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CONSTRUCTION AND PURE INFINITENESS OF C^*-ALGEBRAS ASSOCIATED WITH LAMBDA-GRAPH SYSTEMS

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Abstract

A λ-graph system is a labeled Bratteli diagram with shift transformation. It is a generalization of finite labeled graphs and presents a subshift. In [16] the author has introduced a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ by using groupoid method as a generalization of the Cuntz-Krieger algebras. In this paper, we concretely construct the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ by using both creation operators and projections on a sub Fock Hilbert space associated with $\mathfrak{L}$. We also introduce a new irreducible condition on $\mathfrak{L}$ under which the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ becomes simple and purely infinite.

0. Introduction

For a finite set Σ, a subshift (Λ, σ) is a topological dynamics defined by a closed shift-invariant subset Λ of the compact set Σ^{Z} of all bi-infinite sequences of Σ with shift transformation σ defined by $\sigma((x_i)_{i\in\mathsf{Z}}) = (x_{i+1})_{i\in\mathsf{Z}}$. The author has introduced the notions of symbolic matrix system and λ-graph system as presentations of subshifts ([15]). They are generalized notions of symbolic matrix and λ-graph (= labeled graph) for sofic subshifts. We henceforth denote by Z_+ and by N the set of all nonnegative integers and the set of all positive integers respectively. A symbolic matrix system $(\mathcal{M}, I)$ over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1})$, $l \in \mathsf{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. They satisfy the following commutation relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathsf{Z}_+.$$

It is required that each row of $I_{l,l+1}$ has at least one 1 and each column of $I_{l,l+1}$ has exactly one 1. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota\ (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathsf{Z}_+$. It naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The labeled edges from a vertex $v_i^l \in V_l$ to a vertex $v_j^{l+1} \in V_{l+1}$ are given by

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the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of $\mathcal{M}_{l,l+1}$. The map ι $(= \iota_{l,l+1})$ is defined by $\iota_{l,l+1}(v_j^{l+1}) = v_i^l$ precisely if $I_{l,l+1}(i, j) = 1$. The symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts by gathering label sequences appearing in the labeled Bratteli diagrams of the λ-graph systems. Conversely we have a canonical method to construct a symbolic matrix system and a λ-graph system from an arbitrary subshift [15].

In [16], the author has constructed C^*-algebras from λ-graph systems as groupoid C^*-algebras by using continuous graphs in the sense of Deaconu (cf. [5], [19]) and studied their structure. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. Let $\{v_1^l, \dots, v_{m(l)}^l\}$ be the vertex set V_l. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is generated by partial isometries S_α corresponding to the symbols $\alpha \in \Sigma$ and projections E_i^l corresponding to the vertices $v_i^l \in V_l$, $i = 1, \dots, m(l)$, $l \in \mathbb{Z}_+$. It is realized as a universal unique C^*-algebra subject to certain operator relations among $S_\alpha, \alpha \in \Sigma$ and E_i^l, $i = 1, \dots, m(l)$, $l \in \mathbb{Z}_+$ encoded by the structure of $\mathfrak{L}$. A condition on $\mathfrak{L}$, called condition (I), has been introduced ([16]). Irreducibility and aperiodicity for $\mathfrak{L}$ have been also defined so that if $\mathfrak{L}$ satisfies condition (I) and is irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is shown to be simple. It is also proved that if in particular $\mathfrak{L}$ is aperiodic, $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([16, Theorem 4.7 and Proposition 4.9]).

In this paper, we will first introduce a new construction of the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$. We will construct a sub Fock Hilbert space associated with a λ graph system $\mathfrak{L}$ and define creation operators and sequence of projections. We will then show that $\mathcal{O}_{\mathfrak{L}}$ is canonically isomorphic to the quotient C^*-algebra of the C^*-algebra generated by the creation operators and the projections by an ideal (Theorem 2.6). This construction is a generalization of a construction of Cuntz-Krieger algebras [3] by [7], [8] and C^*-algebras associated with subshifts [14]. We will next introduce a new irreducible condition and new condition (I) on $\mathfrak{L}$ such that $\mathcal{O}_{\mathfrak{L}}$ becomes simple and purely infinite (Theorem 3.9). The new conditions are called λ-irreducible condition and λ-condition (I) respectively. In the previously proved result [16, Theorem 4.7 and Proposition 4.9], we needed aperiodicity condition on $\mathfrak{L}$ for $\mathcal{O}_{\mathfrak{L}}$ to be simple and purely infinite. It is well-known that the Cuntz-Krieger algebra $\mathcal{O}_A$ is simple and purely infinite if the matrix A is irreducible with condition (I). Since the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ are a generalization of the Cuntz-Krieger algebras $\mathcal{O}_A$, the aperiodicity condition on $\mathfrak{L}$ is too strong such that $\mathcal{O}_{\mathfrak{L}}$ becomes simple and purely infinite. From this point of view, the λ-irreducible condition with λ-condition (I) on $\mathfrak{L}$ is an exact generalization of the irreducible condition with condition (I) on the nonnegative matrices A.

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1. Review of C^*-algebras associated with λ-graph systems

For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, the vertex sets V_l, $l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}$, $l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1}. Every vertex in V has outgoing edges and every vertex in V, except V_0, has incoming edges. The label of an edge $e \in E$ means $\lambda(e) \in \Sigma$. It is then required that there exists an edge in $E_{l,l+1}$ with label α and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and its terminal is $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$\begin{aligned} E^{\iota}(u, v) &= \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},\\ E_{\iota}(u, v) &= \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}. \end{aligned}$$

Then there exists a bijective correspondence between $E^{\iota}(u, v)$ and $E_{\iota}(u, v)$ that preserves labels for every pair $(u, v) \in V_{l-1} \times V_{l+1}$. This property is called the local property of the λ-graph system. A finite sequence $(e_1, e_2, \dots, e_n)$ of edges such that $t(e_i) = s(e_{i+1})$, $i = 1, 2, \dots, n-1$ is called a path. We put $\Sigma_i = \Sigma$ and define

$$\Lambda_{\mathfrak{L}}^{+} = \left\{(\lambda(e_1), \lambda(e_2), \dots) \in \prod_{i \in \mathbb{N}} \Sigma_i \mid e_i \in E_{i-1,i}, t(e_i) = s(e_{i+1}),\ i \in \mathbb{N}\right\}$$

and

$$\Lambda_{\mathfrak{L}} = \left\{(\alpha_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \Sigma_i \mid (\alpha_i, \alpha_{i+1}, \dots) \in \Lambda_{\mathfrak{L}}^{+},\ i \in \mathbb{Z}\right\}.$$

