

ANOTHER WAY TO SAY SUBSOLUTION: THE MAXIMUM PRINCIPLE AND SUMS OF GREEN FUNCTIONS

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Abstract

Consider an elliptic second order differential operator L with no zeroth order term (for example the Laplacian $L = -\Delta$). If $Lu \leq 0$ in a domain U , then of course u satisfies the maximum principle on every subdomain $V \subset U$.

We prove a converse, namely that $Lu \leq 0$ on U if on every subdomain V , the maximum principle is satisfied by $u + v$ whenever v is a finite linear combination (with positive coefficients) of Green functions with poles outside \bar{V} .

This extends a result of Crandall and Zhang for the Laplacian.

We also treat the heat equation, improving Crandall and Wang’s recent result. The general parabolic case remains open.

1. Introduction

We will extend the following “maximum principle characterization” of subharmonic functions, given by Crandall and Zhang in their paper *Another way to say harmonic* [5], to a wide class of second order elliptic operators with no zeroth order terms.

Write $K(r) = 1/r^{n-2}$ for the radially symmetric fundamental solution of Laplace’s equation, when $n \neq 2$, with $K(r) = \log(1/r)$ when $n = 2$. Also write $K(x) = K(|x|)$ when $x \in \mathbf{R}^n$.

THEOREM 1.1 ([5], Theorem 2.4). *Let $u : U \rightarrow \mathbf{R}$ be upper semicontinuous on an open set $U \subset \mathbf{R}^n$, $n \geq 1$. Suppose $u + v$ satisfies the maximum principle on V whenever V is a bounded open set with $\bar{V} \subset U$ and v has the form*

$$v(x) = \sum_{m=1}^n a_m K(x - \xi_m)$$

for some positive real numbers a_1, \dots, a_n and some points $\xi_1, \dots, \xi_n \in \mathbf{R}^n \setminus \bar{V}$. (In other words $\max_{\bar{V}}(u + v) = \max_{\partial V}(u + v)$ for all such v and V .)

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Then u is subharmonic.

Note that v is a positive linear combination of n fundamental solutions with distinct singularities outside \bar{V} . The number “ n ” of terms here is minimal, as Crandall and Zhang observed.

The converse to Theorem 1.1 is trivial: if u is subharmonic on V then so is $u + v$, because v is harmonic on V , and thus $u + v$ does indeed satisfy the maximum principle on V .

Our main goal is to find a similar characterization of subsolutions in the elliptic case. This we do in Theorem 2.6 of Section 2. Our Theorem 2.6 is worse than Crandall and Zhang’s Theorem 1.1 because our linear combinations v employ arbitrarily many fundamental solutions, rather than just n of them. Perhaps this is unavoidable, because there is no explicit formula for the fundamental solution in the general elliptic case and hence there is much less information to work with. Instead we proceed by a potential theoretic line of proof that is less explicit and more flexible.

Theorem 2.6 is in other ways better than Theorem 1.1, because it uses Green functions (of which fundamental solutions on all of space are just a special case, when $n \geq 3$).

Our second goal, in Section 3, is to strengthen Theorem 1.1 for subharmonic functions – we consider $u - v$ as well as $u + v$, and we localize the singularities.

In Section 4 we strengthen Crandall and Wang’s analogous characterization [4] of subsolutions of the heat equation, using linear combinations of just n Gaussian fundamental solutions in \mathbf{R}^{n+1} (whereas [4] needed $n + 1$ Gaussians). A natural question is whether subsolutions of more general parabolic equations can be characterized similarly, in terms of the maximum principle and Green functions. Technical difficulties have so far prevented us from dealing with this parabolic case, and so the question remains open.

Incidentally, this line of research began when subsolutions of the ∞ -Laplace equation were characterized by Crandall, Evans and Gariepy [3] in terms of the maximum principle and cone functions:

if $u + v$ satisfies the maximum principle on V whenever V is a bounded open set with $\bar{V} \subset U$ and $v(x) = -a|x - \xi|$ for some $a > 0$ and $\xi \in \mathbf{R}^n \setminus \bar{V}$, then u is ∞ -subharmonic, meaning $-Du \cdot (D^2u)Du \leq 0$.

Here Du is the gradient vector of u and D^2u is the Hessian. For a strikingly simple proof of this fact, see [5, Section 2]. It remains an open problem to similarly characterize subsolutions of the p -Laplace equation, for $p \neq 2, \infty$. Partial results are in [5, Section 3]. For the ∞ -heat equation, see [4, Section 3]. The p -heat equation seems not to have been investigated in this regard, for $p \neq 2, \infty$.

2. Subsolutions of elliptic equations

Our characterization of subsolutions comes in Theorem 2.6 below. But first we establish some notation and lemmas.

NOTE. We will not consider dimension $n = 1$ because the potential theory there is different (for example the Green function is finite on the diagonal) and thus would require special treatment.

2.1. Definitions

Let Λ be an open subset of \mathbb{R}^n , $n \geq 2$. We consider in Λ the elliptic operator L with real coefficients and no zeroth order term:

$$Lu = - \sum_{i,j=1}^n a_{ij} D_i D_j u + \sum_{i=1}^n b_i D_i u,$$

where $a_{ij} \in C^{2+\delta}(\Lambda)$, $b_i \in C^{1+\delta}(\Lambda)$, and $a_{ij} = a_{ji}$. The ellipticity condition is

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for all $x \in \Lambda$ and $\xi \in \mathbb{R}^n$. These assumptions on the coefficients are sufficient for the potential theoretic arguments we use later on.

We shall give a maximum principle characterization of the subsolutions associated with the operator L . The known results that we shall use appeared in [2] and [6]. We follow the potential theoretic terminology of those references.

DEFINITION 2.1. A function $u : U \rightarrow [-\infty, +\infty)$ defined on an open subset U of Λ is called *L-hypoharmonic* if it satisfies the following conditions:

- (i) u is upper semicontinuous;
- (ii) for each L -regular open set $D \Subset U$ we have the inequality

$$u(x) \leq \int u d\rho_x^D \quad \text{for every } x \in D,$$

where ρ_x^D denotes the L -harmonic measure relative to D and x (see [2, p. 62]).

An L -hypoharmonic function on U is called *L-subharmonic* if it is finite at some point in each component of U . A function u is called *L-superharmonic* if $-u$ is L -subharmonic.

The point of L -hypoharmonic is that on every L -regular subdomain D , the function u is majorized by the L -harmonic function equalling u on ∂D (if there is one).

REMARK. Littman [8] studied *weakly L -subharmonic functions*, which are locally integrable functions u on U satisfying

$$\int u(x)L^*v(x) dx \leq 0$$

whenever $v \geq 0$ and $v \in C_c^2(U)$, where L^* is the formal adjoint of L . Note that every weakly L -subharmonic function is equal *a.e.* to an L -subharmonic function, and conversely, as demonstrated in [12] (with the additional, but superfluous, hypothesis that L is self-adjoint).

We shall use the local criterion for L -hypoharmonic given in [2, p. 72]. Thus u is L -hypoharmonic if $u : U \rightarrow [-\infty, +\infty)$ is upper semicontinuous and the inequality in (ii) holds for each D in some base for the Euclidean topology on U that consists entirely of L -regular open sets.

We employ a particular type of base for the topology of U . Fix a bounded convex domain B that contains the origin, fix $\epsilon > 0$, and put

$$(1) \quad \mathbf{B}(U) = \mathbf{B}_{B,\epsilon}(U) = \{x + \delta B : x \in \mathbf{R}^n, 0 < \delta < \epsilon, \overline{x + \delta B} \subset U\}.$$

Then $\mathbf{B}(U)$ is a base which consists entirely of L -regular sets, because every convex domain is Δ -regular and regularity does not vary with the coefficients of the operator, under our conditions on the a_{ij} and b_i (see [6] or [11]).

Let V be an open set, and let u be upper semicontinuous and upper finite on \overline{V} . We say that u *satisfies the maximum principle* on V if

$$\max_{\overline{V}} u = \max_{\partial V} u.$$

Note that L -hypoharmonic functions satisfy the maximum principle on every $V \Subset U$, by [2, p. 72].

2.2. Preparatory lemmas

The first lemma below indicates the direction in which we shall go.

