MATH. SCAND. 97 (2005), 161–187

DEGENERATIONS OF (1, 7)-POLARIZED ABELIAN SURFACES

F. MELLIEZ and K. RANESTAD*

Abstract

The moduli space of $(1, 7)$-polarized abelian surfaces with a level structure was shown by Manolache and Schreyer to be rational with compactification the variety of powersum presentations of the Klein quartic curve. In this paper the possible degenerations of the abelian surfaces corresponding to degenerations of powersum presentations are classified.

1. Introduction

The moduli space $A(1, 7)$ of $(1, 7)$-polarized abelian surfaces with a level structure was shown by Manolache and Schreyer to be rational with compactification $V(\mathscr{K}_4)$ a Fano 3-fold V_{22} [13]. Gross and Popescu obtain the same compactification of $A(1, 7)$ with a different approach [10], but in neither case is the boundary $V(\mathscr{K}_4) \setminus A(1, 7)$ discussed. The purpose of this paper is to describe this boundary. We show that every point on $V(\mathscr{K}_4)$ correspond to a surface in P^6 invariant under the action of a group G_7, and we give a precise description of these surfaces.

More precisely, the $(1, 7)$-polarized abelian surface A with its level structure is embedded in $\mathsf{P}^6 = \mathsf{P}V_0$, where V_0 is the Schrödinger representation of the Heisenberg group H_7 of level 7. The embedding is invariant under the action of G_7, an extension of H_7 by an involution. The fixed points of this involution and its conjugates in G_7 form an H_7-orbit of planes P^+_2 and 3-spaces P^-_3. The 3-fold $V(\mathscr{K}_4)$ parameterizes, what we denote by generalized G_7-embedded abelian surfaces (cf. 2.9). Every such surface intersects P^+_2 in a finite subscheme of length six, which we may classify by its type, namely the length of its components.

In this paper we prove the

Theorem 1.1. *Let A be a generalized G_7-embedded abelian surface in $\mathsf{P}V_0$, then according to the type of $\zeta_A = A \cap \mathsf{P}^+_2$ the surface A is*

* The first author was partially supported by the European Commision through HPRN-CT-2000-00099.

Received May 4, 2004; in revised form December 3, 2004.

type of ζ_A	*description*
$(1, 1, 1, 1, 1, 1)$	*smooth and abelian*
$(2, 1, 1, 1, 1)$	*translation scroll* $(E, \pm\sigma)$ *with* $2 \cdot \sigma \neq 0$
$(3, 1, 1, 1)$	*tangent scroll* $(E, 0)$
$(2, 2, 2)$	*double translation scroll* $(E, \pm\sigma)$ *with* $2 \cdot \sigma = 0$ *and* $\sigma \neq 0$
$(2, 2, 1, 1)$	*union of seven quadrics*
$(4, 2)$	*union of seven double projective planes*
$(2, 2, 2)_c$	*union of fourteen projective planes*

For the translation and tangent scrolls, E is a septimic (this is the term used by Sylvester)*, i.e. of degree seven, elliptic curve with an origin and the translation defined by the specified point σ.*

The two distinct $(2, 2, 2)$ cases (abusively denoted by $(2, 2, 2)$ and $(2, 2, 2)_c$) are described in Figure 1 of the appendix.

In the first section we recall some basic facts on $(1, 7)$-polarized abelian surfaces with level structure, and construct a compactification of their moduli space. In fact, we consider a rational map $\kappa : \mathsf{P}V_0 \dashrightarrow \mathsf{P}^6$ defined by G_7-invariant hypersurfaces of degree 7, which maps any general surface $A \in A(1, 7)$ to a six-secant plane to a certain Veronese surface $S \subset \mathsf{P}^6$ of degree 9. The surface S is the image by κ of P_2^+, so the six points of intersection $S \cap \kappa(A)$ is the image of $A \cap \mathsf{P}_2^+$. It turns out that distinct surfaces A are mapped to distinct planes $\kappa(A)$, so the variety of six-secant planes to S form a natural compactification of $A(1, 7)$. The variety of planes in P^6 that intersects S in a subscheme of length six is the Fano 3-fold $V(\mathscr{K}_4)$. Its name originates from the fact that the finite subschemes $A \cap \mathsf{P}_2^+$ form polar hexagons to a certain Klein quartic curve $\mathscr{K}_4 \subset \mathsf{P}_2^{+*}$, while $V(\mathscr{K}_4)$ form the compact variety of *apolar* subschemes of length six to $\mathscr{K}_4$, cf. [17]. It is in this interpretation that Gross and Popescu identifies $V(\mathscr{K}_4)$ as a compactification of $A(1, 7)$, cf. [10]. The variety $V(\mathscr{K}_4)$ may also be identified with the variety of twisted cubic curves apolar to a certain "Kleinian" net of quadric surfaces. This interpretation is the key to the original approach of Manolache and Schreyer. Although our approach is slightly different from these approaches in the interpretation of $V(\mathscr{K}_4)$, the main technical argument appears in their papers.