Then $\Lambda_{\mathfrak{L}}$ is a subshift over Σ called the subshift presented by $\mathfrak{L}$. A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of $\mu_j \in \Sigma$ that appears in $\Lambda_{\mathfrak{L}}$ is called an admissible word of $\mathfrak{L}$ of length $|\mu| = k$. Denote by $\Lambda_{\mathfrak{L}}^{k}$ the set of all admissible words of length k of $\mathfrak{L}$ and put $\Lambda_{\mathfrak{L}}^{*} = \cup_{k=0}^{\infty} \Lambda_{\mathfrak{L}}^{k}$ where $\Lambda_{\mathfrak{L}}^{0}$ denotes the empty word $\emptyset$.

We briefly review the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$, that has been originally constructed in [16] to be a groupoid C^*-algebra of a groupoid of a continuous graph obtained by $\mathfrak{L}$ (cf. [5], [6], [19]).

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving λ-graph system over Σ, that is, for $e, e' \in E$, $\lambda(e) = \lambda(e')$, $t(e) = t(e')$ implies $e = e'$. The vertex set V_l is denoted by $\{v_1^l, \dots, v_{m(l)}^l\}$. Define the transition matrices $A_{l,l+1}$, $I_{l,l+1}$ of $\mathfrak{L}$ by setting for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, $\alpha \in \Sigma$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1},\\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l,\\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as the universal unital C^*-algebra generated by partial isometries S_α, $\alpha \in \Sigma$ and projections E_i^l, $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathfrak{L})$:

$$\sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1, \tag{1.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1}, \tag{1.2}$$

$$S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \tag{1.3}$$

$$S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \beta, j) E_j^{l+1}, \tag{1.4}$$

for $\beta \in \Sigma$, $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$.

For a vertex $v_i^l \in V_l$, we denote by $\Gamma^+(v_i^l)$ the set

$$\Gamma^+(v_i^l) = \left\{(\lambda(e_1), \lambda(e_2), \dots) \in \Lambda_{\mathfrak{L}}^+ \mid s(e_1) = v_i^l, t(e_j) = s(e_{j+1}), j \in \mathbb{N}\right\}$$

of all infinite label sequences in $\mathfrak{L}$ starting at v_i^l. We say that $\mathfrak{L}$ satisfies condition (I) if for each $v_i^l \in V$, the set $\Gamma^+(v_i^l)$ contains at least two distinct label sequences.

THEOREM 1.1 ([16]). *Suppose that $\mathfrak{L}$ satisfies condition (I). Let $\widehat{S}_\alpha$, $\alpha \in \Sigma$ and $\widehat{E}_i^l$, $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$ be another family of nonzero partial isometries and nonzero projections satisfying the relations* $(\mathfrak{L})$. *Then the map $S_\alpha \to \widehat{S}_\alpha$, $E_i^l \to \widehat{E}_i^l$ extends to an isomorphism from $\mathcal{O}_{\mathfrak{L}}$ onto the C^*-algebra $\widehat{\mathcal{O}}_{\mathfrak{L}}$ generated by $\widehat{S}_\alpha$, $\alpha \in \Sigma$ and $\widehat{E}_i^l$, $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$.*

Hence the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ under the condition that $\mathfrak{L}$ satisfies condition (I) is the unique C^*-algebra subject to the above relations $(\mathfrak{L})$. By the uniqueness of $\mathcal{O}_{\mathfrak{L}}$, the correspondence $S_\alpha \to zS_\alpha$, $E_i^l \to E_i^l$ for $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ yields an action $\alpha_{\mathfrak{L}}$ of $\mathbb{T}$ called the gauge action. Let $\mathcal{F}_k^l$ be the finite dimensional C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by $S_\mu E_i^l S_\nu^*$, $\mu, \nu \in \Lambda_{\mathfrak{L}}^k$, $i = 1, 2, \dots, m(l)$. Let $\mathcal{F}_{\mathfrak{L}}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the algebras $\mathcal{F}_k^l$, $k \le l$. It is an AF-algebra realized as the fixed point algebra $\mathcal{O}_{\mathfrak{L}}^{\alpha_{\mathfrak{L}}}$ of $\mathcal{O}_{\mathfrak{L}}$ under $\alpha_{\mathfrak{L}}$.

A λ-graph system $\mathfrak{L}$ is said to be *irreducible* if for a vertex $v_i^l \in V_l$ and a sequence $(u^0, u^1, \dots)$ of vertices $u^n \in V_n$ with $\iota_{n,n+1}(u^{n+1}) = u^n$, $n \in \mathbb{Z}_+$, there exists a path starting at v_i^l and terminating at u^{l+N} for some $N \in \mathbb{N}$. $\mathfrak{L}$ is said to be *aperiodic* if for a vertex $v_i^l \in V_l$ there exists an $N \in \mathbb{N}$ such that there exist paths starting at v_i^l and terminating at all vertices of V_{l+N}. These properties for λ-graph systems are generalizations of the corresponding properties for finite directed graphs.

THEOREM 1.2 ([16], Proposition 4.9). *Suppose that a λ-graph system $\mathfrak{L}$ satisfies condition (I). If $\mathfrak{L}$ is irreducible, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple. If in particular $\mathfrak{L}$ is aperiodic, $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite.*

In what follows, we fix a left-resolving λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ.

