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ROUCHÉ TYPE THEOREMS AND A THEOREM OF ADAMYAN, AROV AND KREIN

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Abstract

We show Rouché type theorems using a theorem of Adamyan, Arov and Krein. As applications, we obtain a certain characterization of self-maps of the unit disc in terms of the location of the Denjoy-Wolf point and we study a function in the Smirnov class whose real part is positive.

1. Introduction

Let D be the open unit disc in the complex plane C and let ∂D be the boundary of D. An analytic function in D is said to be of class N if the integrals $\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta$ are bounded for $r < 1$. If f is in N, then $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere on ∂D. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})|\, d\theta,$$

then f is said to be in the Smirnov class N_+. The set of all boundary functions in N or N_+ is also denoted by N or N_+, respectively. For $0 < p \leq \infty$, the Hardy space H^p, is denoted by $N_+ \cap L^p$.

Through out this paper, we use the following notations. We call q in N_+ an inner function if $|q| = 1$ a.e. on ∂D. A function h in N_+ is called outer if it is not divisible in N_+ by a nonconstant inner function. For two inner functions q_1, q_2 we will write $q_1 \succ q_2$ when there exists an outer function h in H^1 such that $\bar{q}_1 q_2 = |h|/h$. If $q_1 \succ q_2$ and $q_1 \prec q_2$ then we will write that $q_1 \sim q_2$. For a nonzero function f in N_+, f has an inner outer factorization: $f = qh$ where q is inner and h is outer. The inner part of f will be written as $q[f]$. If $\bar{q}_1 q_2 = |f|/f$ and f is a nonzero function in H^1, put $f = qh$ where $q = q[f]$ and h is outer. Then $|f|/f = \bar{q}|h|/h = |(1+q)^2 h|/(1+q)^2 h$ because $\bar{q}(1+q)^2 = |1+q|^2$. For a function F in H^1, we will write the

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Herglotz integral f of $|F|$ in the form :

$$f(z) = \frac{1+Q_F(z)}{1-Q_F(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}|F(e^{it})|\,dt \qquad (z \in D)$$

where Q_F is a contractive function in H^∞.

The following two theorems are very well known and they are written in the title of this paper. The first one is called Rouché Theorem and the proof is elementary (see [7, p. 225]). The second one is called Adamyan, Arov and Krein Theorem [1]. The present form is a corollary of [4, Lemma 5.5 in Chapter IV] which was proved by the author [5, Lemma 6].

ROUCHÉ THEOREM. *Let U be a bounded domain in $\mathbb{C}$ whose boundary ∂U consists of finite disjoint closed Jordan curves. Suppose f and g are nonzero functions which are holomorphic on $U \cup \partial U$. If $|f(z)| > |g(z)|$ $(z \in \partial U)$ then $\sharp Z(f) = \sharp Z(f-g)$. Here for any analytic function F on $U \cup \partial U$, $\sharp Z(F)$ denotes the number of zeros of F in U, counted according to their multiplicities.*

ADAMYAN, AROV AND KREIN THEOREM. *Let $\phi = F/|F|$ for some nonzero function F in H^1. Then*

$$\begin{aligned}&\{g \in H^\infty\ ;\ \|\phi - g\|_\infty \le 1\}\\ &\quad = \left\{\frac{f(1-Q_f)(1-w)}{1-Q_f w}\ ;\ w \in H^\infty,\ \|w\|_\infty \le 1 \text{ and } f \in H^1,\ f/F \ge 0\right\}\end{aligned}$$

In this paper we generalize Rouché theorem in case $U = D$ to when f and g are not necessary holomorphic on ∂D. Moreover we give a Rouché type theorem. In fact, we describe $f-g$ where f and g are functions in the Smirnov class with $|f| \ge |g|$ a.e. on ∂D, using Adamyan, Arov and Krein Theorem. As an application, we describe a function whose Denjoy-Wolff point is in ∂D and study a function in N_+ whose real part is nonnegative on ∂D.

2. A generalization of Rouché theorem

Theorem 1 is a generalization of Rouché theorem in the Introduction for the open unit disc. In fact, if f and g are holomorphic on $D \cup \partial D$ and $|f| > |g|$ on ∂D then f and g belong to N_+ and there exists $\varepsilon > 0$ such that $|f| \ge \varepsilon + |g|$ on ∂D. Hence Theorem 1 follows from Rouché Theorem. We prove Theorem 1 using a Toeplitz operator. For a function ϕ in L^∞, T_ϕ denotes the usual Toeplitz operator on H^2 with symbol ϕ (see [3, Chapter 7]).

