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# ON THE STABLE RANK AND REAL RANK OF GROUP $C^*$-ALGEBRAS OF NILPOTENT LOCALLY COMPACT GROUPS

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## Abstract

It is shown that if $G$ is an almost connected nilpotent group then the stable rank of $C^*(G)$ is equal to the rank of the abelian group $G/[G, G]$. For a general nilpotent locally compact group $G$, it is shown that finiteness of the rank of $G/[G, G]$ is necessary and sufficient for the finiteness of the stable rank of $C^*(G)$ and also for the finiteness of the real rank of $C^*(G)$.

## Introduction

For a $C^*$-algebra $A$, the real rank $\mathrm{RR}(A)$ [3] and the stable rank $\mathrm{sr}(A)$ [18] have been defined as numerical invariants giving non-commutative analogues of the real and complex dimension of topological spaces. More precisely, for the continuous functions on a compact Hausdorff space $X$ one has $\mathrm{RR}(C(X)) = \dim X$ and $\mathrm{sr}(C(X)) = \lfloor \frac{1}{2} \dim X \rfloor + 1$, where $\dim X$ is the covering dimension of $X$ [17]. For unital $A$, the stable rank $\mathrm{sr}(A)$ is either $\infty$ or the smallest possible integer $n$ such that each $n$-tuple in $A^n$ can be approximated in norm by $n$-tuples $(b_1, \dots, b_n)$ such that $\sum_{i=1}^n b_i^* b_i$ is invertible. Similarly, the real rank $\mathrm{RR}(A)$ is either $\infty$ or the smallest non-negative integer $n$ such that each $(n+1)$-tuple of self-adjoint elements in $A^{n+1}$ can be approximated in norm by $(n+1)$-tuples $(b_0, b_1, \dots, b_n)$ of self-adjoint elements such that $\sum_{i=0}^n b_i^2$ is invertible. For non-unital $A$, these ranks are defined to be those of the unitization of $A$.

Several authors have computed or estimated the stable and the real rank of group $C^*$-algebras $C^*(G)$ for various classes of locally compact groups $G$ [1], [6], [7], [13], [15], [19], [20], [21], [22], [23], [23], [25], [26]. For example, for simply connected nilpotent Lie groups, Sudo and Takai [25] (following earlier work of Sheu [20]) have shown that $\mathrm{sr}(C^*(G))$ is the complex dimension of the space of characters of $G$. On the other hand, for the free group $\mathsf{F}_2$ on 2 generators it has been shown that $\mathrm{sr}(C^*(\mathsf{F}_2)) = \mathrm{RR}(C^*(\mathsf{F}_2)) = \infty$ [18],

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[15], but $\mathrm{sr}(C_r^*(\mathsf{F}_2)) = \mathrm{RR}(C_r^*(\mathsf{F}_2)) = 1$ [6] (where $r$ indicates the reduced $C^*$-algebra of a non-amenable group).

In Section 1 of this paper, the result of Sudo and Takai mentioned above is extended to almost connected nilpotent groups $G$. To be specific,

$$\mathrm{sr}(C^*(G)) = 1 + \left\lfloor \frac{1}{2} \dim \widehat{G/[G, G]} \right\rfloor = 1 + \left\lfloor \frac{1}{2} \mathrm{rank}(G/[G, G]) \right\rfloor < \infty$$

(Theorem 1.5), where the rank of an abelian group is as defined in Section 1. The method of proof involves structure theory for $G$ together with Nistor's estimate for the stable rank of $C^*$-algebras containing certain continuous trace ideals [16]. As a corollary, it follows that either $\mathrm{RR}(C^*(G)) = \mathrm{rank}(G/[G, G])$ (a sufficient condition for this equality is that $\mathrm{rank}(G/[G, G])$ is odd) or $\mathrm{RR}(C^*(G)) = 1 + \mathrm{rank}(G/[G, G])$.

In Section 2, general nilpotent locally compact groups $G$ are considered. The main result (Theorem 2.8) is that the finiteness of the rank of $G/[G, G]$ is necessary and sufficient for the finiteness of $\mathrm{sr}(C^*(G))$ and also for the finiteness of $\mathrm{RR}(C^*(G))$. In addition to further structural properties of $G$, the proof uses an estimate from Section 1 and also Rieffel's estimate for the stable rank of crossed products by the integers [18].

The results of Sections 1 and 2 suggest that the stable rank of $C^*(G)$, for a nilpotent group $G$, may depend only on the abelian quotient $G/[G, G]$. Further evidence in this direction is provided by [24, Theorem 2], which deals with the case of finitely generated, torsion-free, two-step nilpotent (discrete) groups. On the other hand, we give examples to show that the conclusions of Theorems 1.5 and 2.8 may fail to hold if the hypothesis of nilpotency is replaced by solvability.

## 1. Almost connected nilpotent groups

For any locally compact group $G$, let $G_0$ denote the connected component of the identity. Recall that $G$ is said to be *almost connected* if the quotient group $G/G_0$ is compact.