2. Fock space construction

In this section, we will construct a family of partial isometries and projections satisfying the relations ($\mathfrak{L}$) in a concrete way. Let $\Omega_{\mathfrak{L}}$ be the projective limit

$$\Omega_{\mathfrak{L}} = \left\{ (u^l)_{l \in \mathsf{Z}_+} \in \prod_{l \in \mathsf{Z}_+} V_l \mid \iota_{l,l+1}(u^{l+1}) = u^l, l \in \mathsf{Z}_+ \right\}$$

of the system $\iota_{l,l+1} : V_{l+1} \to V_l$, $l \in \mathsf{Z}_+$. We endow $\Omega_{\mathfrak{L}}$ with the projective limit topology from the discrete topologies on $V_l, l \in \mathsf{Z}_+$ so that it is a compact Hausdorff space. An element u in $\Omega_{\mathfrak{L}}$ is called a vertex. Let $E_{\mathfrak{L}}$ be the set of all triplets $(u, \alpha, w) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ such that there exists $e_{l,l+1} \in E_{l,l+1}$ satisfying $u^l = s(e_{l,l+1})$, $w^{l+1} = t(e_{l,l+1})$ and $\alpha = \lambda(e_{l,l+1})$ for each $l \in \mathsf{Z}_+$ where $u = (u^l)_{l \in \mathsf{Z}_+}$, $w = (w^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}$. The set $E_{\mathfrak{L}} \subset \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ is a continuous graph in the sense of Deaconu ([14, Proposition 2.1]). For $w = (w^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}$ and $\alpha \in \Sigma$, the local property of $\mathfrak{L}$ ensures that if there exists $e_{0,1} \in E_{0,1}$ satisfying $w^1 = t(e_{0,1})$, $\alpha = \lambda(e_{0,1})$, there uniquely exist $e_{l,l+1} \in E_{l,l+1}$ and $u = (u^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}$ satisfying $u^l = s(e_{l,l+1})$, $w^{l+1} = t(e_{l,l+1})$, $\alpha = \lambda(e_{l,l+1})$ for all $l \in \mathsf{Z}_+$. Hence for every $w \in \Omega_{\mathfrak{L}}$, there exist $\alpha \in \Sigma$ and $u \in \Omega_{\mathfrak{L}}$ such that $(u, \alpha, w) \in E_{\mathfrak{L}}$. Let us consider the finite path spaces of the graph $E_{\mathfrak{L}}$ as follows:

$$\begin{aligned}
W_{\mathfrak{L}}^0 &= \Omega_{\mathfrak{L}}, \\
W_{\mathfrak{L}}^1 &= E_{\mathfrak{L}}, \\
W_{\mathfrak{L}}^2 &= \{(u_0, \alpha_1, u_1, \alpha_2, u_2) \mid (u_0, \alpha_1, u_1), (u_1, \alpha_2, u_2) \in E_{\mathfrak{L}}\}, \\
&\dots \\
W_{\mathfrak{L}}^k &= \{(u_0, \alpha_1, u_1, \alpha_2, \dots, \alpha_k, u_k) \mid (u_{i-1}, \alpha_i, u_i) \in E_{\mathfrak{L}}, i = 1, 2, \dots, k\}, \\
&\dots
\end{aligned}$$

We assign to a finite path $\eta \in W_{\mathfrak{L}}^k$ the vector e_η. For each $k \in \mathsf{Z}_+$, let $\mathfrak{F}_{\mathfrak{L}}^k$ be the Hilbert space spanned by the complete orthonomal basis $\{e_\eta \mid \eta \in W_{\mathfrak{L}}^k\}$. The Hilbert space $\mathfrak{H}_{\mathfrak{L}}$ is defined by their direct sums

$$\mathfrak{H}_{\mathfrak{L}} = \oplus_{k=0}^{\infty} \mathfrak{F}_{\mathfrak{L}}^k.$$

We define creation operators T_β for $\beta \in \Sigma$ and projections P_i^l for $v_i^l \in V$ on $\mathfrak{H}_{\mathfrak{L}}$ by setting

$$T_\beta e_{(u_0,\alpha_1,u_1,\alpha_2,\dots,\alpha_k,u_k)} = \begin{cases} e_{(u_{-1},\beta,u_0,\alpha_1,u_1,\alpha_2,\dots,\alpha_k,u_k)} & \text{if there exists } u_{-1} \in \Omega_{\mathfrak{L}} \\ & \text{such that } (u_{-1},\beta,u_0) \in E_{\mathfrak{L}}, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_i^l e_{(u_0,\alpha_1,u_1,\alpha_2,\dots,\alpha_k,u_k)} = \begin{cases} e_{(u_0,\alpha_1,u_1,\alpha_2,\dots,\alpha_k,u_k)} & \text{if } u_0^l = v_i^l, \text{ where} \\ & u_0 = (u_0^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the vertex $u_{-1} \in \Omega_{\mathfrak{L}}$ satisfying $(u_{-1}, \beta, u_0) \in E_{\mathfrak{L}}$ is unique for β and u_0 if it exists, because $\mathfrak{L}$ is left-resolving. It is direct to see that

$$T_\beta^* e_{(u_0,\alpha_1,u_1,\alpha_2,\dots,\alpha_k,u_k)} = \begin{cases} e_{(u_1,\alpha_2,\dots,\alpha_k,u_k)} & \text{if } k \geq 1 \text{ and } \alpha_1 = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.1. *For $\beta \in \Sigma$*

(i) *$T_\beta T_\beta^*$ is the projection onto the subspace spanned by the vectors e_η such that $\eta = (u_0, \alpha_1, u_1, \alpha_2, \dots, \alpha_k, u_k) \in W_{\mathfrak{L}}^k$, $\alpha_1 = \beta$, $k \in \mathsf{N}$,*

(ii) *$T_\beta^* T_\beta$ is the projection onto the subspace spanned by the vectors e_ξ such that $\xi = (u_0, \alpha_1, u_1, \alpha_2, \dots, \alpha_k, u_k) \in W_{\mathfrak{L}}^k$, $k \in \mathsf{Z}_+$, $(u_{-1}, \beta, u_0) \in E_{\mathfrak{L}}$ for some $u_{-1} \in \Omega_{\mathfrak{L}}$.*

Let P_0 denote the projection on $\mathfrak{H}_{\mathfrak{L}}$ onto the subspace $\mathfrak{F}_{\mathfrak{L}}^0$. It is immediate to see that $P_0 T_\beta = 0$ for $\beta \in \Sigma$ and $P_0 P_i^l = P_i^l P_0$ for $v_i^l \in V$. We then have

LEMMA 2.2.

$$\sum_{\alpha \in \Sigma} T_\alpha T_\alpha^* + P_0 = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} P_i^l = 1, \qquad P_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) P_j^{l+1}, \tag{2.2}$$

$$T_\beta T_\beta^* P_i^l = P_i^l T_\beta T_\beta^*, \tag{2.3}$$

$$T_\beta^* P_i^l T_\beta = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\beta,j) P_j^{l+1}, \tag{2.4}$$

for $\beta \in \Sigma$, $i = 1, 2, \dots, m(l)$, $l \in \mathsf{Z}_+$.