THEOREM 1. *Suppose f and g are nonzero functions in N_+ and $|f| \ge \varepsilon + |g|$ a.e. on ∂D for some $\varepsilon > 0$. Then $q[f] \sim q[f-g]$.*

PROOF. Put $q_1 = q[f]$ and $q_2 = q[f - g]$. If $f = q_1 h, h$ is outer and $k = g/h$, then $f - g = h(q_1 - k), k \in H^\infty$ and $q_2 = q[q_1 - k]$. Then $\|k\|_\infty < 1$. For by hypothesis $|h| \geq \varepsilon + |g| \geq \varepsilon$ and so $1 \geq \varepsilon |h|^{-1} + |k| \geq |k|$. If ℓ is the outer part of $q_1 - k$, then $q_1 - k = (1 - \bar{q}_1 k) q_1 = q_2 \ell, \bar{q}_1 q_2 = (1 - \bar{q}_1 k)\ell^{-1}$ and ℓ is an invertible function in H^∞ because $(q_1 - k)^{-1} \in L^\infty$. Since $\|\bar{q}_1 k\|_\infty < 1$, $\|I - T_{1-\bar{q}_1 k}\| = \|T_{\bar{q}_1 k}\| = \|\bar{q}_1 k\|_\infty < 1$ and so $T_{1-\bar{q}_1 k}$ is invertible by a theorem of Widom and Devinatz (see [3, Theorem 7.10 in Chapter 7]). Hence $T_{\bar{q}_1 q_2} = T_{(1-\bar{q}_1 k)} T_{\ell^{-1}}$ is invertible because ℓ is invertible in H^∞. Hence there exist $F_1, F_2 \in H^2$ and $G_1, G_2 \in H_0^2$ such that $\bar{q}_1 q_2 F_1 = 1 + \bar{G}_1$ and $q_1 \bar{q}_2 F_2 = 1 + \bar{G}_2$. Then $\bar{q}_1 q_2 F_1(1 + G_1) = |1 + G_1|^2 = |F_1(1 + G_1)|$ and $q_1 \bar{q}_2 F_2(1 + G_2) = |1 + G_2|^2 = |F_2(1 + G_1)|$. Since $F_1(1 + G_1)$ and $F_2(1 + G_2)$ belong to H^1, $q_1 \sim q_2$.

COROLLARY 1. *In Theorem* 1, *if $q[f]$ is a finite Blaschke product then $q[f - g]$ is also a finite Blaschke product and* $\sharp Z(f) = \sharp Z(f - g)$.

PROOF. Put $Q_1 = q[f]$ and $Q_2 = q[f - g]$ then

$$\bar{Q}_1 Q_2 = \frac{|F|}{F} = \frac{G}{|G|}, \qquad F \in H^1 \text{ and } G \in H^1$$

because $Q_1 \sim Q_2$ by Theorem 1. Hence $\bar{Q}_1 Q_2 F = |F| \geq 0$ and so $q[Q_2 F]$ is a finite Blaschke product with $\deg Q_1 \geq \deg q[Q_2 F] \geq \deg Q_2$. This is a result of 8.4 of Chapter 8 in [2]. Similary we can prove that $\deg Q_1 \leq \deg Q_2$.

3. Rouché type theorems

In Theorem 1, if $\varepsilon = 0$ then the conclusion is not valid. In fact, if $f = z$ and $g = 1$ then $q[f] = z$ and $q[f - g]$ = constant. Hence $q[f] \not\sim q[f - g]$ but $q[f] \succ q[f - g]$. Corollary 2 shows that $q[f] \succ q[f - g]$ is valid in general. Corollary 3 is a result of D. Sarason [8, Proposition 3]. Recall that $(1 + Q_F)/(1 - Q_F)$ denotes the Herglotz integral of $|F|$.

THEOREM 2. *Suppose f and g are nonzero functions in N_+, $|f| \geq |g|$ a.e. on ∂D and $f - g \not\equiv 0$. Then*

$$f - g = \frac{hF(1 - Q)(1 - w)}{1 - wQ}$$

where h is the outer part of f, F is a nonzero function in H^1 with $hF/f \geq 0$ a.e. on ∂D, $Q = Q_F$ and w is a contractive function in H^∞ with $w \not\equiv 1$.