Given an open subset V of Λ , we denote by $H_L(V)$ the class of all L -harmonic functions on V , such a function being a classical solution of $Lu = 0$. We also denote by $C(\overline{V})$ the class of all continuous functions on \overline{V} . If V is a regular set and f is a continuous real-valued function on ∂V , then we denote by H_f^V the function on \overline{V} that belongs to $C(\overline{V}) \cap H_L(V)$ and coincides with f on ∂V ; in other words the solution of the Dirichlet problem on V with boundary data f . By definition of L -harmonic measure [2, p. 62], one has $H_f^V(x) = \int f d\rho_x^V$ for $x \in V$.

LEMMA 2.2. *Let $u : U \rightarrow [-\infty, +\infty)$ be upper semicontinuous on an open subset U of Λ . Suppose either*

- (a) $u + v$ satisfies the maximum principle on V whenever $V \in \mathbf{B}(U)$ and $v \in C(\overline{V}) \cap H_L(V)$; or else
- (b) $u - v$ satisfies the maximum principle on V whenever $V \in \mathbf{B}(U)$ and $v \in C(\overline{V}) \cap H_L(V)$.

Then u is L -hypoharmonic on U .

PROOF OF LEMMA 2.2. We need only prove part (a), since $-v \in C(\overline{V}) \cap H_L(V)$ if and only if $v \in C(\overline{V}) \cap H_L(V)$.

Fix $V \in \mathbf{B}(U)$. Since the restriction of u to ∂V is upper semicontinuous and upper finite, there is a decreasing sequence of real-valued continuous functions $\{f_k\}$ on ∂V such that $\lim_{k \rightarrow \infty} f_k = u$. Since V is regular, for each k we can define $v_k \in C(\overline{V}) \cap H_L(V)$ by putting $v_k = H_{f_k}^V$. Then $v_k \geq u$ on ∂V , and $u - v_k$ is upper semicontinuous and upper finite on \overline{V} , so that the maximum principle hypothesis implies $u \leq v_k$ on V . Since $\{f_k\}$ is decreasing, so is $\{v_k\}$, and hence for all $x \in V$ we have

$$u(x) \leq \lim_{k \rightarrow \infty} v_k(x) = \lim_{k \rightarrow \infty} \int f_k d\rho_x^V = \int u d\rho_x^V.$$

Since this holds for all $V \in \mathbf{B}(U)$, the local criterion for L -hypoharmonic shows u is L -hypoharmonic.

REMARK. The function u by itself might satisfy the maximum principle on every V without u being L -hypoharmonic, because subsolutions of any operator having the same form as L will satisfy the maximum principle. For example, the harmonic function $u(x_1, x_2) = x_1^2 - x_2^2$ satisfies the maximum principle on every subdomain of \mathbb{R}^2 , but is not L -hypoharmonic for $L = -(D_1^2 + 2D_2^2)$ because $Lu = 2 > 0$.

The main result of this section, Theorem 2.6, will confine the test functions v required in Lemma 2.2 to a smaller class. For its proof we require two further lemmas, both of which extend known results for the harmonic case $L = -\Delta$.

The first lemma is a criterion for L -superharmonicity that extends [1, Corollary 3.2.4].

LEMMA 2.3. Let u be an L -superharmonic function on an open set U and let h be an L -superharmonic function on an open subset V of U . If

$$(2) \quad u(y) \leq \liminf_{x \rightarrow y, x \in V} h(x)$$

for all $y \in U \cap \partial V$, then the function

$$w = \begin{cases} \min(h, u) & \text{on } V, \\ u & \text{on } U \setminus V, \end{cases}$$

is L -superharmonic on U .

PROOF OF LEMMA 2.3. Clearly $w > -\infty$ on U , because $u > -\infty$ and $h > -\infty$. Also, w is finite at some point in each component of U , since $w \leq u$. Condition (2) implies that w is lower semicontinuous at every point of $U \cap \partial V$, and therefore on U . To complete the proof, we use the local criterion for L -superharmonicity. Clearly the criterion is satisfied at every point of $U \setminus \partial V$. Let $\xi_0 \in U \cap \partial V$, and let D be a regular open set such that $\xi_0 \in D$ and $\overline{D} \subset U$. Then

$$w(\xi_0) = u(\xi_0) \geq \int u d\rho_{\xi_0}^D \geq \int w d\rho_{\xi_0}^D,$$

so that the local criterion is satisfied at ξ_0 . Hence w is L -superharmonic on U .

Lemma 2.3 is used in the proof of the Extension Lemma below, which extends the harmonic case $L = -\Delta$ in [1, p. 192], by following a broadly similar proof. For this Extension Lemma we require the classical notion of a Green function for the Dirichlet problem relative to the operator L .

DEFINITION 2.4. A *classical L -Green function* on a subdomain Γ of Λ is a continuous, extended-real-valued function $G : \Gamma \times \Gamma \rightarrow (0, \infty]$ with the following properties for each $\xi \in \Gamma$:

- (i) $LG(\cdot, \xi) = 0$ (in the classical sense) on $\Gamma \setminus \{\xi\}$;
- (ii) $G(\xi, \xi) = \infty$;
- (iii) $G(\cdot, \xi)$ can be continuously extended to 0 on $\partial\Gamma$ (including the point at infinity if Γ is unbounded).

Traditionally, a Green function's singularity must also be of a prescribed type (see [9, p. 20], for example), but we don't need that property below.

If Γ is a regular set and $\overline{\Gamma}$ is a compact subset of Λ , then it follows from [9, p. 64] that Γ possesses a classical L -Green function. If $L = -\Delta$ and $n \geq 3$, then $\Lambda = \mathbf{R}^n$ itself possess a classical Green function, namely $G(x, \xi) = |x - \xi|^{2-n}$.

In the proof of the Extension Lemma, it will be convenient to use level sets of the L -Green function $G(\cdot, \xi_0)$ with pole at a point ξ_0 . For any $\xi_0 \in \Gamma$ and $r > 0$ such that $K(r) > 0$, we put

$$B_\Gamma(\xi_0, r) = \{x \in \Gamma : G(x, \xi_0) > K(r)\},$$

and call $B_\Gamma(\xi_0, r)$ the L -Green ball with centre ξ_0 and radius r . Note that if $L = -\Delta$ and $\Lambda = \mathbf{R}^n$ with Green function as above, then $B_\Gamma(\xi_0, r)$ is the Euclidean open ball with centre ξ_0 and radius r .

In general, $B_\Gamma(\xi_0, r)$ is an open set with closure in Γ . It is also connected, because if it had a component U that did not contain ξ_0 , we would have

$G(\cdot, \xi_0) = K(r)$ on ∂U and $LG(\cdot, \xi_0) = 0$ throughout U , which would yield $G(\cdot, \xi_0) = K(r)$ on U , contradicting the definition of $B_\Gamma(\xi_0, r)$.

Further, $B_\Gamma(\xi_0, r)$ expands to fill Γ as $r \rightarrow \infty$ if $n \geq 3$, or as $r \rightarrow 1$ if $n = 2$, since $K(r) \rightarrow 0$.

LEMMA 2.5 (Extension Lemma). *Let Γ be a subdomain of Λ that has a classical L -Green function G . Let C be a convex compact subset of Γ , and let v be an L -superharmonic function on an open set containing C .*

Then given any point $\xi_0 \in \Gamma$, there is an L -superharmonic function \bar{v} on Γ such that $\bar{v} = v$ on a neighbourhood of C , and $\bar{v} = \alpha + \beta G(\cdot, \xi_0)$ on $\Gamma \setminus B_\Gamma(\xi_0, \rho)$ for some real constants α, β, ρ with $\beta, \rho > 0$. Furthermore, $\bar{v} - \alpha > 0$ on Γ .

PROOF OF LEMMA 2.5. Let U be a bounded open set containing C , such that v is L -superharmonic and lower bounded on U . By adding a constant if necessary, we can assume that $v > 0$ on U . Choose $R > 0$ such that $\bar{U} \subset B_\Gamma(\xi_0, R)$.

Our first aim is to show that there is an L -superharmonic function w on $B_\Gamma(\xi_0, R + r)$ such that $w = v$ on a neighbourhood of C , where $r = 1$ if $n \geq 3$ and $r = (1 - R)/2$ if $n = 2$. Let D be a convex domain such that $C \subset D \Subset U$. Put $u = \tilde{R}_v^D$, the regularized reduced function (or balayage) of v relative to D in U (see [2, p. 80]). Then u is L -superharmonic on U by [2, p. 77], and L -harmonic on $U \setminus \bar{D}$, by [2, p. 84], with $0 \leq u \leq v$ on U and $u = v$ on D by [2, p. 81]. (In other words, u solves the obstacle problem with v as the obstacle on D .)