PROOF. We will show the relation (2.4). Other relations are direct. For $\beta \in \Sigma$, $v_i^l \in V$, $(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) \in W_{\mathfrak{L}}^k$, it follows that

$$
\begin{aligned}
&T_\beta^* P_i^l T_\beta e_{(u_0,\alpha_1,u_1,\alpha_2,\ldots,\alpha_k,u_k)} \\
&= \begin{cases} e_{(u_0,\alpha_1,u_1,\alpha_2,\ldots,\alpha_k,u_k)} & \text{if } (u_{-1}, \beta, u_0) \in E_{\mathfrak{L}} \text{ for some } u_{-1} \in \Omega_{\mathfrak{L}} \\ & \text{and } u_{-1}^l = v_i^l \text{ where } u_{-1} = (u_{-1}^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} e_{(u_0,\alpha_1,u_1,\alpha_2,\ldots,\alpha_k,u_k)} & \text{if } s(e) = v_i^l,\ t(e) = u_0^{l+1},\ \lambda(e) = \beta \\ & \text{for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \beta, j) P_j^{l+1} e_{(u_0,\alpha_1,u_1,\alpha_2,\ldots,\alpha_k,u_k)}.
\end{aligned}
$$

Hence the relation (2.4) holds.

For a word $\nu = \alpha_1 \cdots \alpha_k \in \Lambda_{\mathfrak{L}}^*$, we set $T_\nu = T_{\alpha_1} \cdots T_{\alpha_k}$.

LEMMA 2.3. *Every polynomial of T_α, P_i^l, $\alpha \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathsf{Z}_+$ is a finite linear combination of elements of the form $T_\mu P_i^l T_\nu^*$ for $\mu, \nu \in \Lambda_{\mathfrak{L}}^*$, $i = 1, 2, \ldots, m(l)$, $l \in \mathsf{Z}_+$.*

PROOF. It follows that by (2.3) and (2.4)

$$P_i^l T_\alpha = T_\alpha T_\alpha^* P_i^l T_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) T_\alpha P_j^{l+1}$$

and hence

$$T_\alpha^* P_i^l = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) P_j^{l+1} T_\alpha^*.$$

The assertion is immediately seen by these equations.

Let $\mathscr{T}_{\mathfrak{L}}$ be the C^*-algebra on $\mathfrak{H}_{\mathfrak{L}}$ generated by T_α, P_i^l, P_0, $\alpha \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathsf{Z}_+$ and $\mathscr{I}$ the closed two-sided ideal of $\mathscr{T}_{\mathfrak{L}}$ generated by P_0.

LEMMA 2.4. *$\mathscr{I}$ is the closure of the algebra of all finite linear combinations of elements of the form $T_\mu P_i^l P_0 T_\nu^*$ for $\mu, \nu \in \Lambda_{\mathfrak{L}}^*$, $i = 1, 2, \ldots, m(l)$, $l \in \mathsf{Z}_+$.*

PROOF. Since $P_0 T_\beta = 0$, one sees $T_\mu P_i^l T_\nu^* P_0 = P_0 T_\mu P_i^l T_\nu^* = 0$. As the algebra $\mathscr{T}_{\mathfrak{L}}$ is generated by elements of the form $T_\mu P_i^l T_\nu^*$ and P_0, by using the relation $P_0P_i^l = P_i^l P_0$, $\mathscr{T}_{\mathfrak{L}}$ is the closure of the algebra of all linear combinations of elements of the forms $T_\mu P_i^l P_0 T_\nu^*$ and $T_\mu P_i^l T_\nu^*$. Since $\mathscr{I} = \overline{\mathscr{T}_{\mathfrak{L}} P_0 \mathscr{T}_{\mathfrak{L}}}$, one concludes that $\mathscr{I}$ is the closure of the algebra of all finite linear combinations of elements of the form $T_\mu P_i^l P_0 T_\nu^*$.

LEMMA 2.5. $T_\beta, P_i^l \notin \mathscr{I}$.

PROOF. Suppose $T_\beta \in \mathscr{I}$. By Lemma 2.4, there exists a finite linear combination $X = \sum_{\mu,\nu,i,l} c_{\mu,\nu,i,l} T_\mu P_i^l P_0 T_\nu^*$ of $T_\mu P_i^l P_0 T_\nu^*$, $\mu, \nu \in \Lambda_{\mathfrak{L}}^*$, $i = 1, 2, \dots, m(l)$, $l \in \mathsf{Z}_+$ such that $\|X - T_\beta\| < \frac{1}{2}$. Let K denote the maximum length of the words ν that appear in the element $\sum_{\mu,\nu,i,l} c_{\mu,\nu,i,l} T_\mu P_i^l P_0 T_\nu^*$. Take a finite path $\xi = (u_0, \alpha_1, u_1, \alpha_2, \dots, \alpha_{K+1}, u_{K+1}) \in W_{\mathfrak{L}}^{K+1}$ such that there exists a vertex $u_{-1} \in \Omega_{\mathfrak{L}}$ satisfying $(u_{-1}, \beta, u_0) \in E_{\mathfrak{L}}$. We have $Xe_\xi = 0$ and $T_\beta e_\xi = e_{(u_{-1},\beta,u_0,\dots,\alpha_{K+1},u_{K+1})}$ so that

$$\|(X - T_\beta)e_\xi\| = \|e_{(u_{-1},\beta,u_0,\dots,\alpha_{K+1},u_{K+1})}\| = 1,$$

a contradiction.

Suppose next $P_i^l \in \mathscr{I}$. There exists similarly an element $Y = \sum_{\mu,\nu,i,l} c_{\mu,\nu,i,l} T_\mu P_i^l P_0 T_\nu^*$ such that $\|Y - P_i^l\| < \frac{1}{2}$. Take a finite path $\eta = (u_0, \alpha_1, u_1, \alpha_2, \dots, \alpha_{K+1}, u_{K+1}) \in W_{\mathfrak{L}}^{K+1}$ such that $u_0^l = v_i^l$, where $u_0 = (u_0^l)_{l \in \mathsf{Z}_+} \in \Omega_{\mathfrak{L}}$ so that $Ye_\eta = 0$ and $P_i^l e_\eta = e_\eta$ a contradiction.