The variety $V(\mathscr{K}_4)$ is a prime Fano threefold of genus 12. Mukai discovered different interpretations of these threefolds that are carefully explained in [17].

In the second section we present some useful aspects in our situation of these interpretations. In particular, we describe carefully the subvariety $\Delta_{\mathscr{K}_4}$ of $V(\mathscr{K}_4)$ parameterizing apolar subschemes of length six to $\mathscr{K}_4$ which are singular, i.e. do not consist of six distinct points, or equivalently planes in P^6 that intersect S in a singular subscheme of length six.

In the last section we prove Theorem 1.1 by considering the inverse images by κ of planes that belong to $\Delta_{\mathcal{K}_4} \subset V(\mathcal{K}_4)$ as surfaces in $\mathsf{P}V_0$. The final argument consists in verifying that only the planes that belong to $\Delta_{\mathcal{K}_4}$ pull back to singular surfaces.

Note that A. Marini also investigates such degenerations in [14]. His approach uses the interpretation of $V(\mathcal{K}_4)$ as the set of twisted cubic curves apolar to the "Kleinian" net of quadrics.

Notations

The base field is the one of complex numbers C. If R is a vector space, the Veronese map from R to $S^n R$ (as well as its projectivisation) will be denoted by ν_n:

$$R \xrightarrow{\nu_n} S^n R.$$

If $s \in \mathrm{Hilb}(n, \mathsf{P}R)$ the type of s (i.e. the associated length partition of n) will be labeled λ_s:

$$s \xrightarrow{\lambda} \lambda_s.$$

If H is a hypersurface of $\mathsf{P}R$ then $e_H = 0$ is an equation of H.

The irreducible representations of $\mathrm{SL}(2, \mathsf{F}_7)$ will be denoted by C, W_3, $W_3^\vee$, U_4, $U_4^\vee$, W_6, U_6, $U_6^\vee$, W_7, W_8 and U_8. The algebra of representations of the group $\mathrm{SL}(2, \mathsf{F}_7)$ is a quotient of

$$\mathsf{Z}[\mathsf{C}, W_3, W_3^\vee, U_4, U_4^\vee, W_6, U_6, U_6^\vee, W_7, W_8, U_8]$$

where C denotes the trivial representation, W_n denotes an irreducible $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-module of dimension n and U_n denotes an irreducible $\mathrm{SL}(2, \mathsf{F}_7)$-module of dimension n on which $\mathrm{SL}(2, \mathsf{F}_7)$ acts *faithfully*.

The corresponding table of multiplication can be found in [13] and [5] with the following possible identifications

[5]	V_1	$V_3 = V_-$	V_3^*	$V_4 = V_+$	V_4^*	V_6	V_6'	$V_6'^*$	V_7	V_8'	V_8
[13]	I	W	W'	U	U'	T	T_1	T_2	L	M_1	M_2
$\times$	C	W_3	$W_3^\vee$	U_4	$U_4^\vee$	W_6	U_6	$U_6^\vee$	W_7	U_8	W_8
	$\cdot$	P_2^+	$\check{\mathsf{P}}_2^+$	P_3^-	$\check{\mathsf{P}}_3^-$	P_5^+	P_5^-	$\check{\mathsf{P}}_5^-$	P_6^+	P_7^-	P_7^+

Note that what are denoted by P_2^+ and P_3^- are respectively denoted by P_-^2 and P_+^3 in [10].

2. Moduli space: a compactification

In this section we describe our main object, the abelian surfaces with a $(1,7)$-polarization and a level structure, and their moduli space $A(1,7)$. The general member of $A(1,7)$ is embedded in P^6 invariant under a group G_7. The hypersurfaces of degree 7 invariant under this group define a rational map on P^6 which is the key to our approach to a compactification of $A(1,7)$. The first analysis of this map is the main aim of this section.

