_{\gamma\in N_\Gamma^n} S_{\sigma,\gamma} P_H S_{\sigma,\gamma}^*$$

give an approximate unit for $\mathscr{I}_\Lambda$, where

$$P_\Lambda = \sum_{\sigma\in\overline{\Sigma(\Lambda)}} P_\sigma.$$

PROOF. By Lemma 6.2, it suffices to check the lemma for the form $X = S_{\sigma,\gamma} U_g P_{e,f}(a) S_{\tau,\gamma}^*$. One can show that $E_n X = X$ for a sufficiently large n by the same arguments of the proof of [15, Lemma 3.4].

THEOREM 6.4 (cf. Theorem 2.5 in [8] and Theorem 3.5 in [15]). *Let Γ be a graph of groups of finite type with infinitely many ends. Then the map $\Lambda \mapsto \mathscr{I}_\Lambda$ is an inclusion preserving bijection from the set of hereditary subsets of $\widetilde{\Sigma}$ onto the set of two-sided closed ideals of $\mathscr{O}_\Gamma$.*

PROOF. Let $\mathscr{I}$ be a two-sided closed ideal of $\mathscr{O}_\Gamma$ and $C = \{\sigma \in \Sigma \mid P_\sigma \in \mathscr{I}\}$.

We first show that if $\sigma = (e, e', a) \in C$ and $\tau = (f, f', b) \in \Sigma$ with $M(\sigma,\tau) \neq 0$, then $\tau \in C$. We put $X = P_\sigma \sum_{\gamma\in N(\overline{e},\xi)} S_\gamma P_\tau \in \mathscr{I}$ with $s(\xi) = e'$ and $r(\xi) = f$. Then

$$\begin{aligned}\mathrm{Tr}(X^*X) &= \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_{g\in\Gamma_{\overline{e}}} \overline{\chi_a^e(g)}\, \mathrm{Tr}\Big(P_\tau \sum_{\gamma'} S_{\gamma'}^* \cdot U_g \cdot \sum_\gamma S_\gamma P_\tau\Big) \\ &= \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_\gamma \sum_{g\in\Gamma_{\overline{e}}(\gamma)} \overline{\chi_a^e(g)}\, \mathrm{Tr}\left(U_{F_\gamma(g)} P_\tau\right) \\ &= c A_T(\sigma,\tau) \neq 0,\end{aligned}$$

for some non-zero positive constant c. Hence we have

$$X^*X = \frac{d_e(a)}{\sharp\Gamma_{\overline{e}}} \sum_{\gamma} \sum_{g\in\Gamma_{\overline{e}}(\gamma)} U_{F_\gamma(g)} P_\tau \neq 0.$$

Thus $P\tau \in \mathscr{I}$ and $\tau \in C$. One can check that C is saturated by the same argument as in the proof of Lemma 6.2. Therefore C is saturated and contains $\Sigma(\Lambda)$, where $\Lambda = \{[\sigma] \mid \sigma \in C, \sigma \geq \sigma\}$ is a hereditary subset of $\widetilde{\Sigma}$. Moreover we can prove that $C = \overline{\Sigma(\Lambda)}$ as in the same proof of the case of Cuntz-Krieger algebras (cf. [8], [15]).

Now we claim that $\mathscr{I} = \mathscr{I}_\Lambda$. By the same argument of the case of Cuntz-Krieger algebras with Lemma 6.3, we can check that $\mathscr{I}$ and $\mathscr{I}_\Lambda$ contain precisely the same generators $\{P_\sigma \mid \sigma \in \overline{\Sigma(\Lambda)}\} = C$. Then both $\mathcal{O}_\Gamma/\mathscr{I}$ and $\mathcal{O}_\Gamma/\mathscr{I}_\Lambda$ are generated by Cuntz-Krieger Γ-families. Therefore they are canonically $*$-isomorphic by Theorem 3.13. This is only possible if $\mathscr{I} = \mathscr{I}_\Lambda$.

COROLLARY 6.5. *Let Γ be a graph of groups of finite type with infinitely many ends and M be the matrix defined in Definition 5.1. Then the C^*-algebra $\mathcal{O}_\Gamma$ is simple if and only if M is irreducible and not a permutation.*

REMARK 6.6. If $\mathcal{O}_\Gamma$ is simple, then it is purely infinite by [18]. (See also [2].)

7. HNN-extension

We consider the case of HNN-extensions in this section. The fundamental group associated with the graph of Figure 2 in Section 2 is the HNN-extension group

$$\Gamma = G *_H \theta = \langle G, x \mid hx = x\theta(h) \text{ for } h \in H\rangle,$$

where $G = \Gamma_{v_0}$, $H = \iota_x(\Gamma_x)$, $\overline{H} = \iota_{\overline{x}}(\Gamma_{\overline{x}})$ and $\theta : H \to \overline{H}$ given by $\theta = \iota_{\overline{x}} \circ \iota_x^{-1}$. We assume that G is finite. Note that $\partial\Gamma$ has infinitely many points if and only if $\Delta_x \neq \{1\}$. In this case, every element N_Γ^1 has the form either gx or $g\overline{x}$.