DEFINITION. Let $\widehat{\mathscr{O}}_{\mathfrak{L}}$ be the quotient C^*-algebra $\mathscr{T}_{\mathfrak{L}}/\mathscr{I}$ of $\mathscr{T}_{\mathfrak{L}}$ by the ideal $\mathscr{I}$, and the operators $\widehat{S}_\alpha$ and $\widehat{E}_i^l$ the quotient images of T_α and P_i^l in $\widehat{\mathscr{O}}_{\mathfrak{L}}$ respectively.

By Lemma 2.5, the elements $\widehat{S}_\alpha$ and $\widehat{E}_i^l$ are not zeros for each $\alpha \in \Sigma$ and $v_i^l \in V$, and satisfy the relations $(\mathfrak{L})$ by Lemma 2.2. Thus by Theorem 1.1 we obtain

THEOREM 2.6. *Suppose that $\mathfrak{L}$ satisfies condition (I). Then the C^*-algebra $\widehat{\mathscr{O}}_{\mathfrak{L}}$ is canonically isomorphic to the C^*-algebra $\mathscr{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$.*

Define a unitary representation U of the circle group T on the Hilbert space $\mathfrak{H}_{\mathfrak{L}}$ by $U_z e_\eta = z^k e_\eta$ for $\eta \in W_{\mathfrak{L}}^k$. It is easy to see that the automorphisms $Ad(U_z)$, $z \in \mathsf{T}$ on the algebra of all bounded linear operators on $\mathfrak{H}_{\mathfrak{L}}$ leave invariant globally both the algebras $\mathscr{T}_{\mathfrak{L}}$ and $\mathscr{I}$. They give rise to an action on the C^*-algebra $\widehat{\mathscr{O}}_{\mathfrak{L}}$ that is the gauge action $\alpha_{\mathfrak{L}}$ on $\mathscr{O}_{\mathfrak{L}}$.

This construction of the C^*-algebra $\widehat{\mathscr{O}}_{\mathfrak{L}}$ is inspired by the construction of the C^*-algebras of Hilbert C^*-bimodules by [18] and [10] (cf. [9]). Our constructoin can work for the construction of the C^*-algebras of general continuous graphs of Deaconu [5].

3. λ-irreducibility and pure infiniteness

As in Section 1, it has been proved in [16] that if $\mathfrak{L}$ is aperiodic, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ becomes simple and purely infinite. The aperiodic condition on $\mathfrak{L}$ however is too strong such that the algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite. In fact, the Cuntz-Krieger algebra $\mathcal{O}_A$ is simple and purely infinite if the matrix A is irreducible with condition (I). In this section, we introduce a new irreducible condition along with a new condition (I) on $\mathfrak{L}$ under which the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite. The new conditions are called λ-irreducible condition and λ-condition (I) respectively. They are exact generalization of the corresponding conditions on a finite square matrix A with entries in $\{0, 1\}$.

DEFINITION. A λ-graph system $\mathfrak{L}$ is *λ-irreducible* if for an ordered pair of vertices $v_i^l, v_j^l \in V_l$, there exists a number $L_l(i, j) \in \mathsf{N}$ such that for a vertex $v_h^{l+L_l(i,j)} \in V_{l+L_l(i,j)}$ with $\iota^{L_l(i,j)}(v_h^{l+L_l(i,j)}) = v_i^l$, there exists a path γ in $\mathfrak{L}$ such that

$$s(\gamma) = v_j^l, \qquad t(\gamma) = v_h^{l+L_l(i,j)},$$

where $\iota^{L_l(i,j)}$ means the $L_l(i, j)$-times compositions of ι, and $s(\gamma)$, $t(\gamma)$ denote the source vertex, the terminal vertex of γ respectively. It is obvious that if $\mathfrak{L}$ is λ-irreducible, then it is irreducible in the sense of Section 1. Let G be a finite directed graph and $\mathfrak{L}_G$ the associated λ-graph system defined in [16, Section 7]. It is then immediate that G is irreducible if and only if $\mathfrak{L}_G$ is λ-irreducible.

The following lemma is direct from the local property of λ-graph system.

LEMMA 3.1. *Suppose that a λ-graph system $\mathfrak{L}$ is λ-irreducible. For a vertex $v_i^l \in V_l$, let L be the number $L_l(i, i)$ as in the definition of λ-irreducible for the pair (v_i^l, v_i^l).*

(i) *For a number $k \in \mathsf{N}$ and a vertex $v_j^{l+kL} \in V_{l+kL}$ with $\iota^{kL}(v_j^{l+kL}) = v_i^l$, there exists a path π in $\mathfrak{L}$ such that $s(\pi) = v_i^l$ and $t(\pi) = v_j^{l+kL}$.*

(ii) *If every path π in $\mathfrak{L}$ of length L with $s(\pi) = v_i^l$ must satisfy $\iota^L(t(\pi)) = v_i^l$, then every path γ in $\mathfrak{L}$ of length kL for some $k \in \mathsf{N}$ with $s(\gamma) = v_i^l$ must satisfy $\iota^{kL}(t(\gamma)) = v_i^l$.*

We will introduce λ-condition (I).

DEFINITION. A λ-graph system $\mathfrak{L}$ is said to satisfy *λ-condition* (I) if for a vertex $v_i^l \in V_l$ there exist two distinct paths γ_1, γ_2 in $\mathfrak{L}$ such that

$$s(\gamma_1) = s(\gamma_2) = v_i^l, \qquad t(\gamma_1) = t(\gamma_2), \qquad \lambda(\gamma_1) \neq \lambda(\gamma_2).$$

It is obvious that if $\mathfrak{L}$ satisfies λ-condition (I), it satisfies condition (I) in the sense of Section 1. One immediately sees that the adjacency matrix of a finite directed graph G satisfies condition (I) in the sense of Cuntz-Krieger [3] if and only if $\mathfrak{L}_G$ satisfies λ-condition (I).