Let A be an abelian surface, i.e. a projective complex torus C^2/Λ where Λ is a (maximal) lattice of $\mathsf{C}^2 \simeq \mathsf{R}^4$. Then the variety $\mathrm{Pic}^0(A)$ is an abelian surface as well (isomorphic to $(\mathsf{C}^2)^\vee/\Lambda^\vee$); this latter one is called the *dual* abelian surface of A and will be denoted by $A^\vee$. As additive group, the surface A acts on itself by translation, if $x \in A$ we will denote by τ_x the corresponding translation.

A line bundle of type $(1,7)$ on A is the data of an ample line bundle $\mathscr{L}$ such that the kernel of the isogeny

$$\varphi_{\mathscr{L}} : A \longrightarrow A^\vee, \qquad x \longrightarrow \tau_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$$

is isomorphic to $\mathsf{Z}_7 \times \mathsf{Z}_7$.

A $(1,7)$-polarization on A is an element of

$$\{(A, \varphi_{\mathscr{L}}) \mid \mathscr{L} \text{ is of type } (1,7)\}.$$

Thanks to Mumford, a coarse moduli space of $(1,7)$-polarized abelian surfaces exists, we will denote it by $M(1,7)$.

Now choose a *generic* $(1,7)$-polarized abelian surface, say A, then $V_0 = \mathrm{H}^0(A, \mathscr{L})$ is of dimension 7. The group $\ker(\varphi_{\mathscr{L}}) \simeq \mathsf{Z}_7 \times \mathsf{Z}_7$ becomes a subgroup of $\mathsf{P}\,\mathrm{SL}(V_0)$. It is certainly safer to work with linear representations rather than projective ones so we need to lift the action of $\mathsf{Z}_7 \times \mathsf{Z}_7$ on $\mathsf{P}V_0$ to an action of one of its central extensions on V_0. The Schur multiplier of $\mathsf{Z}_7 \times \mathsf{Z}_7$ is known to be μ_7 so any projective representation of $\mathsf{Z}_7 \times \mathsf{Z}_7$ is induced by a linear representation of what is called the “Heisenberg group of level 7” and denoted by H_7: that is to say for all $n \in \mathsf{N}^*$ and all projective representations ρ we get a Cartesian diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] & \mathrm{SL}(n,\mathsf{C}) \arrow[r] & \mathsf{P}\,\mathrm{SL}(n,\mathsf{C}) \arrow[r] & 1 \\
1 \arrow[r] & \mu_7 \arrow[r] \arrow[u] & H_7 \arrow[r] \arrow[u] & \mathsf{Z}_7 \times \mathsf{Z}_7 \arrow[r] \arrow[u, "\rho"'] & 1
\end{tikzcd}$$

In this way V_0 becomes a H_7-module (of rank 7), this representation is called the “Schrödinger” representation of H_7. We now have a way to identify *all* the vector spaces $\mathrm{H}^0(A', \mathscr{L})$ for any abelian surface $A' \in M(1,7)$ as they are all isomorphic to V_0 as H_7-modules. This looks too good to be true. So what is wrong? We implicitly made an identification between $\ker(\varphi_{\mathscr{L}})$ and $\mathsf{Z}_7 \times \mathsf{Z}_7$ and this is certainly defined up to $\mathrm{SL}(2, \mathsf{F}_7)$ only! So the construction is only invariant under $N_7 = H_7 \rtimes \mathrm{SL}(2, \mathsf{F}_7)$ which turns out to be the normalizer of H_7 in $\mathrm{SL}(7, \mathsf{C}) \simeq \mathrm{SL}(V_0)$.

So to any basis s of $\ker(\varphi_{\mathscr{L}})$ corresponds an embedding

$$\Phi_s : A \longrightarrow \mathsf{P}V_0.$$

The group $\mathrm{SL}(2, \mathsf{F}_7)$ acts on the set of bases of $\ker(\varphi_{\mathscr{L}})$ and we immediately get another complication (which will turn out to be quite nice after all):

$$\Phi_s(A) = \Phi_{-s}(A).$$

Let us denote by $G_7 = H_7 \rtimes \{-1, 1\} \subset N_7$. This group (after killing μ_7) is in general the full group of automorphisms of the surface $\Phi_s(A)$: if b is any element of $\mathsf{Z}_7 \times \mathsf{Z}_7$ and $\tau_b : \mathsf{P}V_0 \longrightarrow \mathsf{P}V_0$ is the involution induced by the corresponding "-1" of G_7, then τ_b leaves $\Phi_s(A)$ (globally) invariant and is induced by the "opposite" map $x \mapsto -x$ on A for a good choice of the image of the origin on $\Phi_s(A)$. In other words, $\tau_b \cdot \Phi_s = \Phi_{-s}$.