PROOF. Let $f = q_1 h$ where $q_1 = q[f]$ and h is outer. If $k = g/h$ then $k \in H^\infty$, $\|k\|_\infty \leq 1$ and $f - g = h(q_1 - k)$. Since $q_1 = (1 + q_1)^2/|1 + q_1|^2$

and $(1+q_1)^2 \in H^1$, by Adamyan, Arov and Krein Theorem in the Introduction

$$\{\ell\ ;\ \ell \in H^\infty, \|q_1 - \ell\|_\infty \leq 1\}$$
$$= \left\{ \frac{F(1-Q)(1-w)}{1-Qw}\ ;\ w \in H^\infty, \|w\|_\infty \leq 1, F \text{ is a nonzero function in } H^1 \text{ with } \bar{q}_1 F \geq 0 \text{ a.e. on } \partial D \text{ and } Q = Q_F \right\}.$$

Since $\|q_1 - (q_1 - k)\|_\infty \leq 1$, by the equality above $q_1 - k$ has the form: $F(1-Q)(1-w)/(1-Qw)$. Then $w \not\equiv 1$ because $f \not\equiv g$. This implies the theorem.

COROLLARY 2. *If f and g are nonzero functions in N_+, $|f| \geq |g|$ a.e. on ∂D and $f - g \not\equiv 0$, then $q[f] \succ q[f-g]$. It may not happen that $q[f] \sim q[f-g]$.*

PROOF. By Theorem 2, $f - g = hF(1-Q)(1-w)/(1-wQ)$. Then $q[f-g] = q[F]$ because $h(1-Q)(1-w)/(1-wQ)$ is outer. Since $\overline{q[f]}q[F]\ell = hF/f \geq 0$ a.e. on ∂D, $\overline{q[f]}q[F] = |\ell|/\ell$ where ℓ is the outer part of F. This implies that $q[f] \succ q[f-g]$. The second part was proved in the remark above Theorem 2.

COROLLARY 3. *If f is a finite Blaschke procuct and g is a contractive function in H^∞ then* $\deg(f) \geq \deg q[f-g]$.

4. Denjoy Wolff point

A point λ of D is called a Denjoy-Wolff point of the holomorphic self-map ϕ of D if λ is in D and $\phi(\lambda) = \lambda$, or if λ is in ∂D, and ϕ has λ as its nontangential limit at λ, and ϕ has an angular derivative at λ satisfying $|\phi'(\lambda)| \leq 1$. By Denjoy-Wolff Theorem (cf. [8]), any holomorphic self-map ϕ of D, other than the identity map, has a unique Denjoy-Wolff point.

The following lemma was proved by Sarason [8, Proposition 2 and Corollary 1].

LEMMA 1. *Let ϕ be a holomorphic self-map of D, not the function z, and let λ be its Denjoy-Wolff point. λ is in ∂D if and only if $z - \alpha\phi$ is an outer function for some constant α with $|\alpha| = 1$.*

THEOREM 3. *Let ϕ be a holomorphic self-map of D which is not the function z and let λ be its Denjoy-Wolff point. Put*

$$\Phi(a, w) = \frac{((2+|a|^2)z + 2a)w - (|a|^2 z + 2a)}{(2+|a|^2+2\bar{a}z) - (|a|^2 + 2\bar{a}z)w}$$

where $a \in \mathbf{C}$ with $|a| \leq 1$, and $w \in H^\infty$ with $\|w\|_\infty \leq 1$ and $w \not\equiv 1$.

(1) λ *is in* ∂D *if and only if* ϕ *can be written as* $b\Phi(a, w)$ *for some* $| a |=| b |= 1$ *and some function* w *as above.*

(2) λ *is in* D *if and only if* $\phi = \Phi(a, w)$ *for some* $|a| < 1$.

PROOF. Apply Theorem 2 to $f = z$ and $g = \phi$, then

$$z - \phi = \frac{F(1-Q)(1-w)}{1-Qw}$$

where F is a nonzero function in H^1 with $\bar{z}F \geq 0$ a.e. on ∂D, $Q = Q_F$ and w is a contractive function in H^∞ with $w \not\equiv 1$. Since $\bar{z}F \geq 0$ a.e. on ∂D, $F = \gamma(z+a)(1+\bar{a}z)$ where $a \in \mathbf{C}$ and $|a| \leq 1$. By the proof of [5, Lemma 6], we may assume $\gamma = 1$. The Herglotz integral of $|z+a|^2$ is $1 + |a|^2 + 2\bar{a}z$ and so $Q = (|a|^2 + 2\bar{a}z)/(2 + |a|^2 + 2\bar{a}z)$. By simple calculations (see [6]),

$$\phi = z - \frac{F(1-Q)(1-w)}{1-Qw} = z\frac{1-Q}{1-\bar{Q}}\,\frac{w-\bar{Q}}{1-Qw}.$$