Let $A_{l,l+1}, I_{l,l+1}$ be the transition matrices of $\mathfrak{L}$ as in Section 1. Define the matrices $A_{l,l+k}, I_{l,l+k}$ for $k \in \mathsf{N}$ by setting for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+k)$, $\mu \in \Lambda_{\mathfrak{L}}^k$,

$$A_{l,l+k}(i, \mu, j) = \begin{cases} 1 & \text{if } s(\gamma) = v_i^l, \lambda(\gamma) = \mu, t(\gamma) = v_j^{l+k} \\ & \text{for some path } \gamma \text{ in } \mathfrak{L}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+k}(i, j) = \begin{cases} 1 & \text{if } \iota^k(v_j^{l+k}) = v_i^l, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda(\gamma) = \lambda(\gamma_1) \cdots \lambda(\gamma_k)$ for $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_i \in E$, $1 \le i \le k$.

LEMMA 3.2. *Suppose that $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I). For a vertex $v_i^l \in V_l$, let L be the number as in Lemma 3.1. Then one of the following two conditions holds:*

(1) *There exist a word $\eta \in \Lambda_{\mathfrak{L}}^L$ and a vertex $v_j^{l+L} \in V_{l+L}$ such that $A_{l,l+L}(i, \eta, j) = 1$, $I_{l,l+L}(i, j) = 0$.*
(2) *There exists $k \in \mathsf{N}$ such that $I_{l,l+kL}(i, h) = 1$ implies $A_{l,l+kL}(i, \mu, h) = 1$ for some $\mu \in \Lambda_{\mathfrak{L}}^{kL}$, and there exists $h \in \{1, \dots, m(l+L)\}$ such that $\sum_{\mu \in \Lambda_{\mathfrak{L}}^{kL}} A_{l,l+kL}(i, \mu, h) \ge 2$.*

PROOF. Suppose that the condition (1) does not hold. As $\mathfrak{L}$ is λ-irreducible, it satisfies the assumption of Lemma 3.1(ii). By the λ-condition (I), we may take a number $k \in \mathsf{N}$ and a vertex $v_h^{l+kL} \in V_{l+kL}$ and two distinct paths γ_1, γ_2 in $\mathfrak{L}$ such that

$$s(\gamma_1) = s(\gamma_2) = v_i^l, \qquad t(\gamma_1) = t(\gamma_2) = v_h^{l+kL}, \qquad \lambda(\gamma_1) \ne \lambda(\gamma_2).$$

Hence we have $A_{l,l+kL}(i, \gamma_1, h) = A_{l,l+kL}(i, \gamma_2, h) = 1$ so that

$$\sum_{\mu \in \Lambda_{\mathfrak{L}}^{kL}} A_{l,l+kL}(i, \mu, h) \ge 2$$

and the condition (2) holds.

PROPOSITION 3.3. *Assume that $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I). For the projection E_i^l in the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ corresponding to the vertex $v_i^l \in V_l$, there exists a number $L \in \mathsf{N}$ such that for every vertex $v_h^{l+L} \in V_{l+L}$*

with $\iota^L(v_h^{l+L}) = v_i^l$, there exists an admissible word $\mu(h)$ in $\Lambda_{\mathfrak{L}}^L$ such that

$$S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* \neq 0 \qquad \textit{and} \qquad \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* < E_i^l.$$

Proof. For $v_i^l \in V_l$, let L be the number as in Lemma 3.1. One of the two conditions (1) and (2) in the preceding lemma holds. Suppose that (1) holds. As $\mathfrak{L}$ is λ-irreducible, for a vertex $v_h^{l+L} \in V_{l+L}$ with $\iota^L(v_h^{l+L}) = v_i^l$, there exists a path $\gamma(h)$ in $\mathfrak{L}$ of length L such that $s(\gamma(h)) = v_i^l$, $t(\gamma(h)) = v_h^{l+L}$. Put $\mu(h) = \lambda(\gamma(h)) \in \Lambda_{\mathfrak{L}}^L$ so that $S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* \neq 0$. By the condition (1), there exists a word $\eta \in \Lambda_{\mathfrak{L}}^L$ such that $A_{l,l+L}(i,\eta,j) = 1$, $I_{l,l+L}(i,j) = 0$ for some $j = 1,\dots,m(l+L)$. Hence one has

$$\begin{aligned} &\sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* + S_\eta E_j^{l+L} S_\eta^* \\ &\leq \sum_{h=1}^{m(l+L)} \sum_{\nu \in \Lambda_{\mathfrak{L}}^L} A_{l,l+L}(i,\nu,h) S_\nu E_h^{l+L} S_\nu^*. \end{aligned}$$

Now $A_{l,l+L}(i,\eta,j) = 1$ so that $S_\eta E_h^{l+L} S_\eta^* \neq 0$. By (1.1), (1.3) and (1.4), the equality

$$\sum_{h=1}^{m(l+L)} \sum_{\nu \in \Lambda_{\mathfrak{L}}^L} A_{l,l+L}(i,\nu,h) S_\nu E_h^{l+L} S_\nu^* = E_i^l \tag{3.1}$$

holds so that

$$\sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* < E_i^l.$$

We next assume that the condition (2) holds. There exists $k \in \mathsf{N}$ such that $I_{l,l+kL}(i,h) = 1$ implies $A_{l,l+kL}(i,\mu,h) = 1$ for some $\mu \in \Lambda_{\mathfrak{L}}^{kL}$, and there exists $h = 1,\dots,m(l+L)$ such that $\sum_{\mu \in \Lambda_{\mathfrak{L}}^{kL}} A_{l,l+kL}(i,\mu,h) \geq 2$. By (3.1) we obtain

$$\sum_{h=1}^{m(l+kL)} I_{l,l+kL}(i,h) S_{\mu(h)} E_h^{l+kL} S_{\mu(h)}^* < E_i^l.$$

Take L as kL so that we get the desired assertion.