As the cardinality of $\mathrm{SL}(2, \mathsf{F}_7)/\{-1, 1\} = \mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$ is 168, each element of $M(1,7)$ will be mapped into $\mathsf{P}V_0$ in 168 ways (distinct in general). We get a brand new moduli space by considering a $(1,7)$-polarized abelian surface together with one of its embeddings, this moduli space will be denoted by $A(1,7)$:

$$A(1,7) = \big\{((A, \varphi_{\mathscr{L}}), s) \mid (A, \varphi_{\mathscr{L}}) \in M(1,7), s \text{ is a basis of } \ker(\varphi_{\mathscr{L}})\big\}/\square$$

in which the equivalence relation $\square$ is the expected one, $(X_1, s_1) \,\square\, (X_2, s_2)$ if $\Phi_{s_1}(X_1) = \Phi_{s_2}(X_2)$ (fortunately, this implies $X_1 = X_2$). The choice of basis (or embedding) is the level structure referred to in the introduction.

Here are some useful remarks:

(1) The surface $\Phi_s(A)$ is of degree 14;

(2) by construction if $x \in A$, then the set of 49 points $\Phi_s(\varphi_{\mathscr{L}}^{-1}(\varphi_{\mathscr{L}}(x)))$ is an orbit under the action of H_7 (or H_7/μ_7 if we want to be precise);

(3) the above construction works as well for elliptic curves, so in particular $\mathsf{P}V_0$ contains naturally G_7-invariant embedded elliptic curves (of degree 7);

(4) if $b \in \mathsf{Z}_7 \times \mathsf{Z}_7$ the involution τ_b induces a $\mathrm{SL}(2, \mathsf{F}_7)$-module structure on V_0, as such a module V_0 splits in $V_0 = W_3 \oplus U_4$ where both W_3 and

U_4 are irreducible $\mathrm{SL}(2, \mathsf{F}_7)$-modules of dimension 3 and 4 respectively (such that $S^3 W_3^\vee \simeq S^2 U_4$). The projective plane $\mathsf{P}W_3$ and the projective space $\mathsf{P}U_4$ in $\mathsf{P}V_0$ are point wise invariant by the involution τ_b. For a given $b \in \mathsf{Z}_7 \times \mathsf{Z}_7$ these two spaces will often be denoted by P_2^+ and P_3^- (the signs come from the following: W_3 is also a $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-module, i.e. $-1 \in \mathrm{SL}(2, \mathsf{F}_7)$ acts trivially on it, but $\mathrm{SL}(2, \mathsf{F}_7)$ acts faithfully on U_4);

(5) if E is a G_7-invariant elliptic curve in $\mathsf{P}V_0$ then the curve E intersects any P_2^+ in one point (corresponding to the image of 0) and any P_3^- in three points (corresponding to its non trivial 2-torsion points);

(6) the latter holds also for abelian surfaces, with decomposition $6 + 10$ corresponding to the odd and even 2-torsion points ([12]);

(7) by adding a finite set of G_7-invariant heptagons to the union of the G_7-invariant embedded elliptic curves of degree 7, one gets a birational model of the Shioda modular surface of level 7. It intersects each P_2^+ in a plane quartic curve $\mathscr{K}_4'$, the so called Klein quartic curve ([15] or [8] which contains original references to Klein).

Following what happens in the $(1, 5)$ case we consider the rational map

$$\kappa : \mathsf{P}V_0 \dashrightarrow \mathsf{P}(\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7})^\vee$$

i.e. the rational transformation of $\mathsf{P}V_0$ by the linear system of G_7-invariant septimics. In what follows, by a 'G_7-invariant septimic' we always mean a septimic in this linear system. Obviously the vector space $\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7}$ is a $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-module. On the other hand $\mathrm{h}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7} = 8$ (cf. [13]), so κ takes, *a priori*, its values in a P^7. There is a unique N_7-invariant septimic hypersurface [15], so the decomposition of the $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-module $\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7}$ must have a 7-dimensional summand. But the only dimensions of non-trivial irreducible $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-modules are 3, 6, 7 and 8, and W_7 is the only one of dimension 7, so this must be the other summand. Therefore $\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7} \simeq W_7 \oplus \mathsf{C}$ as a $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-module.

We will show show that the image of κ is in fact contained in $\mathsf{P}W_7$. First we analyze the base locus of these septimic hypersurfaces.