Let $\{\chi_a\}_{a=1}^s$ be the set of all characters corresponding with all irreducible unitary representations of the finite groups H and $\overline{H}$. We choose a set $\Delta'_{e,f} \subseteq \Delta_f$ of double coset representatives of $\Gamma_e\backslash\Gamma_{v_0}/\Gamma_f$ and we set $\Delta_{e,f} = \Delta'_{e,f}$ if $e = f$ and $\Delta_{e,f} = \Delta'_{e,f} \setminus \{1\}$ if $e = \overline{f}$. Then we define the matrix $M_\Gamma =$

$[M_{e,f}]_{e,f\in\Gamma^1}$, where $M_{e,f}$ is the $s \times s$-matrix given by

$$M_{x,x}(a,b) = \sum_{g\in\Delta_{\overline{x},x}} \langle \chi_a \mid \chi_b^g \rangle_{\overline{H}(gH)},$$
$$M_{x,\overline{x}}(a,b) = \sum_{g\in\Delta_{\overline{x},\overline{x}}} \langle \chi_a \mid \chi_b^g \rangle_{\overline{H}(g\overline{H})},$$
$$M_{\overline{x},x}(a,b) = \sum_{g\in\Delta_{x,x}} \langle \chi_a \mid \chi_b^g \rangle_{H(gH)},$$
$$M_{\overline{x},\overline{x}}(a,b) = \sum_{g\in\Delta_{x,\overline{x}}} \langle \chi_a \mid \chi_b^g \rangle_{H(g\overline{H})},$$

where $H_1(gH_2)$ is the stabilizers of left multiplication of H_1 on gH_2 for $H_1, H_2 = H$ or $\overline{H}$, and $\chi_b^g(\cdot) = \chi_b(g^{-1}\cdot g)$. By Theorem 6.4, the K-groups of $C(\partial\Gamma) \rtimes \Gamma$ are given as follows.

PROPOSITION 7.1. *Let Γ be a* HNN-*extension $G *_H \theta$ such that $G \neq \{e\}$ and $H \neq G$. Let M_Γ be as above. Then the K-groups of the crossed product $C(\partial\Gamma) \rtimes \Gamma$ are given by*

$$K_0 = \operatorname{coker}(I - M_\Gamma^t),$$
$$K_1 = \ker(I - M_\Gamma^t).$$

We next give a sufficient condition for simplicity of $C(\partial\Gamma) \rtimes \Gamma$.

PROPOSITION 7.2 (cf. Corollary 6.4 in [19]). *Let Γ be the HNN-extension $G *_H \theta$, where $H \subsetneqq G$ are finite groups and θ is an endomorphism from H into G. If the following condition (★) holds, then the C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ is simple and purely infinite.*

$$\bigcap_{g\in G} gHg^{-1} \cap g\overline{H}g^{-1} = \{1\}, \qquad (★).$$

PROOF. It suffices to show that $\{U_h S_\gamma\}_{h\in H}$, $(\{U_{\overline{h}} S_\gamma\}_{\overline{h}\in\overline{H}}$, $\{U_h S_\gamma\}_{h\in H}$, $\{U_{\overline{h}} S_\gamma\}_{\overline{h}\in\overline{H}})$ are mutually orthogonal ranges for some $\gamma = (\xi, \mu)$, (see the proof of [19, Corollary 6.4]). Let $1 \neq g \in \Delta_x$. If $gHg^{-1} \cap H = \{1\}$, then it is enough to set $\xi = x$ and $\mu = (g, 1)$. Now we assume that there is $(1 \neq) h \in gHg^{-1} \cap H$. Namely $h \in gx\overline{H}x^{-1}g^{-1} \cap H$. By the condition (★), there is $g_1 \in G$ such that (i) $x^{-1}g^{-1}hgx \notin g_1 H g_1^{-1}$, or (ii) $\notin g_1 \overline{H} g_1^{-1}$ with $g_1 \notin \overline{H}$. In the case (i), we have $hgxg_1x \notin gxg_1x\overline{H}$. Hence

$h \notin gxg_1x\overline{H}(gxg_1x)^{-1} \cap H$. We put $\gamma_1 = (\xi_1, \mu_1)$ where $\xi_1 = (x, x)$ and $\mu_1 = (g, g_1)$. Then we obtain

$$\gamma_1\overline{H}\gamma_1^{-1} \cap H \subsetneqq gx\overline{H}(gx)^{-1} \cap H.$$