The next step in the construction of w is to let E be a convex compact set such that $\bar{D} \subset E^\circ$ and $E \subset U$. Put $A = B_\Gamma(\xi_0, R + r) \setminus E$, so that A is a kind of “annular” region. The convexity of E ensures every point of ∂E is a regular point for the L -Dirichlet problem on A , because every such point is regular for Laplace’s equation and regularity does not vary with the coefficients of the operator, as remarked earlier. (Incidentally, here we use exterior regularity of the convex set. Earlier we used interior regularity.) Furthermore, because $G(\cdot, \xi_0) - K(R + r)$ is a positive L -superharmonic function on A that tends to zero at every point of $\partial B_\Gamma(\xi_0, R + r)$, every such boundary point is also regular for the L -Dirichlet problem on A , by the “barrier” criterion in [2, p. 118]. Hence A is a regular set.

Next put $g_1 = u$ on ∂E and $g_1 = 0$ on $\partial B_\Gamma(\xi_0, R + r)$, and put $g_2 = 0$ on ∂E and $g_2 = 1$ on $\partial B_\Gamma(\xi_0, R + r)$. Let $h_k = H_{g_1}^A - kH_{g_2}^A$ for each $k \in \mathbb{N}$. Note that u is continuous on ∂E because u is L -harmonic on $U \setminus \bar{D}$. Since $B_\Gamma(\xi_0, R + r)$ is connected and E is convex, A is connected. Therefore $H_{g_2}^A > 0$ on A , so that $\{h_k\}$ decreases to $-\infty$ in A as $k \rightarrow \infty$. Choose m such that $h_m \leq 0$ on ∂U . Since A is regular, $h_m(x) \rightarrow u(y)$ as $x \rightarrow y$, for every $y \in \partial E$. Since we

also have $h_m \leq 0$ on ∂U and $u \geq 0$ on U , the minimum principle ensures that $h_m \leq u$ on $U \setminus E$.

Now define $w = u$ on E , $w = h_m$ on A . An application of Lemma 2.3 (with $V = U \setminus E$ and $h = h_m$) shows that w is L -superharmonic on U , and hence on $B_\Gamma(\xi_0, R + r)$. Clearly w is L -harmonic on A , with $w = u = v$ on D , and w tends to $-m$ on $\partial B_\Gamma(\xi_0, R + r)$.

Our second aim in the proof is to extend w from $B_\Gamma(\xi_0, R + r)$ to all of Γ , thereby obtaining \bar{v} . Since w is L -harmonic on A , it is continuous on $\partial B_\Gamma(\xi_0, R)$ and so it has a maximum value $M(R)$ there. We choose $\beta > 0$ such that

$$\beta(K(R) - K(R + r)) \geq M(R) + m,$$

so that

$$\beta K(R) - M(R) \geq \beta K(R + r) + m.$$

Then we choose $\alpha \in \mathbf{R}$ such that

$$\beta K(R) - M(R) \geq -\alpha \geq \beta K(R + r) + m,$$

so that

$$\alpha + \beta K(R) \geq M(R) \quad \text{and} \quad \alpha + \beta K(R + r) \leq -m.$$

Thus

$$\begin{aligned} \alpha + \beta G(\cdot, \xi_0) &\geq M(R) && \text{on } \partial B_\Gamma(\xi_0, R), \\ \alpha + \beta G(\cdot, \xi_0) &\leq -m && \text{on } \partial B_\Gamma(\xi_0, R + r). \end{aligned}$$

Now we define

$$\bar{v} = \begin{cases} w & \text{on } \overline{B_\Gamma(\xi_0, R)}, \\ \min(w, \alpha + \beta G(\cdot, \xi_0)) & \text{on } B_\Gamma(\xi_0, R + r) \setminus \overline{B_\Gamma(\xi_0, R)}, \\ \alpha + \beta G(\cdot, \xi_0) & \text{on } \Gamma \setminus B_\Gamma(\xi_0, R + r), \end{cases}$$

and note that $\bar{v} = w = v$ on D . An application of Lemma 2.3 (with $U = B_\Gamma(\xi_0, R + r)$, $V = B_\Gamma(\xi_0, R + r) \setminus \overline{B_\Gamma(\xi_0, R)}$, $u = w$ and $h = \alpha + \beta G(\cdot, \xi_0)$) shows that \bar{v} is L -superharmonic on $B_\Gamma(\xi_0, R + r)$. Another application of Lemma 2.3 (with $U = \Gamma \setminus B_\Gamma(\xi_0, R)$, $V = B_\Gamma(\xi_0, R + r) \setminus \overline{B_\Gamma(\xi_0, R)}$, $u = \alpha + \beta G(\cdot, \xi_0)$ and $h = w$) shows that \bar{v} is also L -superharmonic on $\Gamma \setminus \overline{B_\Gamma(\xi_0, R)}$, and hence \bar{v} is L -superharmonic on Γ . Finally, since $\bar{v} - \alpha = \beta G(\cdot, \xi_0) > 0$ outside a compact subset of Γ , we conclude $\bar{v} - \alpha > 0$ on Γ .

2.3. *The characterization of subsolutions*

We now come to our main result, Theorem 2.6, in which Lemma 2.2 is improved by replacing the arbitrary $v \in C(\bar{V}) \cap H_L(V)$ with a finite linear combination (having positive coefficients) of L -Green functions with poles outside \bar{V} .

THEOREM 2.6. *Let $u : U \rightarrow [-\infty, +\infty)$ be upper semicontinuous on an open subset U of Λ . Suppose U is contained in a domain $\Gamma \subset \Lambda$ which has a classical L -Green function G . Assume either*

- (a) $u + v$ satisfies the maximum principle on V whenever $V \in \mathbf{B}(U)$ and v has the form

$$(3) \quad v = \sum_{m=1}^M a_m G(\cdot, \xi_m)$$

for some $M \in \mathbf{N}$, some positive real numbers a_1, \dots, a_M , and some distinct points $\xi_1, \dots, \xi_M \in U \setminus \bar{V}$; or else

- (b) $u - v$ satisfies the maximum principle on V whenever $V \in \mathbf{B}(U)$ and v has the form (3).

Then u is L -hypoharmonic on U .

For example if $L = -\Delta$ is the Laplacian then one could take the Green function to be the fundamental solution $K(x - \xi)$ on all of space, provided $n \geq 3$. (When $n = 2$, the fundamental solution fails to be a Green function because it is negative near infinity.) Thus Theorem 2.6 implies the Crandall-Zhang result Theorem 1.1 on subharmonic functions, when $n \geq 3$, except using linear combinations v that have arbitrarily many terms. See Section 3 for more on this subharmonic case.

PROOF OF THEOREM 2.6. *Part (a).* In view of Lemma 2.2(a), it suffices to prove that $u + v$ satisfies the maximum principle on V whenever $V \in \mathbf{B}(U)$ and $v \in C(\bar{V}) \cap H_L(V)$. We prove the contrapositive. Suppose that we can find $V \in \mathbf{B}(U)$ and $v \in C(\bar{V}) \cap H_L(V)$ such that

$$\max_{\bar{V}}(u + v) > \max_{\partial V}(u + v).$$

We will show the same inequality holds on a slightly smaller set $V_1 \in \mathbf{B}(U)$.

Define $w = u + v$ on \bar{V} . Then there is $y_0 \in V$ such that $w(y_0) > w(y)$ for all $y \in \partial V$. Since w is upper semicontinuous and ∂V is compact, $w(y_0) > w(y)$ for all y in some neighborhood of ∂V . Hence we can find $V_1 \in \mathbf{B}(U)$ such that $y_0 \in V_1 \Subset V$ and $w(y_0) > w(y)$ for all $y \in \partial V_1$ (for instance one could write $V = x + \delta B$ as in the definition (1) and then put $V_1 = x + \delta_1 B$ where

$\delta_1 \in (0, \delta)$ is sufficiently close to δ , using here that B is convex and contains the origin). Thus

$$\max_{\overline{V_1}}(u + v) > \max_{\partial V_1}(u + v),$$

and v is defined on an open set that contains $\overline{V_1}$.

We now choose ε such that

$$0 < \varepsilon < (u + v)(y_0) - \max_{\partial V_1}(u + v).$$

By the Extension Lemma (with $C = \overline{V_1}$), there is an L -superharmonic function \bar{v} on Γ such that $\bar{v} = v$ on some bounded open set W with $\overline{V_1} \subset W \subset V$, and $\bar{v} - \alpha > 0$ on Γ for some $\alpha \in \mathbf{R}$. We now take the regularized reduced function (or balayage) of $\bar{v} - \alpha$ relative to W in Γ : let $s = \widehat{\mathbf{R}}_{\bar{v} - \alpha}^W$ so that $s = \bar{v} - \alpha = v - \alpha$ on W and s is L -harmonic on $\Gamma \setminus \overline{W}$ (cf. the proof of the Extension Lemma). Hence s is L -harmonic on $\Gamma \setminus \partial W$. Since W is relatively compact, s is an L -potential on Γ by [2, p. 94].