Let $N_h^{l+n,n}$ be the number of paths γ in $\mathfrak{L}$ starting at a vertex in V_l and terminating at v_h^{l+n}. As $\mathfrak{L}$ is left-resolving, it is the number of admissible words

μ in $\Lambda_{\mathfrak{L}}$ of length n such that $S_\mu E_h^{l+n} S_\mu^* \neq 0$. It satisfies the equality

$$N_h^{l+n,n} E_h^{l+n} = \Bigl(\sum_{\mu\in\Lambda_{\mathfrak{L}}^n} S_\mu^* S_\mu\Bigr) E_h^{l+n}.$$

By the local property of λ-graph system, we have $N_h^{l+n,n} = N_k^{l+n,n}$ if $\iota^n(v_h^{l+n}) = \iota^n(v_k^{l+n})$. For a vertex $v_h^{l+n} \in V_{l+n}$, define a projection $P_h^{l+n,n}$ by setting

$$P_h^{l+n,n} = \frac{1}{N_h^{l+n,n}} \sum_{\mu,\nu\in\Lambda_{\mathfrak{L}}^n} S_\mu E_h^{l+n} S_\nu^*.$$

LEMMA 3.4. *Take $\mu \in \Lambda_{\mathfrak{L}}^n$ satisfying $S_\mu E_h^{l+n} S_\mu^* \neq 0$. Then there exists a partial isometry $U_{h,\mu}^{l+n}$ in $\mathcal{O}_{\mathfrak{L}}$ such that*

$$U_{h,\mu}^{l+n} U_{h,\mu}^{l+n*} = U_{h,\mu}^{l+n*} U_{h,\mu}^{l+n} = \sum_{\nu\in\Lambda_{\mathfrak{L}}^n} S_\nu E_h^{l+n} S_\nu^*,$$

$$U_{h,\mu}^{l+n} P_h^{l+n,n} U_{h,\mu}^{l+n*} = S_\mu E_h^{l+n} S_\mu^*.$$

PROOF. The elements $S_\xi E_h^{l+n} S_\eta^*$, $\xi,\eta \in \Lambda_{\mathfrak{L}}^n$ form a matrix units of the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by $S_\xi E_h^{l+n} S_\eta^*$, $\xi,\eta\in\Lambda_{\mathfrak{L}}^n$ that is isomorphic to the full matrix algebra of size $N_h^{l+n,n}$. As $P_h^{l+n,n}$ is a projection of rank one in the subalgebra, one can find a desired partial isometry by elementary linear algebra.

The following lemma is straightforward.

LEMMA 3.5. *Put $V_L = \frac{1}{\sqrt{N_h^{l+L,L}}} \sum_{\mu\in\Lambda_{\mathfrak{L}}^L} S_\mu E_h^{l+L}$. Then we have*

$$V_L^* V_L = 1, \qquad V_L E_i^l V_L^* = \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) P_h^{l+L,L}.$$

PROPOSITION 3.6. *Assume that $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I). Then the projection E_i^l for $v_i^l \in V$ is an infinite projection in $\mathcal{O}_{\mathfrak{L}}$.*

PROOF. Suppose that the number $m(l)$ of the vertex set V_l is one for all $l \in \mathbb{Z}_+$. Then we have $E_i^l = 1$. Since $\mathfrak{L}$ satisfies λ-condition (I), the alphabet Σ is not singleton. Now $1 = \sum_{\alpha\in\Sigma} S_\alpha S_\alpha^*$ and $A_{l,l+1}(i,\alpha,j) = 1$ for all i,α,j. Hence we see by the relations $(\mathfrak{L})$,

$$S_\alpha^* S_\alpha = \sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1.$$

This implies that the unit 1 is an infinite projection. In this case, the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is isomorphic to the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$ the number of Σ.

Suppose next that there exists $l_0 \in \mathsf{Z}_+$ such that $m(l_0) \geq 2$. Hence $m(l) \geq 2$ for $l \geq l_0$. For a projection E_i^l with $l \geq l_0$, by Proposition 3.3 for $h = 1, \dots, m(l+L)$ with $I_{l,l+L}(i,h) = 1$, there exists an admissible word $\mu(h)$ in $\Lambda_{\mathfrak{L}}^L$ such that

$$S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* \neq 0 \qquad \text{and} \qquad \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* < E_i^l.$$

Let V_L be the isometry as in Lemma 3.5 and $U_{h,\mu(h)}^{l+L}$ the partial isometry as in Lemma 3.4. Then we set $W_i^l = \big(\sum_{h=1}^{m(l+L)} U_{h,\mu(h)}^{l+L}\big) V_L$. As $U_{k,\mu(k)}^{l+L} P_h^{l+L} = 0$ for $k \neq h$, it follows that $W_i^{l*} W_i^l = 1$ and

$$\begin{aligned} W_i^l E_i^l W_i^{l*} &= \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) U_{h,\mu(h)}^{l+L} P_h^{l+L} {U_{h,\mu(h)}^{l+L}}^* \\ &= \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h) S_{\mu(h)} E_h^{l+L} S_{\mu(h)}^* < E_i^l. \end{aligned}$$

Hence E_i^l is an infinite projection.

LEMMA 3.7. *Assume that $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I). Then for the projection $E_i^l \in \mathcal{O}_{\mathfrak{L}}$ for $v_i^l \in V$, there exists an element $U \in \mathcal{O}_{\mathfrak{L}}$ such that $UU^* = 1$ and $UE_i^lU^* = 1$.*

PROOF. Assume that $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I), so that $\mathfrak{L}$ is irreducible and satisfies condition (I). Hence $\mathcal{O}_{\mathfrak{L}}$ is simple. By [4, Lemma V.5.4] with Proposition 3.6, the unit 1 of $\mathcal{O}_{\mathfrak{L}}$ is equivalent to a subprojection of E_i^l. Take an element $U \in \mathcal{O}_{\mathfrak{L}}$ such that $UU^* = 1$ and $U^*U \leq E_i^l$. This implies $UE_i^lU^* = 1$.