LEMMA 2.1. *A G_7-invariant septimic hypersurface of $\mathsf{P}V_0$ contains any of the fortynine projective spaces P_3^-.*

PROOF. Consider the restriction to any projective space $\mathsf{P}_3^- = \mathsf{P}U_4$ of G_7-invariant septimic hypersurfaces. Then we get a map

$$\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathsf{P}U_4}(7)) = S^7 U_4^\vee$$

which needs to be $\mathrm{SL}(2, \mathsf{F}_7)$-equivariant (the entire collection of P_3^-'s being invariant under the action of G_7). But U_4 is a faithful module for $\mathrm{SL}(2, \mathsf{F}_7)$ and 7 is odd, so the map is the zero map.

Using Bezout's theorem we get

COROLLARY 2.2. *A G_7-invariant septimic hypersurface of $\mathsf{P}V_0$ contains any G_7-invariant elliptic curve of $\mathsf{P}V_0$ as well as its translation scroll by a non trivial 2-torsion point.*

Notice that our forty nine P_2^+ constitute an orbit under G_7, so it makes sense to consider *the* surface $\kappa(\mathsf{P}_2^+)$.

COROLLARY 2.3. *'The' plane P_2^+ is mapped by κ to a Veronese surface S of degree nine in $\mathsf{P}W_7$.*

PROOF. The restriction of n G_7-invariant septimic hypersurface to P_2^+ contains, by the previous corollary and the last item (7) above, the Klein quartic curve $\mathscr{K}_4'$. The residual factor is a cubic, so the image of the restriction map $\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathsf{P}W_3^\vee}(7)) = S^7 W_3^\vee$ factors through $W_7 \subset S^3 W_3^\vee = W_7 \oplus W_3$. Therefore the restriction of κ to P_2^+ is defined by $W_7 \subset S^3 W_3^\vee$ which forms a basepoint free linear system of cubics, and the corollary follows.

REMARK 2.4. This phenomenon holds also in the $(1,5)$ case where P_2^+ is mapped by the linear system of G_5-invariant quintic hypersurfaces to a (projected) Veronese surface of degree 25 in a Grassmannian $\mathrm{Gr}(1, \mathsf{P}_3) \subset \mathsf{P}_5$ known as the bisecants variety of a certain rational sextic curve in P_3. The image of the blow-up of 'the' line P_1^- is the sextic complex in $\mathrm{Gr}(1, \mathsf{P}_3)$ of lines contained in a dual sextic of planes in P_3. In this case, any G_5-embedded $(1,5)$-polarized abelian surface is mapped to a ten-secant plane to the image of P_2^+ (which intersects the sextic complex along six lines).

Although the same kind of results are expected in our situation, here is a difference between the two cases. In the $(1,5)$ case the vector space of G_5-invariant quintics is spanned by determinants of socalled Moore matrices.

REMARK 2.5. The vector space $\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7}$ is not spanned by determinants of (symmetric) Moore matrices ([10]). For this, let us recall first what a Moore matrix is; there is a nice isomorphism of irreducible N_7-modules (defined up to homothety) $S^2 V_4 = U_4 \otimes V_0$ which induces a map $U_4 \longrightarrow S^2 V_4 \otimes V_0^\vee$. For a good choice of basis in $V_0^\vee$ we get a 7×7 matrix with coefficients in $V_0^\vee$ which is called a (symmetric) "Moore matrix". Now considering determinants, i.e. the map $S^2 V_4 \xrightarrow{\text{“}S^2\Lambda^7\text{”}} \mathsf{C}$) we get a first map

$$S^7 U_4 \longrightarrow S^7 V_0^\vee = \mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7)),$$

which composed with the projection to the invariant part yields a map

$$S^7U_4 \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbf{P}V_0}(7))^{G_7}.$$

This latter one is certainly zero: the action of $-1 \in \mathrm{SL}(2, \mathsf{F}_7)$ cannot be trivial on any $\mathrm{SL}(2, \mathsf{F}_7)$-invariant subspace of the vector space S^7U_4.

Nevertheless *anti-symmetric* Moore matrices play a fundamental role in the $(1, 7)$ case. They are defined by the isomorphism of irreducible N_7-modules $\Lambda^2 V_4 = W_3^\vee \otimes V_0$. The locus (in $\mathbf{P}V_0$) where such a matrix drops its rank is a Calabi Yau threefold (see [10]) and will appear in subsection 4.3.