Let $G$ be a nilpotent locally compact group. Then $G^c$, the set of all compact elements of $G$, is a closed normal subgroup of $G$ [10, Corollary 3.5.1 and Lemma 3.8]. Moreover, $G^c$ is compact whenever $G$ is compactly generated [10, Theorem 9.7]. From this it can easily be deduced that $G/G^c$ is compact-free (see [13, Remark 1]). In particular, $(G/G^c)_0$, the connected component of $G/G^c$, is a simply connected nilpotent Lie group. Also, $G_0G^c$ is open in $G$ and $G/G_0G^c$ is torsion-free [10, Theorem 8.3]. Since $G_0G^c/G^c$ is connected and open in $G/G^c$, $G_0G^c/G^c = (G/G^c)_0$. Hence $G/G^c$ is a Lie group. When $G$ is discrete, $G^c$ is just the subgroup consisting of all elements of finite order which is usually denoted $G^t$ and called the torsion subgroup of $G$. Finally, recall that if $G$ is a torsion-free nilpotent group, then all the subquotients $Z_{j+1}(G)/Z_j(G)$ arising from the upper central series of $G$ are torsion-free as well [2, Corollary 2.11].

We next introduce the group theoretical *rank* of a locally compact abelian group. For a discrete torsion-free abelian group $D$, rank $D$ means the torsion-free rank of $D$ (see [11]), that is, rank $D$ is the maximal number of independent elements of $D$ when this number is finite and rank $D = \infty$ otherwise. Let $G$ be an arbitrary locally compact abelian group. Then $G/G^c = \mathsf{R}^k \times D$, where $D$ is torsion-free discrete, and the rank of $G$ is defined to be $k + \text{rank } D$. Note that rank $G < \infty$ whenever $G/G^c$ is compactly generated. On the other hand, the additive group of rational numbers has rank 1.

Throughout the paper, we shall frequently use the fact that if $G$ is a locally compact abelian group and $H$ is a closed subgroup of $G$, then $\text{rank}(G/H) \leq \text{rank}\, G$. This is easily seen as follows. Define a closed subgroup $K$ of $G$ by $K \supseteq H$ and $K/H = (G/H)^c$. Then $G/G^c = \mathsf{R}^m \times D$ and $G/K = (G/H)/(G/H)^c = \mathsf{R}^n \times E$, where $m, n \in \mathsf{N}_0$ and $D$ and $E$ are torsion-free abelian discrete groups. Since $G^c \subseteq K$, the quotient homomorphism $G \to G/H$ induces a homomorphism $q : G/G^c \to G/K$. It follows that $q(\mathsf{R}^m) = \mathsf{R}^n$, whence $n \leq m$, and hence $q$ gives rise to a homomorphism from $D$ onto $E$. Now, it is immediate from the definition of the torsion-free rank that rank $E \leq$ rank $D$. Thus

$$\text{rank}(G/H) = n + \text{rank } E \leq m + \text{rank } D = \text{rank}\, G.$$

Denoting by $\widehat{G}$ the dual group of $G$, we have

$$\text{RR}(C^*(G)) = \text{RR}(C_0(\widehat{G})) = \dim \widehat{G} = \text{rank}\, G$$

and

$$\text{sr}(C^*(G)) = \text{sr}(C_0(\widehat{G})) = 1 + \left\lfloor \frac{1}{2} \dim \widehat{G} \right\rfloor = 1 + \left\lfloor \frac{1}{2} \text{rank}\, G \right\rfloor$$

(see [1, Section 2] and the references therein).

In passing, we note that if $J$ is a closed ideal of a $C^*$-algebra $A$ then $\text{sr}(J), \text{sr}(A/J) \leq \text{sr}(A)$ [18, Section 4] and similarly for the real rank [9, Théorème 1.4].

LEMMA 1.1. *Let $G$ be a projective limit of groups $G_\alpha = G/K_\alpha$, where each $K_\alpha$ is a compact normal subgroup of $G$. Then*

(i) $\text{sr}(C^*(G)) = \sup_\alpha \text{sr}(C^*(G_\alpha))$ *and* $\text{sr}(C^*_r(G)) = \sup_\alpha \text{sr}(C^*_r(G_\alpha))$.

(ii) $\text{RR}(C^*(G)) = \sup_\alpha \text{RR}(C^*(G_\alpha))$ *and* $\text{RR}(C^*_r(G)) = \sup_\alpha \text{RR}(C^*_r(G_\alpha))$.

PROOF. Let $K$ be any compact normal subgroup of $G$, and let $q : G \to G/K$ denote the quotient homomorphism and $\mu_K$ normalized Haar measure on $K$. Then $\mu_K$ is a central idempotent measure, and the map $\phi : f \to f \circ q$ establishes an isomorphism between $L^1(G/K)$ and the closed ideal $L^1(G) * \mu_K$ of $L^1(G)$. For $\pi \in \widehat{G}$ and $f \in L^1(G/K)$, $\pi(f \circ q) = \pi(f \circ q)\pi(\mu_K) = 0$ whenever $\pi \notin \widehat{G/K} \circ q$. Notice that if $\pi = \sigma \circ q$ with $\sigma \in \widehat{G/K}$, then $\pi \in \widehat{G}_r$ if and only if $\sigma \in (\widehat{G/K})_r$. This implies that $\|f\|_{C^*(G/K)} = \|f \circ q\|_{C^*(G)}$ and $\|f\|_{C_r^*(G/K)} = \|f \circ q\|_{C_r^*(G)}$, and hence $\phi$ extends uniquely to isomorphisms from $C^*(G/K)$ onto the closed ideal $\overline{L^1(G) * \mu_K}$ of $C^*(G)$ and from $C_r^*(G/K)$ onto the closed ideal $\overline{L^1(G) * \mu_K}^r$ of $C_r^*(G)$.