In the case (ii), we obtain $hgxg_1x^{-1} \notin gxg_1x^{-1}H$. Thus $h \notin gxg_1x^{-1}\overline{H}(gxg_1 x^{-1})^{-1} \cap H$. We put $\gamma_1 = (\xi_1, \mu_1)$ where $\xi_1 = (x, \overline{x})$ and $\mu_1 = (g, g_1)$. Then we obtain

$$\gamma_1 H\gamma_1^{-1} \cap H \subsetneqq gx\overline{H}(gx)^{-1} \cap H.$$

Since H is finite, by repeating this argument, we eventually obtain $\gamma = (\xi, \mu)$ such that $\gamma H\gamma^{-1} \cap H = \{1\}$. Hence we have proved that $\{U_h S_\gamma\}_{h\in H}$ are mutually orthogonal ranges. One can also show the other cases. Therefore, thanks to Theorem 3.11, the C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ is simple.

Finally we consider several certain examples of HNN-extensions.

EXAMPLE 7.3. Let $H = \mathsf{Z}_q \subseteq G = \mathsf{Z}_{pq}$ with $p \neq 1$ and $\theta = \mathrm{id}$, where $\mathsf{Z}_n = \mathsf{Z}/n\mathsf{Z}$ for positive integer n. We consider the HNN-extension $\Gamma = \mathsf{Z}_{pq} *_{\mathsf{Z}_q} \mathrm{id}$. Then the corresponding C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ is $*$-isomorphic to

$$\underbrace{\mathcal{O}_A \oplus \cdots \oplus \mathcal{O}_A}_{q \text{ times}},$$

where $\mathcal{O}_A$ is the simple Cuntz-Krieger algebra associated with the $(p+1) \times (p+1)$-matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

More generally, let H be a finite group, $G = \mathsf{Z}_p \times H$, θ the natural inclusion of H into G and Γ the corresponding HNN-extension. Then the C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ is $*$-isomorphic to

$$\mathcal{O}_A \otimes C^*(H).$$

EXAMPLE 7.4. We put $G = \mathsf{Z}_2 \times \mathsf{Z}_2 = H_l \times H_r$. Let Γ_2 be the HNN-extension given by two inclusions of Z_2 into $H_l, H_r \subseteq G$. Then $C(\partial\Gamma_2) \rtimes \Gamma_2$ is a purely infinite simple unital nuclear C^*-algebra, which is called a unital *Kirchberg algebra*, with $K_0 = K_1 = \mathsf{Z}^2$ and satisfies the universal coefficient theorem (UCT). By the classification theorem of E. Kirchberg and N. C. Phillips [20], we obtain

$$C(\partial\Gamma_2) \rtimes \Gamma_2 \simeq C(\partial\mathsf{F}_2) \rtimes \mathsf{F}_2,$$

where F_2 is the free group with two generators. More generally, for $n \geq 3$, let $G = \mathsf{Z}_n \times \mathsf{Z}_n$ with inclusions of Z_n into the left and right hand sides of G. We denote by Γ_n the corresponding HNN-extension. Then the K-groups of $C(\partial\Gamma_n) \rtimes \Gamma_n$ are $K_0 = \mathsf{Z}_{n-2} \oplus \mathsf{Z}^{n-1}$ and $K_1 = \mathsf{Z}^{n-1}$. Again by the classification theorem, we moreover obtain

$$C(\partial\Gamma_3) \rtimes \Gamma_3 \simeq C(\partial\Gamma_2) \rtimes \Gamma_2 \simeq C(\partial\mathsf{F}_2) \rtimes \mathsf{F}_2.$$

When $n > 3$, the C^*-algebra $C(\partial\Gamma_n) \rtimes \Gamma_n$ is *the* unital Kirchberg algebra satisfying the UCT with $(K_0, [1]_0, K_1) = (\mathsf{Z}_{n-2} \oplus \mathsf{Z}^{n-1}, 0, \mathsf{Z}^{n-1})$. Notice that $C(\partial\Gamma_n) \rtimes \Gamma_n$ is stably $*$-isomorphic to $C(\partial\mathsf{F}_{n-1}) \rtimes \mathsf{F}_{n-1}$, i.e.,

$$(C(\partial\Gamma_n) \rtimes \Gamma_n) \otimes \mathsf{K} \simeq (C(\partial\mathsf{F}_{n-1}) \rtimes \mathsf{F}_{n-1}) \otimes \mathsf{K},$$

where K is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space, but they are not $*$-isomorphic.

EXAMPLE 7.5. Let $\Gamma = \mathfrak{S}_4 *_{\mathfrak{S}_3} \mathrm{id}$, where $\mathfrak{S}_4$, $\mathfrak{S}_3$ are the symmetric groups. Then one can check that $C^*(\partial\Gamma) \rtimes \Gamma$ is *the* unital Kirchberg algebra satisfying the UCT with $(K_0, [1]_0, K_1) = (\mathsf{Z}^4, 0, \mathsf{Z}^4)$.

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