The L -potentials on Γ with point support are proportional, by [6, p. 563], so that there is a nonnegative Radon measure μ on Γ such that

$$s(x) = \int_{\Gamma} G(x, \xi) d\mu(\xi)$$

for all $x \in \Gamma$, by [6, p. 481]. Since s is L -harmonic on $\Gamma \setminus \partial W$, we know μ is supported on ∂W in view of [6, pp. 464-5]. For each fixed $\xi \in \Gamma$ the function $G(\cdot, \xi)$ is an L -potential with point support, and hence G is continuous off the diagonal by [6, p. 480]. Also G is finite off the diagonal, by definition. It follows that G is uniformly continuous on $\overline{V_1} \times \partial W$, so that we can find $\eta > 0$ such that

$$|G(x, \zeta) - G(x, \xi)| < \frac{\varepsilon}{2\mu(\partial W)}$$

whenever $x \in \overline{V_1}$ and $\xi, \zeta \in \partial W$ with $|\xi - \zeta| < \eta$. (Note $\mu(\partial W) > 0$ because otherwise $s \equiv 0$, contradicting that $s = \bar{v} - \alpha > 0$ on W .) We now partition ∂W into finitely many disjoint subsets E_1, \dots, E_M , such that $\mu(E_m) > 0$ and $\text{diam}(E_m) < \eta$ for all $m = 1, \dots, M$. We choose points ξ_1, \dots, ξ_M such that $\xi_m \in E_m$ for all m , and put

$$v_1(x) = \sum_{m=1}^M \mu(E_m)G(x, \xi_m)$$

for all $x \in \Gamma$. Then v_1 has the form (3) relative to V_1 , because ξ_1, \dots, ξ_M lie outside $\overline{V_1}$. Clearly v_1 is a Riemann sum approximation to s , and indeed for

all $x \in \overline{V_1}$ we have

$$\begin{aligned}
 |s(x) - v_1(x)| &= \left| \sum_{m=1}^M \int_{E_m} (G(x, \xi) - G(x, \xi_m)) d\mu(\xi) \right| \\
 &\leq \sum_{m=1}^M \int_{E_m} |G(x, \xi) - G(x, \xi_m)| d\mu(\xi) \\
 &< \sum_{m=1}^M \frac{\varepsilon}{2\mu(\partial W)} \mu(E_m) \\
 (4) \qquad \qquad &= \frac{\varepsilon}{2}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \max_{\partial V_1} (u + v_1) &\leq \max_{\partial V_1} (u + s) + \frac{\varepsilon}{2} && \text{by (4)} \\
 &= \max_{\partial V_1} (u + v - \alpha) + \frac{\varepsilon}{2} && \text{since } s = v - \alpha \text{ on } W \\
 &< (u + v)(y_0) - \alpha - \frac{\varepsilon}{2} && \text{by choice of } \varepsilon \\
 &= (u + s)(y_0) - \frac{\varepsilon}{2} && \text{since } s = v - \alpha \text{ on } W \\
 &< (u + v_1)(y_0) && \text{by (4)}.
 \end{aligned}$$

Thus we have found $V_1 \in \mathbf{B}(U)$ and v_1 of the form (3) such that $u + v_1$ does not satisfy the maximum principle on V_1 . This proves the contrapositive and establishes the theorem.

Part (b). Simply make the obvious changes to the proof of part (a), using $u - v$ instead of $u + v$, and using Lemma 2.2(b) at the beginning instead of Lemma 2.2(a).

3. Subharmonic functions

3.1. The characterization

The characterization of subsolutions in Theorem 2.6 takes a particularly attractive form for the Laplacian $L = -\Delta$, because one can fix the number of terms in the linear combinations to be $M = n$ provided one employs the fundamental solution instead of a Green function. This is the content of Theorem 1.1, due to Crandall and Zhang.

We make two improvements to that result, in Theorem 3.1 below. First, we handle $u - v$ as well as $u + v$ in the hypotheses – this does require proof.

Second, we localize the singularities ξ_m to belong to U (whereas Crandall and Zhang considered all ξ_m in \mathbf{R}^n , in their hypotheses). It is natural to seek localized singularities because subharmonicity is itself a local property. Note that the singularities are already localized to U in our elliptic characterization theorem, Theorem 2.6.

We also simplify Crandall and Zhang's proof: see the remarks after the theorem below.

Recall the fundamental solution $K(x) = K(|x|)$ of the Laplacian from Section 1.

THEOREM 3.1. *Let $u : U \rightarrow \mathbf{R}$ where U is an open set in \mathbf{R}^n , $n \geq 1$, with neighborhood base $\mathcal{B}(U)$. Suppose $u \in C^2(U)$. Assume either*

- (a) *$u + v$ satisfies the maximum principle on V whenever $V \in \mathcal{B}(U)$ and v has the form*

$$(5) \quad v = \sum_{m=1}^n a_m K(\cdot - \xi_m)$$

for some positive real numbers a_1, \dots, a_n , and some distinct points $\xi_1, \dots, \xi_n \in U \setminus \bar{V}$; or else

- (b) *$u - v$ satisfies the maximum principle on V whenever $V \in \mathcal{B}(U)$ and v has the form (5).*

Then u is subharmonic in U .

COROLLARY 3.2. *Theorem 3.1 still holds if the hypotheses $u : U \rightarrow \mathbf{R}$ and $u \in C^2(U)$ are weakened to $u : U \rightarrow [-\infty, \infty)$ being upper semicontinuous with $u \not\equiv -\infty$.*

REMARKS ON THEOREM 3.1. 1. The neighborhood base $\mathcal{B}(U)$ here is arbitrary, unlike in Theorem 2.6 where each $V \in \mathcal{B}(U)$ must be convex.

2. In one dimension ($n = 1$) the fundamental solution is $K(x) = -|x|$. Then $v(x) = a_1 K(x - \xi_1)$ is linear on the interval V , and can be either increasing or decreasing depending on whether the singularity ξ_1 is to the right or left of V . Thus in one dimension, Theorem 3.1 says u is convex on an interval U if for every subinterval $V \subset U$ and every linear function v on V , the sum $u + v$ attains its maximum at an endpoint of V . This statement is easily proved by the contrapositive.

3. To prove the theorem and corollary below, we follow Crandall and Zhang's method. But we introduce a new parameter λ to allow localization of the singularities. And we perturb the standard basis $\{e_1, \dots, e_n\}$ for \mathbf{R}^n to the basis $\{q_1, \dots, q_n\}$, in estimate (9) in the proof, thereby eliminating the tricky "non-generic" case of Crandall and Zhang.

A further simplification is the removal of the case $Du(x_0) = 0$, which required separate treatment in [5, p. 251, 253]. In Theorem 3.1 we remove this case by a simple perturbation. In Corollary 3.2 we avoid it by employing a weaker-than-usual notion of viscosity subsolution, namely a definition that has the restriction $D\phi(x_0) \neq 0$ imposed on the test functions [7, Definition 2.5].

4. Crandall and Zhang also proved a version of Theorem 1.1 in which the n -th term in the linear combination $v(x)$ is changed from $a_n K(x - \xi_n)$ to $x \cdot \xi_n$ and the coefficients are $a_m = 1$ for $m = 1, \dots, n - 1$. The proof is similar but easier, and our proof below can be adapted to this version also.

PROOF OF THEOREM 3.1. *Part (a).* To establish the contrapositive we assume u is *not* subharmonic, meaning $\Delta u(x_0) < 0$ at some point $x_0 \in U$. We may suppose the gradient vector $Du(x_0) \neq 0$, just by shifting x_0 slightly if necessary; this is possible since $Du \equiv 0$ in a neighborhood of x_0 would imply $\Delta u(x_0) = 0$.