Now, in the situation of the lemma, let $I_\alpha$ and $J_\alpha$ denote the closure of $L^1(G) * \mu_{K_\alpha}$ in $C^*(G)$ and $C_r^*(G)$, respectively. Then $\cup_\alpha I_\alpha$ is dense in $C^*(G)$ and $\cup_\alpha J_\alpha$ is dense in $C_r^*(G)$ since $\cup_\alpha L^1(G) * \mu_{K_\alpha}$ is dense in $L^1(G)$. Since the sets $\{I_\alpha\}$ and $\{J_\alpha\}$ are suitably directed, it follows from [18, Theorem 5.1] and [13, Lemma 4(i)] that

$$\mathrm{sr}(C^*(G)) \leq \sup_\alpha \mathrm{sr}(I_\alpha) = \sup_\alpha \mathrm{sr}(C^*(G_\alpha)) \leq \mathrm{sr}(C^*(G))$$

and

$$\mathrm{RR}(C^*(G)) \leq \sup_\alpha \mathrm{RR}(I_\alpha) = \sup_\alpha \mathrm{RR}(C^*(G_\alpha)) \leq \mathrm{RR}(C^*(G)),$$

and similarly for the reduced $C^*$-algebras. This yields (i) and (ii).

Alternatively, to prove Lemma 1.1, we could exploit Proposition 2.2 of [14].

LEMMA 1.2. *Let $G$ be a Lie group such that $G_0$ is nilpotent and $G/G_0$ is finite. Then*

$$\mathrm{sr}(C^*(G)) \leq \max\left\{2, 1 + \left\lfloor \frac{1}{2}\mathrm{rank}(G_0/[G_0, G_0]) \right\rfloor\right\}.$$

PROOF. The connected nilpotent Lie group $G_0$ is type I, and every irreducible representation of $G_0$ is either 1-dimensional or infinite dimensional. This follows from the fact that every irreducible representation of a connected nilpotent Lie group is induced from a character of some closed subgroup [4]. Since $G/G_0$ is finite, $G$ is type I and $\dim \pi \leq [G : G_0]$ for every finite dimensional irreducible representation $\pi$ of $G$. Indeed, for any such $\pi$, $\pi \leq \mathrm{ind}_{G_0}^G \chi$ for some character $\chi$ of $G_0$. In particular, the set $\widehat{G}_{fin}$ consisting of all finite dimensional representations in $\widehat{G}$ is closed in $\widehat{G}$. Since $\widehat{G}_{fin}$ has the property that if $\pi, \rho \in \widehat{G}_{fin}$ then every irreducible subrepresentation of $\pi \otimes \bar{\rho}$ belongs to $\widehat{G}_{fin}$, it follows that $\widehat{G}_{fin} = \widehat{G/N}$ for some closed normal subgroup $N$ of $G$.

Actually, $N = [G_0, G_0]$. In fact, if $x \in [G_0, G_0]$ then $\chi(x) = 1$ for all characters $\chi$ of $G_0$ and hence $\pi(x)$ is the identity operator for every $\pi \in \widehat{G}_{fin}$, and conversely, if $x \in N$ then $\mathrm{ind}_{G_0}^G \chi(x)$ is the identity operator, whence $x \in G_0$ and $\chi(x) = 1$ for every character $\chi$ of $G_0$.

We claim that $C^*(G)$ has a composition series of finite length in which the successive quotients have continuous trace. By a result of Dixmier [5], the $C^*$-algebra of any simply connected nilpotent Lie group has this property. Hence the same is true for the $C^*$-algebra of any connected nilpotent Lie group because its $C^*$-algebra is a quotient of the $C^*$-algebra of its simply connected covering group. Since $G/G_0$ is finite, it follows that $C^*(G)$ has a composition series of finite length in which all the successive quotients have continuous trace [8, Corollary 1].

Now, let $I$ be the closed ideal of $C^*(G)$ such that $\widehat{I} = \widehat{G} \setminus \widehat{G}_{fin}$. Then $I$ is a separable type I $C^*$-algebra all of whose irreducible representations are infinite dimensional. By intersecting the ideals in the composition series for $C^*(G)$ with $I$, we obtain a sequence

$$I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{r+1} = \{0\}$$

of closed ideals of $C^*(G)$ such that $I_j/I_{j+1}$ has continuous trace for $0 \leq j \leq r$. Applying Lemma 2 of [16] repeatedly, we get

$$\begin{aligned} \mathrm{sr}(C^*(G)) \leq \max\{2, \mathrm{sr}(C^*(G)/I_r)\} &\leq \max\{2, \max\{2, \mathrm{sr}(C^*(G)/I_{r-1})\}\} \\ &\leq \ldots \leq \max\{2, \mathrm{sr}(C^*(G)/I)\}. \end{aligned}$$

Recall that $C^*(G)/I = C^*(G/N) = C^*(G/[G_0, G_0])$. Since $G_0/[G_0, G_0]$ is an abelian normal subgroup of finite index in $G/[G_0, G_0]$ and $G$ is second countable, by [19, Corollary 3.3]

$$\mathrm{sr}(C^*(G/N)) \leq \mathrm{sr}(C^*(G_0/[G_0, G_0])) = 1 + \left\lfloor \frac{1}{2} \mathrm{rank}(G_0/[G_0, G_0]) \right\rfloor,$$

which finishes the proof.