PROPOSITION 2.6. *The image by the map κ of a G_7-embedded $(1, 7)$-polarized abelian surface is (generically) a projective plane and we have a factorization*

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A)$$

where the surface $A^\flat$ is the blowup of the surface A along its intersection with the base locus of the G_7-invariant septimics, and $K^\flat_{A^\vee}$ is the quotient of $A^{\vee\flat}$ by the involution, i.e. in general its Kummer surface.

PROOF. Assume the $(1, 7)$-polarization of $A \in M(1, 7)$ is given by a very ample line bundle, then from the G_7-equivariant resolution of the surface A in $\mathbf{P}V_0$ which can be found in [13, appendix], one can check that $\dim(\mathrm{H}^0(\mathcal{O}_A(7))^{G_7}) = 3$.

If the map $\kappa|_A$ is finite, then we have a factorization

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A).$$

The first two maps (as well as their degree) come from the construction itself, the degree of the last one follows by Bezout's theorem.

If $\kappa|_A$ is not finite, then it is composed with a pencil. We may assume that $\mathrm{Pic}(A)$ has rank 1, i.e. all curves are hypersurface sections or translates thereof. But no such curve is G_7-invariant unless the curve is a possible translate of a septimic hypersurface section, so $\kappa|_A$ has at most isolated base points. Therefore the linear system defining $\kappa|_A$ is a subsystem of $|7 \cdot h|$. The linear system is a net, so if it is composed with a pencil each member is reducible. In fact the general member must be the reducible union of seven hyperplane sections through the base locus. The intersection of one of these hyperplane sections with the base locus is a finite set whose stabilizer in G_7 has order at least 14. Therefore the hyperplane itself must have stabilizer of order at least 14. But there are only finitely many such hyperplanes, so this is impossible. Thus the map κ is finite on A and the proposition follows.

Let us denote by $A(1, 7)^v$ the (open) subset of $A(1, 7)$ corresponding to $(1, 7)$-polarized abelian surfaces for which the polarization is given by a very ample line bundle and $\kappa(A)$ is a plane. The association

$$A \mapsto \kappa(A)$$

maps $A(1, 7)^v$ into the variety of six-secant planes to $\kappa(\mathsf{P}_2^+)$. Notice that the six points $\kappa(A) \cap \kappa(\mathsf{P}_2^+) = \kappa(A \cap \mathsf{P}_2^+)$. Gross and Popescu in [10] prove that $A \cap \mathsf{P}_2^+$ is a polar hexagon to the Klein quartic curve. On the other hand six points in P_2^+ form a polar hexagon to the Klein curve precisely if all four cubics in their ideal is contained in $W_7 \subset S^3 W_3^\vee$ i.e. when their span on the Veronese surface $\kappa(\mathsf{P}_2^+)$ is a plane. The variety of planes that intersect $\kappa(\mathsf{P}_2^+)$ in a finite scheme of length six therefore define a natural compactification $\overline{A(1, 7)^v}$. In this compactification, an abelian surface $A \in A(1, 7)^v$ is the proper transform of a six-secant plane of the Veronese surface by κ^{-1}.

Moreover, any $(1, 7)$-polarized abelian surface is mapped into the hyperplane $\mathsf{P}W_7$ of $\mathsf{P}\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7}$ so their union is contained in a septimic hypersurface of $\mathsf{P}V_0$. Therefore we have

Corollary 2.7. *The compactification* $\overline{A(1, 7)^v}$ *is isomorphic to the unique prime Fano threefold of genus* 12 *which admits* $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$ *as its automorphisms group. The universal* $(1, 7)$*-polarized abelian surface with level* 7 *structure is birational to the unique* N_7*-invariant septimic hypersurface of* $\mathsf{P}V_0$.

Proof. Let us denote by X_7 the unique N_7-invariant septimic hypersurface of $\mathsf{P}V_0$ and by B_κ the base locus of the G_7-invariant septimic hypersurfaces. Put $Y_4 = \overline{\kappa(X_7 \backslash B_\kappa)} \subset \mathsf{P}W_7$ and consider the diagram

$$\begin{tikzcd} & & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_7 \backslash B_\kappa \arrow[r, "\kappa"] & Y_4 & & \mathsf{G}(3, W_7) \end{tikzcd}$$

where $I \subset \mathsf{P}W_7 \times \mathsf{G}(3,\ W_7)$ denotes the graph of the incidence correspondence between $\mathsf{P}W_7$ and the (projective) fibers of the tautological sheaf over $\mathsf{G}(3,\ W_7)$ and where p_1 and p_2 are the natural projections. In order to prove birationality we just need to prove that a general point of X_7 is contained in one (and only one) abelian surface. One first needs to remark, using representation theory for instance, that both the hypersurfaces X_7 and Y_4 are irreducible.