By a translation we may further suppose $x_0 = 0$, so that we have

$$\Delta u(0) < 0 \quad \text{and} \quad Du(0) \neq 0.$$

We will choose $a_m > 0$ and $\xi_m \in U$ distinct in such a way that

$$v(x) = \sum_{m=1}^n a_m K(x - \xi_m)$$

satisfies

$$(6) \quad D(u + v)(0) = 0,$$

$$(7) \quad D^2(u + v)(0) < 0.$$

These two conditions imply the origin is a strict local maximum point for $u + v$, and so $u + v$ fails to satisfy the maximum principle on V whenever $V \in \mathcal{B}(U)$ contains the origin and is sufficiently small (using that v has the desired form (5) as soon as V is small enough to avoid the singularities at ξ_1, \dots, ξ_n). This failure of the maximum principle establishes the contrapositive of part (a) of the theorem.

The next portion of the proof, in which we choose a_m and ξ_m , will appear unmotivated and should really be read backwards from the end. But in the interests of clear logic, we build the proof carefully from the ground up.

Write

$$D^2u = [D_j D_k u]_{j,k=1}^n$$

for the Hessian matrix of u , so that D^2u is real and symmetric at each x . In particular the Hessian $D^2u(0)$ at the origin has n real eigenvalues (counted

with respect to multiplicity), which we denote by $\lambda_1, \dots, \lambda_n$. Choose a number $\lambda > 0$ large enough that $\lambda > \lambda_m$ for all m , or $1 - \lambda_m/\lambda > 0$ for all m . Notice that

$$\begin{aligned} \sum_{m=1}^n \frac{1 - \lambda_m/\lambda}{n} &= 1 - (\lambda_1 + \dots + \lambda_n)/\lambda n = 1 - \text{Trace } D^2u(0)/\lambda n \\ &= 1 - \Delta u(0)/\lambda n > 1 \end{aligned}$$

since $\Delta u(0) < 0$. Therefore it is possible to choose numbers $\alpha_1, \dots, \alpha_n > 0$ such that $\sum_{m=1}^n \alpha_m = 1$ and

$$\frac{1 - \lambda_m/\lambda}{n} > \alpha_m, \quad m = 1, \dots, n.$$

These n inequalities imply the diagonal matrix inequality

$$E - \lambda I + \sum_{m=1}^n \alpha_m \lambda (n e_m e_m^T) < 0,$$

where E is the diagonal matrix of eigenvalues, with entries $\lambda_1, \dots, \lambda_n$, and where $e_m = (0, \dots, 1, \dots, 0)^T$ is the m^{th} unit column vector. Hence

$$(8) \quad E + \sum_{m=1}^n \alpha_m \lambda (-I + n q_m q_m^T) < 0$$

whenever q_1, \dots, q_n are vectors sufficiently close to e_1, \dots, e_n , respectively, where we have used also here that $\sum_{m=1}^n \alpha_m = 1$.

Now take p_1, \dots, p_n to be an orthonormal collection of eigenvectors of $D^2u(0)$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$, so that $P^T D^2u(0) P = E$ where $P = [p_1 \dots p_n]$ is the matrix with the p_m as its columns. We may suppose $-p_m^T D u(0) \geq 0$ for each m , just by replacing p_m with $-p_m$ if necessary. That is,

$$-P^T D u(0) \cdot e_m \geq 0, \quad m = 1, \dots, n.$$

Geometrically, this says the vector $-P^T D u(0)$ lies in the closed first quadrant of \mathbb{R}^n , and is nonzero (since $D u(0) \neq 0$). Clearly an orthonormal system q_1, \dots, q_n exists such that

$$(9) \quad -P^T D u(0) \cdot q_m > 0, \quad m = 1, \dots, n,$$

and that the q_1, \dots, q_n are a small enough perturbation of e_1, \dots, e_n for (8) to hold.

Write z_1, \dots, z_n for the columns of PQ , meaning $PQ = [z_1 \cdots z_n]$, and let

$$\xi_m = -\frac{(P^T Du(0) \cdot q_m)}{\alpha_m \lambda} z_m,$$

so that the ξ_m are all distinct (since the z_m are orthonormal) and

$$(10) \quad \xi_m = |\xi_m| z_m \quad \text{where} \quad |\xi_m| = -\frac{(P^T Du(0) \cdot q_m)}{\alpha_m \lambda} > 0.$$

We will show the singularities ξ_m lie in U whenever λ is chosen sufficiently large. Clearly $\frac{1-\lambda_m/\lambda}{n} < \frac{2}{2n-1}$ for all large λ , and for such λ our choice of $\alpha_m < \frac{1-\lambda_m/\lambda}{n}$ guarantees $\alpha_m < \frac{2}{2n-1}$ for all m . Thus

$$\alpha_\ell = 1 - \sum_{m \neq \ell} \alpha_m > 1 - (n-1) \frac{2}{2n-1} = \frac{1}{2n-1}$$

for each ℓ . Hence

$$|\xi_\ell| = -\frac{P^T Du(0) \cdot q_\ell}{\alpha_\ell \lambda} < \frac{|P^T Du(0)|}{\frac{1}{2n-1} \lambda} \rightarrow 0$$

as $\lambda \rightarrow \infty$, so that by choosing λ sufficiently large we ensure the singularities ξ_m all lie close to $x_0 = 0$ and inside U .

Now define

$$(11) \quad \begin{aligned} a_m &= -(Du(0) \cdot z_m) |\xi_m|^{n-1} \\ &= -(P^T Du(0) \cdot q_m) |\xi_m|^{n-1} && \text{since } z_m = Pq_m \\ &= \alpha_m \lambda |\xi_m|^n && \text{by (10)} \\ &> 0. \end{aligned}$$

This completes our definitions. Now to prove (6) and (7), we compute

$$\begin{aligned} v_{x_j}(x) &= \sum_{m=1}^n a_m (-1) \frac{x_j - (\xi_m)_j}{|x - \xi_m|^n}, \\ v_{x_j}(0) &= \sum_{m=1}^n a_m \frac{(\xi_m)_j}{|\xi_m|^n}, \\ v_{x_j x_k}(0) &= \sum_{m=1}^n a_m \frac{1}{|\xi_m|^n} \left(-\delta_{jk} + n \frac{(\xi_m)_j (\xi_m)_k}{|\xi_m|^2} \right), \end{aligned}$$

and hence

$$(12) \quad Dv(0) = \sum_{m=1}^n \alpha_m \lambda \xi_m = \sum_{m=1}^n \frac{a_m}{|\xi_m|^{n-1}} z_m,$$

$$(13) \quad D^2v(0) = \sum_{m=1}^n \frac{a_m}{|\xi_m|^n} (-I + n z_m z_m^T),$$

since $z_m = \xi_m / |\xi_m|$. Using (12) and the definition of a_m , we find

$$Dv(0) = - \sum_{m=1}^n (Du(0) \cdot z_m) z_m = -Du(0)$$

because the z_m form an orthonormal basis (notice P and Q are orthogonal matrices). Thus $D(u+v)(0) = 0$, which is (6).

Conjugating the Hessian inequality (7) with P shows that it is equivalent to

$$P^T \{D^2u(0) + D^2v(0)\} P < 0.$$

Then substituting in the formula (13) for the Hessian of v , and invoking relation (11), reduces this last inequality to (8) (recalling $P^T z_m = q_m$). This proves (7).

Part (b). It remains to prove u is subharmonic if $u - v$ (instead of $u + v$) satisfies the maximum principle on V whenever $V \in \mathcal{B}(U)$ and v has the form (5). The goal is now to prove (6) and (7) with v replaced by $-v$. Modifying the proof above, we choose $\lambda > 0$ large enough that $\lambda > -\lambda_m$ for all m , or $1 + \lambda_m/\lambda > 0$ for all m . Then

$$\sum_{m=1}^n \frac{1 + \lambda_m/\lambda}{n} = 1 + \Delta u(0)/\lambda n < 1$$

since $\Delta u(0) < 0$. Therefore it is possible to choose numbers $\alpha_1, \dots, \alpha_n > 0$ such that $\sum_{m=1}^n \alpha_m = 1$ and

$$\frac{1 + \lambda_m/\lambda}{n} < \alpha_m, \quad m = 1, \dots, n.$$

Hence the analogue of (8) is

$$(14) \quad E - \sum_{m=1}^n \alpha_m \lambda (-I + n q_m q_m^T) < 0.$$

Continuing with the proof like earlier, we may suppose $P^T Du(0) \cdot e_m \geq 0$ for each m . Thus an orthonormal system q_1, \dots, q_n exists such that $P^T Du(0) \cdot$

$q_m > 0$ for each m , and that the q_1, \dots, q_m are a small enough perturbation of e_1, \dots, e_m for (14) to hold.