Proposition 1.3. *Let $G$ be an almost connected locally compact group such that $G_0$ is nilpotent. Then*

$$\mathrm{sr}(C^*(G)) \leq \max\left\{2, 1 + \left\lfloor \frac{1}{2} \mathrm{rank}(G_0/[G_0, G_0]) \right\rfloor\right\}.$$

Proof. By [12] $G$ is a projective limit of Lie groups $G/K_\alpha$, and by Lemma 1.1

$$\mathrm{sr}(C^*(G)) = \sup_\alpha \mathrm{sr}(C^*(G/K_\alpha)).$$

Thus it suffices to show that if $K$ is any compact normal subgroup of $G$ such that $G/K$ is a Lie group, then

$$\operatorname{sr}(C^*(G/K)) \le \max\left\{2, 1 + \left\lfloor \frac{1}{2}\operatorname{rank}(G_0/[G_0, G_0])\right\rfloor\right\}.$$

Observe next that $(G/K)_0 = G_0K/K$. Indeed, $G_0K/K$ is connected and $(G/K)/(G_0K/K) = G/G_0K$ is a totally disconnected Lie group and hence discrete. Also

$$(G/K)_0/[(G/K)_0, (G/K)_0] = (G_0K/K)/[G_0K/K, G_0K/K],$$

which is a quotient of $G_0/[G_0, G_0]$. Thus

$$\operatorname{rank}((G/K)_0/[(G/K)_0, (G/K)_0]) \le \operatorname{rank}(G_0/[G_0, G_0]).$$

Since $(G/K)_0$ is nilpotent and has finite index in $G/K$, Lemma 1.2 yields that

$$\begin{aligned}\operatorname{sr}(C^*(G/K)) &\le \max\left\{2, 1 + \left\lfloor \frac{1}{2}\operatorname{rank}((G/K)_0/[(G/K)_0, (G/K)_0])\right\rfloor\right\}\\ &\le \max\left\{2, 1 + \left\lfloor \frac{1}{2}\operatorname{rank}(G_0/[G_0, G_0])\right\rfloor\right\},\end{aligned}$$

as required.

When $G$ itself rather than just $G_0$ is nilpotent, we can prove a considerably stronger result (Theorem 1.5) which generalizes the result of Sudo and Takai mentioned in the introduction. For that, we need the following fairly simple lemma.

LEMMA 1.4. *Let G be an almost connected nilpotent locally compact group. Then*

$$\operatorname{rank}(G/[G, G]) = \operatorname{rank}(G_0/[G_0, G_0]).$$

PROOF. Choose any compact normal subgroup $K$ of $G$ such that $G/K$ is a Lie group. Since

$$(G/K)/[G/K, G/K] = (G/K)/([G, G]K)/K = G/[G, G]K$$

and $[G, G]K/[G, G]$ is compact, we obtain

$$\operatorname{rank}((G/K)/[G/K, G/K]) = \operatorname{rank}(G/[G, G]).$$

Replacing $G$ with $G_0$ and $K$ with $K \cap G_0$, we get

$$\begin{aligned}\operatorname{rank}(G_0/[G_0, G_0]) &= \operatorname{rank}(G_0/(G_0 \cap K)/[G_0/(G_0 \cap K), G_0/(G_0 \cap K)]) \\ &= \operatorname{rank}((G_0K/K)/[G_0K/K, G_0K/K]) \\ &= \operatorname{rank}((G/K)_0/[(G/K)_0, (G/K)_0])\end{aligned}$$

(compare the proof of Proposition 1.3). Thus it suffices to prove the lemma when $G$ is a Lie group, so that $G/G_0$ is finite.

To that end, we observe first that if $H$ is a torsion-free nilpotent group having an abelian normal subgroup of finite index, then $H$ is abelian. This is shown by induction on the length of the upper central series of $H$. Since $H/Z(H)$ is torsion-free and has an abelian subgroup of finite index, it is abelian. Thus $H$ is 2-step nilpotent, torsion-free and has an abelian normal subgroup $A$ of finite index $d$, say. For $x, y \in H$, it follows that

$$[x, y]^{d^2} = [x, y^d]^d = [x^d, y^d] = e.$$

Since $H$ is torsion-free, we obtain that $[H, H] = \{e\}$.

Now define a closed normal subgroup $N$ of $G$ by $N \supseteq [G_0, G_0]$ and $N/[G_0, G_0] = (G/[G_0, G_0])^c$. Then, since $(N \cap G_0)/[G_0, G_0]$ has only compact elements,

$$\operatorname{rank}(G_0/[G_0, G_0]) = \operatorname{rank}(G_0/(N \cap G_0)) = \operatorname{rank}(G_0N/N).$$

Moreover, $G_0N/N$ is an abelian subgroup of finite index in the compact-free nilpotent group $G/N$. By the above observation, $G/N$ is abelian. Thus $N \supseteq [G, G] \supseteq [G_0, G_0]$ and $N/[G, G]$ has only compact elements. This implies

$$\operatorname{rank}(G/[G, G]) = \operatorname{rank}(G/N) = \operatorname{rank}(G_0N/N) = \operatorname{rank}(G_0/[G_0, G_0]),$$

which concludes the proof.

Before proceeding we mention that if $G$ is an almost connected nilpotent locally compact group and $G$ is non-abelian, then $\operatorname{rank}(G/[G, G]) \geq 2$. This can be seen as follows.