THEOREM 3.8. *If $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I), for any nonzero $X \in \mathcal{O}_{\mathfrak{L}}$ there exist $A, B \in \mathcal{O}_{\mathfrak{L}}$ such that $AXB = 1$.*

PROOF. Let $E : \mathcal{O}_{\mathfrak{L}} \to \mathcal{F}_{\mathfrak{L}}$ be the canonical conditional expectation given by

$$E(X) = \int_{\mathsf{T}} (\alpha_{\mathfrak{L}})_t(X)\, dt, \qquad X \in \mathcal{O}_{\mathfrak{L}}.$$

Since E is faithful, we may assume that $\|E(X^*X)\| = 1$. Let $\mathcal{P}_{\mathfrak{L}}$ be the $*$-algebra generated algebraically by the generators $S_\alpha, E_i^l, \alpha \in \Sigma, v_i^l \in V$. For any $0 < \epsilon < \frac{1}{4}$, we may find $0 \le Y \in \mathscr{P}_{\mathfrak{L}}$ such that $\|X^*X - Y\| < \frac{\epsilon}{2}$ so that $\|E(Y)\| > 1 - \frac{\epsilon}{2}$. As in the discussion in [16, Section 3], the element Y is expressed as

$$Y = \sum_{|\nu|\ge 1} Y_{-\nu} S_\nu^* + Y_0 + \sum_{|\mu|\ge 1} S_\mu Y_\mu \qquad \text{for some} \quad Y_{-\nu}, Y_0, Y_\mu \in \mathscr{F}_{\mathfrak{L}} \cap \mathscr{P}_{\mathfrak{L}}.$$

Take $k \le l$ such that $Y_{-\nu}, Y_0, Y_\mu \in \mathscr{F}_k^l$ for all μ, ν in the above expression. Now $\mathfrak{L}$ satisfies condition (I). By [16, Lemma 3.1 and Lemma 4.2] there exists a projection Q_k^l in the diagonal algebra of $\mathscr{F}_{\mathfrak{L}}$ for $k \le l$ satisfying the following properties

(1) Q_k^l commutes with $\mathscr{F}_k^l$.

(2) The map $X \in \mathscr{F}_k^l \to Q_k^l X Q_k^l \in Q_k^l \mathscr{F}_k^l Q_k^l$ is an isomorphism.

(3) $Q_k^l S_\mu Q_k^l = Q_k^l S_\nu^* Q_k^l = 0$ for $1 \le |\mu|, |\nu| \le k$.

As $E(Y) = Y_0$, it follows that by (1) and (3),

$$Q_k^l Y Q_k^l = \sum_{|\nu|\ge 1} Y_{-\nu} Q_k^l S_\nu^* Q_k^l + Q_k^l Y_0 Q_k^l + \sum_{|\mu|\ge 1} Q_k^l S_\mu Q_k^l Y_\mu = Q_k^l E(Y) Q_k^l.$$

Since $Q_k^l E(Y) Q_k^l \in \mathscr{F}_{\mathfrak{L}}$, there exists $0 \le Z \in \mathscr{F}_{k'}^{l'}$ for some $k' \le l'$ such that $\|Q_k^l E(Y) Q_k^l - Z\| < \frac{\epsilon}{2}$. By (2), we note $\|Q_k^l E(Y) Q_k^l\| = \|E(Y)\|$ so that

$$\|Z\| \ge \|E(Y)\| - \frac{\epsilon}{2} > 1 - \epsilon$$

and

$$\|Z\| < \|Q_k^l E(Y) Q_k^l\| + \frac{\epsilon}{2} = \|E(Y)\| + \frac{\epsilon}{2} \le \|E(X^*X)\| + \frac{\epsilon}{2} + \frac{\epsilon}{2} < 1 + \epsilon.$$

As the algebra $\mathscr{F}_{k'}^{l'}$ is finite dimensional, we have spectral decomposition $Z = \sum_{i=1}^s \lambda_i R_i$ of Z for some real numbers $\lambda_i \ge 0$ and minimal projections $R_i \in \mathscr{F}_{k'}^{l'}$. Since $1 - \epsilon < \|Z\| < 1 + \epsilon$, we may find i_0 such that $1 - \epsilon < \lambda_{i_0} < 1 + \epsilon$, and may assume that $R_{i_0} = S_{\mu_0} E_{i_0}^{l'} S_{\mu_0}^*$ for some $|\mu_0| = k'$ and $v_{i_0}^{l'} \in V_{l'}$. By Lemma 3.7, there exists $U \in \mathscr{O}_{\mathfrak{L}}$ such that $UU^* = 1$, $U E_{i_0}^{l'} U^* = 1$. Put $A = U S_{\mu_0}^* R_{i_0} Q_k^l$. It follows that

$$\begin{aligned}\|AX^*XA^* - 1\| \le{}& \|AX^*XA^* - AYA^*\| \\ &+ \|AYA^* - U S_{\mu_0}^* R_{i_0} Z R_{i_0} S_{\mu_0} U^*\| + \|U S_{\mu_0}^* R_{i_0} Z R_{i_0} S_{\mu_0} U^* - 1\|.\end{aligned}$$

One then sees

$$\|AX^*XA^* - AYA^*\| \le \|X^*X - Y\| < \frac{\epsilon}{2},$$

$$2\|AYA^* - US^*_{\mu_0}R_{i_0}ZR_{i_0}S_{\mu_0}U^*\| = \|US^*_{\mu_0}R_{i_0}(Q^l_kYQ^l_k - Z)R_{i_0}S_{\mu_0}U^*\| < \frac{\epsilon}{2},$$

$$US^*_{\mu_0}R_{i_0}ZR_{i_0}S_{\mu_0}U^* = \lambda_{i_0}UE^{l'}_{i_0}U^* = \lambda_{i_0}.$$

Thus we obtain

$$\|AX^*XA^* - 1\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} + |\lambda_{i_0} - 1| < 2\epsilon < \frac{1}{4}.$$

Hence AX^*XA^* is invertible so that we have an element $C \in \mathcal{O}_{\mathfrak{L}}$ such that $AX^*XA^*C = 1$.

Therefore we conclude by [4, Theorem V.5.5]

THEOREM 3.9. *If $\mathfrak{L}$ is λ-irreducible and satisfies λ-condition (I), then the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite.*

Let A be a finite square matrix with entries in $\{0, 1\}$ and G_A its corresponding directed graph. By considering the associated λ-graph system $\mathfrak{L}_{G_A}$, we have the following well-known result:

COROLLARY 3.10 ([1], [2], [3]). *If A satisfies condition (I) in the sense of Cuntz-Krieger [3] and is irreducible, the Cuntz-Krieger algebra $\mathcal{O}_A$ is simple and purely infinite.*

In [12], [16, Theorem 7.7] and [17], examples of λ-graph systems that are λ-irreducible and satisfy λ-condition (I) are presented and the K-groups for the associated C^*-algebras are computed.

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