Letting

$$\xi_m = \frac{(P^T Du(0) \cdot q_m)}{\alpha_m \lambda} z_m,$$

one finds

$$(15) \quad \xi_m = |\xi_m| z_m \quad \text{where} \quad |\xi_m| = \frac{(P^T Du(0) \cdot q_m)}{\alpha_m \lambda} > 0.$$

To show the singularities ξ_m lie in U whenever λ is chosen sufficiently large, just observe

$$\alpha_m \lambda > \frac{\lambda + \lambda_m}{n} \rightarrow \infty$$

as $\lambda \rightarrow \infty$, which implies by (15) that $|\xi_m| \rightarrow 0$ as $\lambda \rightarrow \infty$. (Notice this part of the proof is easier than in part (a)).

To complete the proof, only a few more sign changes need be made: define $a_m = (Du(0) \cdot z_m) |\xi_m|^{n-1} = \alpha_m \lambda |\xi_m|^n > 0$, and use (12) and the definition of a_m to show $Dv(0) = Du(0)$, giving $D(u - v)(0) = 0$. Finally, after conjugating the Hessian inequality with P and changing v to $-v$, the desired inequality is $P^T \{D^2u(0) - D^2v(0)\} P < 0$, which we prove like before.

3.2. Proof of Corollary 3.2

Let $u : U \rightarrow [-\infty, \infty)$ be upper semicontinuous with $u \not\equiv -\infty$. We will prove only part (a), since the proof of part (b) is the same except with v changed to $-v$.

To prove the contrapositive of part (a), suppose u is not subharmonic in U . Then u is not a viscosity subsolution of Laplace's equation in U , by the equivalence of viscosity subsolutions and the usual potential theoretic subharmonic functions (see [7, Theorem 2.7] with $p = 2$, and the references therein).

Hence by definition of viscosity subsolution (see [7, Definition 2.5]), there exists a point $x_0 \in U$ and a function $\phi \in C^2(U)$ such that $u(x_0) = \phi(x_0)$, $u(x) < \phi(x)$ for all $x \in U \setminus \{x_0\}$, and $D\phi(x_0) \neq 0$ and $\Delta\phi(x_0) < 0$. By applying the proof of Theorem 3.1 to the function ϕ , we find a function v of the form (5) such that

$$D(\phi + v)(x_0) = 0, \quad D^2(\phi + v)(x_0) < 0.$$

Then clearly $u + v = (u - \phi) + (\phi + v)$ has a strict local maximum at x_0 , so that $u + v$ fails to satisfy the maximum principle on all sufficiently small domains $V \in \mathcal{B}(U)$ containing the point $x_0 \in V$. That is, condition (a) in Theorem 3.1 fails, completing the proof of the contrapositive.

4. Subtemperatures

4.1. The characterization

Now we characterize subtemperatures. As explained later, our results improve on recent work of Crandall and Wang [4].

Write the fundamental solution of the heat (or diffusion) operator in \mathbf{R}^{n+1} , $n \geq 1$, as

$$K(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad x \in \mathbf{R}^n, t > 0.$$

Call a set $W \subset \mathbf{R}^{n+1}$ a *cylinder* if it has the form $W = V \times (t_1, t_2)$ where $V \subset \mathbf{R}^n$ is open and $t_1 < t_2$.

Let U be an open set in \mathbf{R}^{n+1} , $n \geq 1$. Suppose $\mathcal{B}_p(U)$ (where the “ p ” stands for parabolic) is a collection of cylinders contained in U such that for each $(x, t) \in U$ and $\varepsilon > 0$ there exists a cylinder $W = V \times (t_1, t)$ in $\mathcal{B}_p(U)$ with $x \in V \subset B(x, \varepsilon)$ and $0 < t - t_1 < \varepsilon$.

For example, $\mathcal{B}_p(U)$ could consist of all cylinders in which V is a ball with radius in $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ and with $\overline{W} \subset U$.

Write $C^{2,1}(U)$ for the class of functions with two spatial derivatives and one temporal derivative in U , with all these derivatives being continuous. For $u \in C^{2,1}(U)$ to be called a *subtemperature* means that $\Delta u \geq u_t$, where Δ denotes the Laplacian in the spatial variables. In Corollary 4.2 below one should instead use the potential theoretic definition of a subtemperature (see [7, Definition 4.4] with $p = 2$).

With these definitions, we can now state our characterization of subtemperatures.

THEOREM 4.1. *Let $u : U \rightarrow \mathbf{R}$ where U is an open set in \mathbf{R}^{n+1} , $n \geq 1$, with associated collection of cylinders $\mathcal{B}_p(U)$ as above. Suppose $u \in C^{2,1}(U)$. Assume either*

- (a) $u + w$ satisfies the parabolic maximum principle on W whenever $W \in \mathcal{B}_p(U)$ and w has the form

$$(16) \quad w(x, t) = \sum_{m=1}^n b_m K(x - \xi_m, t - \tau_m)$$

for some positive real numbers b_1, \dots, b_n , and some distinct points $(\xi_m, \tau_m) \in U$ with $\tau_m < t_1$; or else

- (b) $u - w$ satisfies the parabolic maximum principle on W whenever $W \in \mathcal{B}_p(U)$ and w has the form (16), and also $u_t \leq 0$ in every subregion of U on which u is independent of x .

Then u is a subtemperature in U .

COROLLARY 4.2. *Theorem 4.1(a) still holds if the hypotheses $u : U \rightarrow \mathbb{R}$ and $u \in C^{2,1}(U)$ are weakened to $u : U \rightarrow [-\infty, \infty)$ being upper semicontinuous with u finite in a dense subset of U .*

The parabolic maximum principle for a function ψ means $\max_{\overline{W}} \psi = \max_{\partial_p W} \psi$, where

$$\partial_p W = (V \times \{t_1\}) \cup (\partial V \times [t_1, t_2])$$

denotes the parabolic boundary of W .

REMARKS. 1. The function w in formula (16) is a positive linear combination of n fundamental solutions with singularities occurring before the time t_1 at which \overline{W} begins. Clearly w is a temperature on \overline{W} (meaning $\Delta w = w_t$ there). Thus the converse of Theorem 4.1 is easy: if a function u is a subtemperature on U then $u \pm w$ is a subtemperature on \overline{W} , and so $u \pm w$ satisfies the parabolic maximum principle there.

2. One cannot remove the requirement in condition (b) that $u_t \leq 0$ in every subregion of U on which u is independent of x . For example in dimension $n = 1$ suppose $u(x, t) = t$, which is not a subtemperature. Notice u is independent of x with $u_t > 0$, so that u fails the second requirement of (b). However u does satisfy the first requirement of (b); indeed $u - w = t - b_1 K(x - \xi_1, t - \tau_1)$ satisfies the parabolic maximum principle on every cylinder $W = V \times (t_1, t_2)$, $\tau_1 < t_1$, because the Gaussian $x \mapsto e^{-x^2}$ has no local minimum.

3. Corollary 4.2 covers only part (a) of Theorem 4.1. We believe part (b) should hold also, when $u : U \rightarrow [-\infty, \infty)$ is upper semicontinuous and finite in a dense subset of U , but we do not see how to prove this.

4. Crandall and Wang [4, Theorem 3] recently proved Theorem 4.1(a) and Corollary 4.2, except that they needed $n + 1$ terms in the definition of w , whereas we need only n terms. The first reason we do better here is that when proving Theorem 4.1(a), we perturb (x_0, t_0) so as to get $D_x u(x_0, t_0) \neq 0$: this is possible except when u depends only on t , which we treat separately. Once one has $D_x u(x_0, t_0) \neq 0$, one needs only n terms in w to complete the proof (as Crandall and Wang also observed). The second reason we need only n terms is that when proving Corollary 4.2 we use a nominally weaker notion of viscosity subsolution (a notion going back to Ohnuma and Sato, thence to Ishii and Souganidis), which gives more control over the test function ϕ at points where its spatial gradient vanishes. This additional control at points where $D_x \phi(x_0, t_0) = 0$ enables us to again use only n terms, when constructing w .

5. We learned of Crandall and Wang's work only after writing this paper. Our methods are very similar to theirs, due to the common starting point (the subharmonic characterization Theorem 1.1 by Crandall and Zhang).

6. Our proof of Theorem 4.1(a) is simpler than the proof of Crandall and Wang in that it avoids the “non-generic” case of [4, p. 8] (the case with \tilde{S}), by means of our perturbation of the standard basis vector e_m to the vector q_m , in the proof below.

7. Crandall and Wang do not localize their singularities, although presumably they could modify their proof a little to enforce $(\xi_m, \tau_m) \in U$, like in our work.