By Lemma 1.4, we can assume that $G$ is connected. Moreover, we can assume that $G$ is a Lie group since for any compact normal subgroup $C$ of $G$, $(G/C)/[G/C, G/C]$ is a quotient of $G/[G, G]$. Let $H$ denote the simply connected covering group of $G$ and $q : H \to G$ the covering homomorphism. Let $Z_{n+1}(G) = G$ and $Z_n(G) \neq G$. Then $[G, G] \subseteq Z_n(G)$ and $q^{-1}(Z_j(G)) = Z_j(H)$, $1 \leq j \leq n+1$. Now, it is well-known that since $H$ is simply connected, $H/Z_n(H) = \mathsf{R}^d$ for some $d \geq 2$. So the above claim follows.

THEOREM 1.5. *Let $G$ be an almost connected nilpotent locally compact group. Then*

$$\mathrm{sr}(C^*(G)) = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G/[G,G]) \right\rfloor = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0,G_0]) \right\rfloor = \mathrm{sr}(C^*(G_0)) < \infty.$$

PROOF. We can assume that $G$ is non-abelian. By Proposition 1.3, Lemma 1.4 and the fact that $C^*(G/[G,G])$ is a quotient of $C^*(G)$,

$$\mathrm{sr}(C^*(G)) \le 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0,G_0]) \right\rfloor = 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G/[G,G]) \right\rfloor = \mathrm{sr}(C^*(G/[G,G])) \le \mathrm{sr}(C^*(G)).$$

This proves the first equality. Replacing $G$ with $G_0$ and applying Lemma 1.4 again we obtain the remaining equalities. Finally, note that $G$ is compactly generated and that the rank of any compactly generated abelian group is finite.

COROLLARY 1.6. *Let $G$ be an almost connected nilpotent locally compact group.*

(i) *If* $\mathrm{rank}(G/[G,G])$ *is odd, then*

$$\mathrm{RR}(C^*(G)) = \mathrm{rank}(G/[G,G]).$$

(ii) *If* $\mathrm{rank}(G/[G,G])$ *is even, then*

$$\mathrm{rank}(G/[G,G]) \le \mathrm{RR}(C^*(G)) \le 1 + \mathrm{rank}(G/[G,G]).$$

PROOF. Both (i) and (ii) are immediate consequences of Theorem 1.5 and the estimate $\mathrm{RR}(A) \le 2\,\mathrm{sr}(A) - 1$ which holds for arbitrary $C^*$-algebras $A$ [3, Proposition 1.2].

In [26, Example 4.2] the so-called split oscillator group $G$ (a semidirect product of $\mathsf{R}$ with the Heisenberg group) was considered. It was shown that $\mathrm{sr}(C^*(G)) = 2$ although $\mathrm{rank}(G/[G,G]) = 1$. So the last equality of Theorem 1.5 does not hold in general for simply connected solvable Lie groups of type I.

The following example shows that all the other equalities of Theorem 1.5 may also fail for solvable almost connected groups.

EXAMPLE 1.7. Let $G$ be the semidirect product $G = \mathsf{Z}_2 \ltimes \mathsf{R}^n$, where $\mathsf{Z}_2 = \{1, -1\}$ acts on $\mathsf{R}^n$ by coordinatewise multiplication. Then $G_0 = [G,G] = \mathsf{R}^n$

and hence

$$\mathrm{sr}(C^*(G_0)) = 1+\left\lfloor \frac{n}{2} \right\rfloor, \mathrm{rank}(G_0/[G_0, G_0]) = n \text{ and } \mathrm{rank}(G/[G, G]) = 0.$$

On the other hand, since every non-trivial character of $\mathsf{R}^n$ induces a 2-dimensional irreducible representation of $G$, $C^*(G)$ has a closed ideal $J$ isomorphic to $C_0((\mathsf{R}^n \setminus \{0\})/\mathsf{Z}_2, M_2(\mathsf{C}))$ with quotient $C^*(G)/J = \mathsf{C}^2$. Since $n$ is the maximal dimension of compact subsets of the quotient space $(\mathsf{R}^n \setminus \{0\})/\mathsf{Z}_2$,

$$\begin{aligned}\mathrm{sr}(C^*(G)) &= \max\left\{\mathrm{sr}(C_0((\mathsf{R}^n \setminus \{0\})/\mathsf{Z}_2), M_2(\mathsf{C})), \mathrm{sr}(\mathsf{C}^2)\right\}\\ &= 1+\left\lceil \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \right\rceil\end{aligned}$$

(see [1, Lemmas 1.3 and 1.2]).

## 2. Finiteness of the ranks for nilpotent groups

In this section we are going to characterize, for a general nilpotent locally compact group $G$, the finiteness of both $\mathrm{sr}(C^*(G))$ and $\mathrm{RR}(C^*(G))$ in terms of a simple and purely group-theoretic condition, the finiteness of the rank of the abelian locally compact group $G/[G, G]$ (Theorem 2.8). To establish this result, we need to show that if $G$ is a (discrete) nilpotent group and $G/[G, G]$ has finite rank then all the subquotients arising from the lower central series of $G$ also have finite rank. The analogous conclusion is known to be true when the finite rank condition is replaced by finite generation. The arguments in that case have influenced our proofs (in particular, that of Lemma 2.3). Note that in Lemmas 2.3 and 2.5 we have temporarily suspended the convention that $G_0$ is the connected component of the identity of a locally compact group. Instead, we use the notation $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{n+1} = \{e\}$ for the lower central series of an $(n+1)$-step nilpotent discrete group.