Hence

$$\begin{aligned}\phi &= z\frac{1-Q}{1-Q}\,\frac{w-Q}{1-Qw}\\ &= z\frac{1-\frac{|a|^2+2\bar{a}z}{2+|a|^2+2\bar{a}z}}{1-\frac{|a|^2+2a\bar{z}}{2+|a|^2+2a\bar{z}}}\,\frac{w-\frac{|a|^2+2a\bar{z}}{2+|a|^2+2a\bar{z}}}{1-\frac{|a|^2+2\bar{a}z}{2+|a|^2+2\bar{a}z}w}\\ &= \frac{((2+|a|^2)z+2a)w-(|a|^2z+2a)}{2+|a|^2+2\bar{a}z-(|a|^2+2\bar{a}z)w}\end{aligned}$$

Moreover $\lambda \in \partial D$ if and only if F is outer if and only if $|a| = 1$. Now Theorem 3 follows from Lemma 1.

Suppose that $\phi = \Phi(a, w)$ in Theorem 3 and λ is in $D \cup \partial D$. Then $\phi(\lambda) = \lambda$ if and only if $w(\lambda) = 1$ or $\lambda = -a$. For if $\phi(\lambda) = \lambda$ then $\Phi(a, w)(\lambda) = \lambda$ and so $((2+|a|^2)\lambda + 2a)w(\lambda) - (|a|^2\lambda + 2a) = \{(2+|a|^2+2\bar{a}\lambda) - (|a|^2 + 2\bar{a}\lambda)w(\lambda)\}\lambda$. Hence $(\bar{a}\lambda^2 + (1+|a|^2)\lambda + a)w(\lambda) = \bar{a}\lambda^2 + (1+|a|^2)\lambda + a$. Therefore $w(\lambda) = 1$ or $(\bar{a}\lambda + 1)(\lambda + a) = 0$. Thus $w(\lambda) = 1$ or $\lambda = -a$. The converse is clear.

5. Functions f in N_+ with $\mathrm{Re}\, f \geq 0$ on ∂D

If f is a function in H^1 with $\mathrm{Re} f \geq 0$ on ∂D then $\mathrm{Re}\, f \geq 0$ on $\bar{D}$. This is well known and it is easy to see. In fact, we can use the Poisson integral representation of f. Unfortunately we can not do it when f is not in H^1. In the previous paper [6], we started to study functions in N_+ whose real parts are nonnegative on ∂D. In this section, applying Theorems 1 and 2 we prove Theorem 4 which is a generalization of [6, Theorem 14].

THEOREM 4. *If f is a function in N_+ with* $\operatorname{Re} f \geq 0$ *on ∂D, then $f = (g+k)/(g-k)$ where g is inner, k is a contractive function in H^∞ and $g-k$ is outer. Moreover $q[f] \prec g$ and $q[f+\lambda] \sim g$ for any λ in $\mathbf{C}$ with $\lambda > 0$.*

PROOF. The first part is just Proposition 10 in [6]. By Corollary 2, $q[g+k] \prec g$ and so $q[f] \prec g$. For any $\lambda > 0$

$$\frac{g+k}{g-k} + \lambda = (1+\lambda)\frac{g + \frac{1-\lambda}{1+\lambda}k}{g-k}$$

Since

$$|g| = 1 \geq \frac{2\lambda}{1+\lambda} + \left|-\frac{1-\lambda}{1+\lambda}k\right|,$$

by Theorem 1

$$q\left[g + \frac{1-\lambda}{1+\lambda}k\right] \sim g \qquad \text{and so} \qquad q\left[\frac{g+k}{g-k} + \lambda\right] \sim g$$

for any $\lambda > 0$.

COROLLARY 4. *In Theorem* 4, *if $q[f+\lambda]$ is a finite Blaschke product for some $\lambda > 0$ then g is also a finite Blaschke procuct with* $\deg g = \deg q[f+\lambda]$ *and* $\deg q[f+\lambda] = \deg q[f+\gamma]$ *for any $\gamma > 0$.*

By Corollary 4, if f is a nonzero function in N_+ with $\operatorname{Re} f \geq 0$ on ∂D and $f + \lambda$ is outer for $\lambda > 0$ then $f = (1+k)/(1-k)$ and f is also outer. It will be interesting to study the following special function: $f = (z+k)/(z-k)$.

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