8. Theorem 4.1(b), in which we treat $u - w$ instead of $u + w$, has no analogue in Crandall and Wang’s paper.

9. Fundamental solutions are not the only choice for characterizing subtemperatures by means of the maximum principle. Indeed the simpler class of functions $w(x, t) = \frac{1}{2}x^T Ax + p \cdot x + t \text{Trace } A$ was shown to suffice by Crandall and Wang [4, Remark 6], where A is a symmetric $n \times n$ matrix and $p \in \mathbb{R}^n$.

PROOF OF THEOREM 4.1. *Part (a).* We establish the contrapositive. So assume u is not a subtemperature, meaning $\Delta u < u_t$ at some point $(x_0, t_0) \in U$. Assume $D_x u(x_0, t_0) \neq 0$, and consider later the case $D_x u(x_0, t_0) = 0$. (Here D_x denotes the spatial gradient operator, taken with respect to the x -variables and keeping t fixed.)

By a translation we may suppose $(x_0, t_0) = (0, 0)$, so that $\Delta u(0, 0) < u_t(0, 0)$ and $D_x u(0, 0) \neq 0$. Define $\varepsilon > 0$ by

$$\varepsilon = u_t(0, 0) - \Delta u(0, 0).$$

We will choose $b_m > 0$, $\xi_m \in \mathbb{R}^n$ distinct and $\tau_m < 0$ in such a way that

$$(17) \quad w(x, t) = \sum_{m=1}^n b_m K(x - \xi_m, t - \tau_m)$$

satisfies

$$(18) \quad D_x(u + w)(0, 0) = 0,$$

$$(19) \quad D_x^2(u + w)(0, 0) < 0,$$

$$(20) \quad (u + w)_t(0, 0) > 0.$$

These imply the origin is a strict local maximum point for $u + w$ with respect to any sufficiently small cylinder $W = V \times (t_1, 0)$ having $0 \in V$ and $t_1 < 0$. Thus $u + w$ does not satisfy the parabolic maximum principle, whenever this cylinder W belongs to $\mathcal{B}_p(U)$ and is sufficiently small (noting that w has the desired form (16) provided $t_1 > \max_m \tau_m$). This failure of the maximum principle establishes the contrapositive of part (a) of the theorem.

Like in the proof of Theorem 3.1, we take $\lambda_1, \dots, \lambda_n$ to be the eigenvalues of the spatial Hessian $D_x^2 u(0, 0)$ at the origin, and we choose $\lambda > 0$ large enough that $\lambda > \lambda_m$ for all m . Then by summing,

$$n\lambda - (\lambda_1 + \dots + \lambda_n) > 0.$$

Hence we may choose a number $\nu > 0$ satisfying

$$(21) \quad n - (\lambda_1 + \dots + \lambda_n)/\lambda > \nu > n - (\lambda_1 + \dots + \lambda_n + \varepsilon)/\lambda.$$

Thus $\nu \rightarrow n$ as $\lambda \rightarrow \infty$.

Notice that

$$\sum_{m=1}^n \frac{1 - \lambda_m/\lambda}{\nu} = \frac{1}{\nu} \left(n - \frac{\lambda_1 + \dots + \lambda_n}{\lambda} \right) > 1$$

by choice of ν . Therefore it is possible to choose numbers $\beta_1, \dots, \beta_n > 0$ such that $\sum_{m=1}^n \beta_m = 1$ and

$$\frac{1 - \lambda_m/\lambda}{\nu} > \beta_m, \quad m = 1, \dots, n.$$

These n inequalities imply the diagonal matrix inequality

$$E - \lambda I + \sum_{m=1}^n \beta_m \lambda (\nu e_m e_m^T) < 0,$$

where E is the diagonal matrix of eigenvalues and the e_m are the standard basis vectors. Hence

$$(22) \quad E + \sum_{m=1}^n \beta_m \lambda (-I + \nu q_m q_m^T) < 0$$

whenever q_1, \dots, q_n are vectors sufficiently close to e_1, \dots, e_n , respectively, where we have used also here that $\sum_{m=1}^n \beta_m = 1$. Write Q for the matrix with the q_m as its column vectors.

As before, we take $P = [p_1 \dots p_n]$ to be an orthogonal matrix whose columns p_m are eigenvectors of $D_x^2 u(0, 0)$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$, so that $P^T D_x^2 u(0, 0) P = E$. We may suppose $-p_m^T D_x u(0, 0) \geq 0$ for each m , or

$$-P^T D_x u(0, 0) \cdot e_m \geq 0, \quad m = 1, \dots, n.$$

This means the vector $-P^T D_x u(0, 0)$ lies in the closed first quadrant of \mathbf{R}^n , and is nonzero (since $D_x u(0, 0) \neq 0$). Again we may choose an orthonormal system q_1, \dots, q_n such that

$$-P^T D_x u(0, 0) \cdot q_m > 0, \quad m = 1, \dots, n,$$

and that the q_1, \dots, q_n are a small enough perturbation of e_1, \dots, e_n for (22) to hold.

Once more write $PQ = [z_1 \cdots z_n]$, and let

$$\xi_m = -\frac{(P^T D_x u(0, 0) \cdot q_m)}{\beta_m \lambda} z_m,$$

so that the ξ_m are all distinct and

$$(23) \quad \xi_m = |\xi_m| z_m \quad \text{where} \quad |\xi_m| = -\frac{(P^T D_x u(0, 0) \cdot q_m)}{\beta_m \lambda} > 0.$$

Put

$$\tau_m = -\frac{|\xi_m|^2}{2\nu} < 0$$

and define

$$b_m = -(D_x u(0, 0) \cdot z_m)(-2\tau_m)/(|\xi_m|K(-\xi_m, -\tau_m)).$$

Notice

$$(24) \quad \frac{b_m}{2(-\tau_m)} |\xi_m| K(-\xi_m, -\tau_m) = -(D_x u(0, 0) \cdot z_m) \\ = -(P^T D_x u(0, 0) \cdot q_m) \quad \text{since } z_m = Pq_m$$

$$(25) \quad = \beta_m \lambda |\xi_m| \\ > 0.$$

This shows $b_m > 0$.

We must show $(\xi_m, \tau_m) \in U$ whenever λ is chosen sufficiently large. For this it is enough to show that $|\xi_m|$ and $|\tau_m|$ approach zero as $\lambda \rightarrow \infty$, since $(x_0, t_0) = (0, 0)$ lies in U .

Since $\nu \rightarrow n$ as $\lambda \rightarrow \infty$, we see $\frac{1-\lambda_m/\lambda}{\nu} < \frac{2}{2n-1}$ for all large λ . Then our choice of $\beta_m < \frac{1-\lambda_m/\lambda}{\nu}$ guarantees that $|\xi_m| \rightarrow 0$ as $\lambda \rightarrow \infty$ by arguing like in the proof of Theorem 3.1. Hence $\tau_m = -|\xi_m|^2/2\nu \rightarrow 0$ as $\lambda \rightarrow \infty$ as we wanted.

Now to prove (18)–(20), we first compute

$$\begin{aligned}
 w_{x_j}(x, t) &= \sum_{m=1}^n b_m (-1)^{x_j - (\xi_m)_j} \frac{1}{2(t - \tau_m)} K(x - \xi_m, t - \tau_m), \\
 (26) \quad w_{x_j}(0, 0) &= \sum_{m=1}^n b_m \frac{(\xi_m)_j}{2(-\tau_m)} K(-\xi_m, -\tau_m), \\
 (27) \quad w_{x_j x_k}(0, 0) &= \sum_{m=1}^n b_m \frac{1}{2(-\tau_m)} \left(-\delta_{jk} + \frac{(\xi_m)_j (\xi_m)_k}{2(-\tau_m)} \right) K(-\xi_m, -\tau_m).
 \end{aligned}$$

Then (26) implies

$$(28) \quad D_x w(0, 0) = - \sum_{m=1}^n (D_x u(0, 0) \cdot z_m) z_m$$

by using (24) and $\xi_m = |\xi_m| z_m$, while (27) implies

$$(29) \quad D_x^2 w(0, 0) = \sum_{m=1}^n \beta_m \lambda (-I + \nu z_m z_m^T)$$

by using (25) and $\xi_m = |\xi_m| z_m$ and $\nu = |\xi_m|^2 / 2(-\tau_m)$.

Now (28) shows $D_x w(0, 0) = -D_x u(0, 0)$ because the z_m form an orthonormal basis. Thus $D_x(u + w)(0, 0) = 0$, which is (18).