LEMMA 2.1. *Let $A$ be an abelian group. Then* rank $A \leq r$ *if and only if every finitely generated subgroup $B$ of $A$ is of the form $B = \mathsf{Z}^k \times F$, where $k \leq r$ and $F$ is a finite group.*

PROOF. Suppose first that rank $A \leq r$ and let $B$ be a finitely generated subgroup of $A$. Then $B = \mathsf{Z}^k \times F$ where $k \geq 0$ and $F$ is finite. Thus $A/A^t$ contains a copy of $\mathsf{Z}^k$ and so $r \geq \mathrm{rank}(A/A^t) \geq k$.

Conversely, let $\{x_1, \ldots, x_l\}$ be a subset of $A$ whose image in $A/A^t$ is an independent set. Assuming the stated condition on finitely generated subgroups of $A$, the subgroup generated by $x_1, \ldots, x_l$ is of the form $\mathsf{Z}^k \times F$ where $k \leq r$ and $F$ is finite. Passing to $A/A^t$, we obtain a copy of $\mathsf{Z}^k$ containing an independent set of $l$ elements. Hence $l \leq k \leq r$, and so rank $A \leq r$.

LEMMA 2.2. *Let $H$ be a nilpotent group and let $\{e\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H$ be a sequence of normal subgroups of $H$ such that $H_j/H_{j-1} = \mathsf{Z}^{d_j} \times F_j$ for some finite group $F_j$ and $d_j \in \mathsf{N}_0$ ($1 \le j \le n$). Then $H$ contains a normal subgroup $N$ such that $H/N$ is finite and $N$ has a composition series $N_0 = \{e\} \subseteq N_1 \subseteq \cdots \subseteq N_n = N$ where $N_{j-1}$ is normal in $N_j$, $N_j/N_{j-1} = \mathsf{Z}^{e_j}$ and $\sum_{j=1}^n e_j \le \sum_{j=1}^n d_j$. In particular, $N$ is generated by $\le \sum_{j=1}^n d_j$ elements.*

PROOF. If $n = 1$, we may simply put $N = \mathsf{Z}^{d_1}$. Now suppose that $n > 1$ and that the result has been established up to stage $n - 1$. Since $H$ is finitely generated, it contains a torsion-free normal subgroup $K$ of finite index [2, Theorem 2.1]. Let $K_j = K \cap H_j$ for $j = 1, \ldots, n$. Then the quotients $K_j/K_{j-1}$ have similar structure to the quotients $H_j/H_{j-1}$, with no increase in the $d_j$. Applying the induction hypothesis to $K/K_1$, we obtain a normal subgroup $L$ of finite index in $K$ and a series

$$K_1 = L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n = L$$

such that $L_{j-1}$ is normal in $L_j$, $L_j/L_{j-1}$ is isomorphic to $\mathsf{Z}^{l_j}$ ($2 \le j \le n$) and $\sum_{j=2}^n l_j \le \sum_{j=2}^n d_j$.

Since $L$ has finite index in $H$, there exists a normal subgroup $N$ of $H$ such that $N \subseteq L$ and $H/N$ is finite. Let $N_j = L_j \cap N$, $1 \le j \le n$. Then $N_j/N_{j-1}$ is isomorphic to a subgroup of $K_j/K_{j-1}$ and hence is isomorphic to $\mathsf{Z}^{e_j}$ where $e_j \le l_j$ ($2 \le j \le n$). Finally, since $N_1 = L_1 \cap N$ is isomorphic to $\mathsf{Z}^{d_1}$, the inductive step is complete.

LEMMA 2.3. *Let $G$ be a nilpotent group and let $G_n$, $n = 0, 1, \ldots$ denote the lower central series of $G$. Suppose that $G_{n+1} = \{e\}$ and that all the quotient groups $G_{k-1}/G_k$, $1 \le k \le n$, are of finite rank. Then $G_n$ has finite rank.*

PROOF. Let $a_1, \ldots, a_m \in G_n = [G, G_{n-1}]$ be given. Then each $a_i$, $1 \le i \le m$, can be written as

$$a_i = \prod_{j=1}^{m_i} [b_{ij}, c_{ij}],$$

where $b_{ij} \in G$, $c_{ij} \in G_{n-1}$, $1 \le j \le m_i$, $m_i \in \mathsf{N}$. Since all the quotients $G_{k-1}/G_k$, $1 \le k \le n$, have finite rank, by Lemma 2.2 there exist $N \in \mathsf{N}$ and elements $u_1, \ldots, u_s \in G$, where $s$ depends only on $G/G_n$, such that, denoting by $U$ the subgroup generated by $u_1, \ldots, u_s$, $b_{ij}^N \in UG_n$ for all $1 \le i \le m$, $1 \le j \le m_i$. Similarly, since $G_{n-1}/G_n$ has finite rank, there exist $M \in \mathsf{N}$ and $v_1, \ldots, v_t \in G_{n-1}$, where $t$ depends only on $G_{n-1}/G_n$, such that $c_{ij}^M \in VG_n$ ($V$ the subgroup of $G_{n-1}$ generated by $v_1, \ldots, v_t$). Thus

$$b_{ij}^N = u_{ij} x_{ij} \qquad \text{and} \qquad c_{ij}^M = v_{ij} y_{ij},$$

where $u_{ij} \in U$, $v_{ij} \in V$, $x_{ij}, y_{ij} \in G_n$. Since $G_n$ is contained in the centre of $G$, we have $[x, y_1 y_2] = [x, y_1][x, y_2]$ for $x \in G$, $y_1, y_2 \in G_{n-1}$ and $[x_1 x_2, y] = [x_1, y][x_2, y]$ for $x_1, x_2 \in G$, $y \in G_{n-1}$. It follows that

$$a_i^{NM} = \left(\prod_{j=1}^{m_i}[b_{ij}, c_{ij}]\right)^{NM} = \prod_{j=1}^{m_i}[b_{ij}^N, c_{ij}^M] = \prod_{j=1}^{m_i}[u_{ij}x_{ij}, v_{ij}y_{ij}] = \prod_{j=1}^{m_i}[u_{ij}, v_{ij}].$$

So, if $a$ is any element of the subgroup $A$ generated by $a_1, \ldots, a_m$, then $a^{NM}$ is contained in the subgroup $H$ of $G_n$ generated by the set of all commutators $[u_k, v_l]$, $1 \leq k \leq s$, $1 \leq l \leq t$.

Since $A = \mathsf{Z}^k \times F$, where $k \geq 0$ and $F$ is finite, and $\{a^{NM} : a \in A\} \subseteq H$, it follows that $H$ contains a subgroup isomorphic to $\mathsf{Z}^k$. Hence $k \leq st$ and so rank $G_n \leq st$ by Lemma 2.1.

COROLLARY 2.4. *Let $G$ be a nilpotent group. If $G/[G, G]$ is of finite rank, then so are all the subquotients arising from the lower central series of $G$.*

PROOF. This follows by induction from Lemma 2.3.

LEMMA 2.5. *Let $G$ be a torsion-free nilpotent group of length $n + 1$ such that $G_j/G_{j+1}$ is of finite rank $r_j$, $0 \leq j \leq n$. Then there exists a sequence $\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{n+1} = G$ of normal subgroups such that $N_{j+1}/N_j$, $0 \leq j \leq n$, is contained in the centre of $G/N_j$ and is torsion-free of rank $s_j$ where $\sum_{j=0}^n s_j \leq \sum_{j=0}^n r_j$.*

PROOF. We prove the lemma by induction on the length $l(G)$ of the lower central series. If $l(G) = 1$, nothing has to be shown. Suppose the statement is true when $l(G) \leq n$, and let $G$ be as in the lemma with $l(G) = n + 1$. Let $Z$ denote the centre of $G$ and let

$$N = \{x \in Z : xG_n \in (Z/G_n)^t\}.$$

Then $Z/N$ is torsion-free and since $G/Z$ is also torsion-free, $G/N$ is torsion-free. Moreover, $l(G/N) \leq n$ since $G_n \subseteq Z$. Let $(G/N)_j$, $j = 0, 1, \ldots$, denote the lower central series of $G/N$. Then $(G/N)_n = \{N\}$ and $(G/N)_j/(G/N)_{j+1}$ is a quotient of $G_j/G_{j+1}$. Thus

$$\operatorname{rank}((G/N)_j/(G/N)_{j+1}) \leq r_j \qquad (0 \leq j \leq n-1).$$

Set $N_1 = N$. By the inductive hypothesis, there exists a sequence of normal subgroups $N_k$ of $G$, $k = 2, \ldots, n+1$, such that $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{n+1} = G$, $N_{k+1}/N_k$ is torsion-free and contained in the centre of $G/N_k$, and $N_{k+1}/N_k$ has rank $s_k$, where $\sum_{k=1}^n s_k \leq \sum_{j=0}^{n-1} r_j$. Since $N_1 \subseteq Z$, it only remains to notice that rank $N = \operatorname{rank} G_n$.

To that end, let $\{x_1, \dots, x_d\}$ be an independent subset of $N$. Then, since $N/G_n$ is a torsion group, $x_1^s, \dots, x_d^s \in G_n$ for some $s \in \mathsf{N}$, and these elements are independent in $G_n$. Thus $d \leq \operatorname{rank} G_n$. This finishes the proof.

PROPOSITION 2.6. *Let $G$ be a nilpotent locally compact group such that $G/G_0 = (G/G_0)^c$. Then*

$$\operatorname{sr}(C^*(G)) \leq \max\left\{2, 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G_0/[G_0, G_0]) \right\rfloor\right\}.$$

PROOF. Let $\mathscr{H}$ denote the collection of all compactly generated open subgroups of $G$. Then $G = \cup_{H \in \mathscr{H}} H$ and, for each $H \in \mathscr{H}$, $H_0 = G_0$ and $H/G_0$ is compact since every compact subset of $G/G_0$ generates a compact subgroup of $G/G_0$. Since $C^*(G)$ is the inductive limit of $C^*$-subalgebras $C^*(H)$, $H \in \mathscr{H}$, it follows from Proposition 1.3 that

$$\begin{aligned}\operatorname{sr}(C^*(G)) \leq \sup_{H \in \mathscr{H}} \operatorname{sr}(C^*(H)) &\leq \sup_{H \in \mathscr{H}} \max\left\{2, 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(H_0/[H_0, H_0]) \right\rfloor\right\} \\ &= \max\left\{2, 1 + \left\lfloor \frac{1}{2} \operatorname{rank}(G_0/[G_0, G_0]) \right\rfloor\right\}.\end{aligned}$$