Conjugating the Hessian inequality (19) with P shows it is equivalent to

$$P^T \{ D_x^2 u(0, 0) + D_x^2 w(0, 0) \} P < 0.$$

Then substituting in the formula (29) for the Hessian of w reduces this inequality to (22) (since $P^T z_m = q_m$), thus completing the proof of (19).

Lastly, to prove (20) we observe

$$\begin{aligned}
 (u + w)_t(0, 0) &= \varepsilon + \Delta(u + w)(0, 0) \quad \text{by choice of } \varepsilon \\
 &= \varepsilon + \text{Trace } D_x^2(u + w)(0, 0) \\
 &= \varepsilon + \text{Trace}[P^T D_x^2 u(0, 0) P] + \text{Trace}[(PQ)^T D_x^2 w(0, 0) (PQ)] \\
 &= \varepsilon + \text{Trace } E + \sum_{m=1}^n \beta_m \lambda \text{Trace}[-I + \nu e_m e_m^T] \\
 &= \varepsilon + (\lambda_1 + \cdots + \lambda_n) + \lambda(-n + \nu) \\
 &> 0
 \end{aligned}$$

by definition of v in (21). This completes the proof in the case where $D_x u(x_0, t_0) \neq 0$.

Now return to the beginning of the proof and consider the case where $\Delta u < u_t$ and $D_x u = 0$ at some point $(x_0, t_0) \in U$. In fact, we can suppose $D_x u \equiv 0$ on a neighborhood of (x_0, t_0) , because otherwise we could instead take (x_0, t_0) to be some nearby point at which $D_x u \neq 0$ and $\Delta u < u_t$, and then apply the previous case. But if $D_x u \equiv 0$ near (x_0, t_0) , then the Hessian vanishes too: $D_x^2 u \equiv 0$ near (x_0, t_0) .

By a translation we can suppose $(x_0, t_0) = (0, 0)$, so that $u_t(0, 0) > \Delta u(0, 0) = 0$. Choose $\xi_m = 0$ and $\tau_m < 0$ for all m , in the definition (17) of w , with the τ_m chosen to be distinct and so small that $(\xi_m, \tau_m) \in U$. And then choose $b_m > 0$ to be so small that $(u + w)_t(0, 0) > 0$, meaning (20) holds. Notice $D_x w(0, 0) = 0$ by (26) (since the Gaussians in w are all centered at the spatial origin $\xi_m = 0$), so that $D_x(u + w)(0, 0) = 0$, which gives (18).

Further,

$$\begin{aligned} D_x^2(u + w)(0, 0) &= D_x^2 w(0, 0) = - \sum_{m=1}^n \frac{b_m}{2(-\tau_m)} K(0, -\tau_m) I && \text{by (27)} \\ &< 0, \end{aligned}$$

which gives (19). Now (18)–(20) suffice to show that the parabolic maximum principle fails for some W , just like before.

Part (b). Again we establish the contrapositive. Assume u is not a subtemperature, meaning $\Delta u < u_t$ at some point $(x_0, t_0) \in U$.

If $D_x u \equiv 0$ on a neighborhood of (x_0, t_0) then $0 \equiv \Delta u < u_t$ near (x_0, t_0) , and so condition (b) fails as desired. Thus we can assume $D_x u \not\equiv 0$ on every neighborhood of (x_0, t_0) . Hence by perturbing the point (x_0, t_0) slightly, we can suppose $D_x u(x_0, t_0) \neq 0$.

Now proceed like in the proof of part (a), except with the following changes. Choose $\lambda > 0$ large enough that $\lambda > -\lambda_m$ for all m . Choose $v > 0$ to satisfy

$$n + (\lambda_1 + \cdots + \lambda_n)/\lambda < v < n + (\lambda_1 + \cdots + \lambda_n + \varepsilon)/\lambda,$$

so that

$$\sum_{m=1}^n \frac{1 + \lambda_m/\lambda}{v} = \frac{1}{v} \left(n + \frac{\lambda_1 + \cdots + \lambda_n}{\lambda} \right) < 1.$$

Then it is possible to choose the $\beta_m > 0$ to satisfy $\sum_{m=1}^n \beta_m = 1$ and

$$\frac{1 + \lambda_m/\lambda}{v} < \beta_m, \quad m = 1, \dots, n,$$

which leads to the matrix inequality

$$E - \sum_{m=1}^n \beta_m \lambda (-I + \nu q_m q_m^T) < 0$$

whenever the q_m are sufficiently close to the e_m .

We may suppose $P^T D_x u(0, 0) \cdot e_m \geq 0$ for each m , and then choose the q_m to be an orthonormal system with $P^T D_x u(0, 0) \cdot q_m > 0$ for each m (using here that $D_x u(0, 0) \neq 0$) and with the q_m sufficiently close to the e_m for the previous paragraph to apply.

Letting

$$\xi_m = \frac{(P^T D_x u(0, 0) \cdot q_m)}{\beta_m \lambda} z_m,$$

one finds the ξ_m are all distinct and $\xi_m = |\xi_m| z_m$ where $|\xi_m| = (P^T D_x u(0, 0) \cdot q_m) / (\beta_m \lambda) > 0$.

Put $\tau_m = -\frac{|\xi_m|^2}{2\nu} < 0$ and $b_m = (D_x u(0, 0) \cdot z_m)(-2\tau_m) / (|\xi_m| K(-\xi_m, -\tau_m))$, so that $b_m > 0$ like before.

To show $|\xi_m| \rightarrow 0$ as $\lambda \rightarrow \infty$ it suffices to observe

$$\beta_m \lambda > \frac{\lambda + \lambda_m}{\nu} \rightarrow \infty \quad \text{as } \lambda \rightarrow \infty.$$

(This part of the proof is easier than in part (a).)

With these preliminaries, one can verify that w satisfies

$$D_x(u - w)(0, 0) = 0,$$

$$D_x^2(u - w)(0, 0) < 0.$$

Furthermore

$$\begin{aligned} (u - w)_t(0, 0) &= \varepsilon + \Delta(u - w)(0, 0) && \text{by choice of } \varepsilon \\ &= \varepsilon + (\lambda_1 + \cdots + \lambda_n) - \lambda(-n + \nu) \\ &> 0 \end{aligned}$$

by definition of ν . From these last three inequalities, we deduce condition (b) fails for some small cylinder W .

4.2. Proof of Corollary 4.2

Let $u : U \rightarrow [-\infty, \infty)$ be upper semicontinuous with u finite in a dense subset of U .

To prove the contrapositive of part (a), suppose u is not a subtemperature in U . Then u is not a relaxed viscosity subsolution of the heat equation in U , by the equivalence between the relaxed viscosity subsolutions and the potential theoretic subtemperatures (see [7, Theorem 4.6] and [7, p. 716], with $p = 2$; and note the references to earlier work, especially that of Ohnuma and Sato [10]).

Hence by definition of relaxed viscosity subsolution (see [7, Definition 4.17]), there exists a function $\phi \in C^2(U)$ and a local maximum point $(x_0, t_0) \in U$ of $u - \phi$ such that $\Delta\phi(x_0, t_0) < \phi_t(x_0, t_0)$ and such that if $D_x\phi(x_0, t_0) = 0$ then $D_x^2\phi(x_0, t_0) = 0$. (We have used here the definition of ‘‘admissibility’’ for ϕ , from [7, Definition 4.16], which ensures that if $D_x\phi(x_0, t_0) = 0$ then $\phi(x, t_0) - \phi(x_0, t_0) = o(|x - x_0|^2)$ and hence that the Hessian of ϕ vanishes at (x_0, t_0) .)

If $D_x\phi(x_0, t_0) \neq 0$ then we can apply the first part of the proof of Theorem 4.1(a) to the function ϕ , while if $D_x\phi(x_0, t_0) = 0$ then $D_x^2\phi(x_0, t_0) = 0$ by assumption and so we can apply the second part of the proof of Theorem 4.1(a) to ϕ . Either way we obtain a function w of the form (16) such that

$$D_x(\phi + w)(x_0, t_0) = 0, \quad D_x^2(\phi + w)(x_0, t_0) < 0, \quad (\phi + w)_t(x_0, t_0) > 0.$$

Then $u + w = (u - \phi) + (\phi + w)$ has a strict local maximum at (x_0, t_0) , relative to times $t < t_0$. Hence $u + w$ fails to satisfy the parabolic maximum principle, on each sufficiently small cylinder $W = V \times (t_1, t_0) \in \mathcal{B}_p(U)$ having $x_0 \in V$ and $t_1 < t_0$. That is, condition (a) in Theorem 4.1 fails.

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