LEMMA 2.7. *Let $N$ be an open normal subgroup of a locally compact group $G$, and suppose that $G/N$ is abelian and torsion-free of finite rank $r$. Then*

$$\operatorname{sr}(C^*(G)) \leq \operatorname{sr}(C^*(N)) + r.$$

PROOF. Let $H$ be a subgroup of $G$ containing $N$ such that $H/N$ is isomorphic to $\mathsf{Z}^m$ for some $m \in \mathsf{N}$. Then $C^*(H)$ can be written as a repeated crossed product

$$C^*(H) = (\dots(C^*(N) \times_{\alpha_1} \mathsf{Z}) \dots) \times_{\alpha_m} \mathsf{Z},$$

and hence Theorem 7.1 of [17] yields that $\operatorname{sr}(C^*(H)) \leq \operatorname{sr}(C^*(N)) + m$.

Let $\mathscr{H}$ be the collection of all subgroups $H$ of $G$ such that $N \subseteq H$ and $H/N$ is finitely generated. Then, for each such $H$, $H/N$ is isomorphic to $\mathsf{Z}^m$, where $m \leq r$, whence $\operatorname{sr}(C^*(H)) \leq \operatorname{sr}(C^*(N)) + r$ by the first paragraph. Finally,

$$\operatorname{sr}(C^*(G)) \leq \sup_{H \in \mathscr{H}} \operatorname{sr}(C^*(H)) \leq \operatorname{sr}(C^*(N)) + r.$$

THEOREM 2.8. *Let $G$ be a nilpotent locally compact group. Then the following conditions are equivalent.*

(i) $\mathrm{sr}(C^*(G)) < \infty$.

(ii) $\mathrm{RR}(C^*(G)) < \infty$.

(iii) *The abelian locally compact group $G/[G, G]$ has finite rank.*

Proof. (i) $\Rightarrow$ (ii) follows from $\mathrm{RR}(A) \leq 2\,\mathrm{sr}(A) - 1$ for any $C^*$-algebra $A$ [3].

Suppose that (ii) holds and let $H = G/[G, G]$. Since $C^*(H)$ is a quotient of $C^*(G)$, $\mathrm{RR}(C^*(H)) < \infty$. Now $H/H^c$ is the direct product of a vector group $\mathsf{R}^n$ and a torsion-free discrete group $D$. By definition, $\mathrm{rank}(H) = n + \mathrm{rank}(D)$. Thus it suffices to observe that $D$ has finite rank. Since $C^*(D)$ is a quotient of $C^*(H)$, we have

$$\mathrm{rank}(D) = \mathrm{RR}(C^*(D)) \leq \mathrm{RR}(C^*(H)) < \infty.$$

To show (iii) $\Rightarrow$ (i), let $q : G \to G/G_0$ denote the quotient homomorphism and let $N = q^{-1}((G/G_0)^c)$. By Proposition 2.6,

$$\mathrm{sr}(C^*(N)) \leq \max\left\{2, 1 + \left\lfloor \frac{1}{2}\,\mathrm{rank}(G_0/[G_0, G_0]) \right\rfloor\right\}.$$

$N$ is open in $G$ because $G/G_0$ is totally disconnected and hence has compact open subgroups. Moreover, $G/N$ is torsion-free. By hypothesis (iii), $(G/N)/[G/N, G/N]$ has finite rank. Then, by Corollary 2.4 and Lemma 2.5, there exists a sequence $N_0 = N \subseteq N_1 \subseteq \cdots \subseteq N_{n+1} = G$ of normal subgroups of $G$ such that each $N_{j+1}/N_j$ is abelian, torsion-free and has finite rank $r_j$, say ($0 \leq j \leq n$). Climbing up the ascending series $N_j$, $j = 1, \ldots, n+1$, and applying Lemma 2.7 at each step, we obtain $\mathrm{sr}(C^*(G)) \leq \mathrm{sr}(C^*(N)) + \sum_{j=0}^{n} r_j$.

The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) of the preceding theorem do not hold for arbitrary locally compact groups. We conclude with a simple example of a discrete group $G$ which has an abelian subgroup of index 2 with the property that $G/[G, G]$ has rank zero, whereas

$$\mathrm{sr}(C^*(G)) = \mathrm{RR}(C^*(G)) = \infty.$$

Example 2.9. Let $N$ denote the direct sum of infinitely many copies of $\mathsf{Z}$ and $G$ the semidirect product $G = \mathsf{Z}_2 \ltimes N$, where the action of $\mathsf{Z}_2 = \{1, -1\}$ on $N$ is given by $(-1) \cdot (x_1, x_2, \ldots) = (-x_1, -x_2, \ldots)$. Then $[G, G]$ consists of all elements $(x_1, x_2, \ldots)$ of $N$ such that all $x_j$ are even. Thus $G/[G, G]$ is the direct sum of infinitely many copies of $\mathsf{Z}_2$ and hence is a torsion group. So $\mathrm{rank}(G/[G, G]) = 0$. However, the finite conjugacy class subgroup of $G$

equals $N$. Then, by Theorem 3.4 of [1],

$$\mathrm{RR}(C^*(G)) \geq \left\lceil \frac{\mathrm{rank}(N)}{2[G:N]-1} \right\rceil = \left\lceil \frac{1}{3}\,\mathrm{rank}(N) \right\rceil = \infty,$$

and hence also $\mathrm{sr}(C^*(G)) = \infty$.

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