Let $A \in A(1, 7)^v$ a G_7-embedded abelian surface. We have:

- the septimic hypersurface X_7 contains the surface A;
- the surface A intersects P_2^+ along a reduced scheme;
- the surface A is not contained in the base locus B_κ.

The only non obvious fact is the third item. But B_κ intersects P_2^+ along a Klein quartic curve $\mathscr{K}_4'$ so if we had $A \subset B_\kappa$ this would imply the non emptiness of $A \cap \mathscr{K}_4'$ and in such cases $A \cap \mathsf{P}_2^+$ admits a double point (see e.g. section 3 below) contradicting the second item. Next the map $A \longmapsto A \cap \mathsf{P}_2^+$ is injective (see [10]) so the plane $\overline{\kappa(A \backslash B_\kappa)}$ entirely characterizes the surface A. Summing up we get that two distinct surfaces A and A' intersect each other either on

- the threefold B_κ (which is of codimension 2 in X_7),
- or on the preimage by κ of the points in $Y_4 \subset \mathsf{P}W_7$ which are contained in more than one six-secant plane to the Veronese surface $\overline{\kappa(\mathsf{P}_2^+ \backslash \mathscr{K}_4')}$.

Since A is not contained in B_κ, it remains to show that A is not contained in the second locus. But one proves easily that the second locus is 2-dimensional, being the preimage of the union of the Veronese surface $\overline{\kappa(\mathsf{P}_2^+ \backslash \mathscr{K}_4')}$ itself and its ruled surface of trisecant lines (for which the base is isomorphic to the Klein quartic curve $\mathscr{K}_4$ of the dual plane $\check{\mathsf{P}}_2^+$).

REMARK 2.8. Notice that one can also prove (using Schubert calculus) that the hypersurface Y_4 has degree four in $\mathsf{P}W_7$ (this is true for any collection of six-secant planes to such a projected Veronese surface).

With this compactification of $A(1,7)^v$, we define

DEFINITION 2.9. A *generalized G_7-embedded abelian surface* is the proper transform by κ^{-1} of a plane that intersects the Veronese surface $\kappa(\mathsf{P}_2^+)$ in a finite scheme of length six.

Notice that to each generalized G_7-embedded abelian surface A one may associate a subscheme $\zeta_A = \kappa(A) \cap \kappa(\mathsf{P}_2^+)$ of length six.

3. Fano threefolds V_{22}

The natural boundary of the compactification $\overline{A(1,7)^v}$ constructed above consists of planes that intersect $\kappa(\mathsf{P}_2^+)$ in nonreduced subschemes of length six. The aim of this section is to describe this boundary in terms of the degrees of the components of these subschemes, but first we need some general facts on this compactification as a prime Fano threefold of genus 12 in its anticanonical embedding.

Recall Mukai's characterization of prime Fano threefolds of genus 12 (cf. [16]).

DEFINITION-PROPOSITION 3.1. *Any Fano threefold of index 1 and genus* 12 *is isomorphic to the variety of sums of powers*

$$VSP(F,6) = \overline{\left\{(\ell_1,\ldots,\ell_6) \in \mathrm{Hilb}_6\, \mathsf{P}W^\vee \mid e_F \equiv \textstyle\sum_{i=1}^6 e_{\ell_i}^4\right\}}$$

of a plane quartic curve F. Conversely, if F is not a Clebsch quartic (i.e. its catalecticant invariant vanishes), then $VSP(F, 6)$ is a Fano threefold of index 1 and genus 12. Its anti-canonical model is denoted by V_{22}.

3.1. Construction

Let W be an irreducible SL(3, $\mathbf{C}$)-module of dimension 3, we have a decomposition of SL(3, $\mathbf{C}$)-modules ([9])

$$S^2(S^2W)^\vee = S^4W^\vee \oplus S^2W$$

generating an exact sequence

$$0 \longrightarrow S^4W^\vee \longrightarrow \mathrm{Hom}(S^2W, S^2W^\vee).$$

So a plane quartic F in $\mathbf{P}W$ whose equation is given by 'an' element e_F of $S^4W^\vee$ gives rise to 'a' morphism $\alpha_F : S^2W \longrightarrow S^2W^\vee$ and a quadric $\mathbf{Q}_F$ in $\mathbf{P}(S^2W)$ of equation $\alpha_F(x) \cdot x = 0$ or even $x \cdot \alpha_F(x) = 0$ by the canonical identification $S^2W = (S^2W^\vee)^\vee$. From the equality $\mathrm{h}^0(\mathscr{I}_{\nu_2(F)}(2)) = 7$, we get a *characterization of this quadric* by the two properties:

(i) the two forms on W defined by $\alpha_F(\nu_2(-))\cdot\nu_2(-)$ and e_F are proportional i.e. the quadric $\mathbf{Q}_F$ and the Veronese surface $\nu_2(\mathbf{P}W)$ intersect along the image of the plane quartic F under ν_2;

(ii) the quadric $\mathbf{Q}_F$ is apolar to the Veronese surface $\nu_2(\mathbf{P}W^\vee)$ of $\mathbf{P}S^2W^\vee$ i.e. apolar to each element of the vector space $\mathrm{H}^0(\mathscr{I}_{\nu_2(\mathbf{P}W^\vee)}(2)) \simeq S^2W^\vee \subset S^2(S^2W)$.

LEMMA 3.2 (Sylvester). *The minimal integer n for which $VSP(F, n)$ is non empty is the rank of α_F (called the catalecticant invariant of the quartic curve).*

PROOF. This well known result of Sylvester (see e.g. Dolgachev and Kanev [6], Elliot [7, page 294]) can be deduced from the following observation: let $n \in \mathbf{N}^*$, then

$$\nu_2(VSP(F, n)) = \{(p_1, \dots, p_n) \in VSP(\mathbf{Q}_F, n) \mid p_\times \in \nu_2(\mathbf{P}W^\vee)\}.$$

Indeed if $s = (\ell_1, \dots, \ell_n) \in VSP(F, n)$ then $e_F \equiv \sum_{i=1}^n e_{\ell_i}^4$ for a good normalization of $e_{l_\times}$ and the quadric $\mathbf{Q} \subset \mathbf{P}S^2W$ of equation

$$e_{\mathbf{Q}} \equiv \sum_{i=1}^n e_{\nu_2(\ell_i)}^2$$

is endowed with the two properties which characterize the quadric Q_F: the second one is a direct consequence of $\mathrm{H}^0(\mathscr{I}_{\nu_2(\mathsf{P}W^\vee)}(2)) \subset \mathrm{H}^0(\mathscr{I}_{\nu_2(s)}(2))$ and the first one arises by construction. Applying ν_2^{-1} we get the required equality.

Define the vector space $Y_\ell \subset S^2W$ such that the line ℓ of the plane $\mathsf{P}W$ induces the exact sequence

$$0 \longrightarrow \mathsf{C} \cdot e_\ell^2 \longrightarrow S^2W^\vee \longrightarrow Y_\ell^\vee \longrightarrow 0,$$

that is to say Y_ℓ is the orthogonal space (in S^2W) of e_ℓ^2.

Definition 3.3. The subscheme $\mathscr{C}_\ell$ of the plane $\mathsf{P}W^\vee$ defined by $\mathscr{C}_\ell = \{x \in \mathsf{P}W^\vee \mid e_x^2 \in \alpha_F(Y_\ell)\} = \nu_2^{-1}(\alpha_F(Y_\ell))$ is called the *anti-polar conic* of the line ℓ (with respect to the quartic F).

Alternatively, if α_F has maximal rank we have obviously

$$\mathscr{C}_\ell = \{x \in \mathsf{P}W^\vee \mid e_x^2 \cdot \alpha_F^{-1}(e_\ell^2) = 0\}.$$

Set $n = \mathrm{rank}(\alpha_F)$; the construction of a point of $VSP(F, n)$ is now very easy by the following corollary, which is a consequence of the classical construction of a point of $VSP(\mathsf{Q}, n)$ when Q is a quadric of rank n.

Corollary 3.4. *A point $(\ell_1, \ldots, \ell_n)$ lies in $VSP(F, n)$ if and only if $\ell_i \in \mathscr{C}_{\ell_j}$ when $i \neq j$,*

We turn to the anti-canonical embedding of V_{22}, in particular to

3.2. *Conics on the anti-canonical model*

Let V be the seven dimensional vector space defined by the exact sequence

$$0 \longrightarrow W \longrightarrow S^3W^\vee \xrightarrow{p_F} V \longrightarrow 0$$

where the second map is induced by $F \in S^4W^\vee \subset \mathrm{Hom}(W, S^3W^\vee)$ and denote by $\mathcal{V}'_{2,9}$ the image of $\mathsf{P}W^\vee$ in $\mathsf{P}V$ by the Veronese embedding ν_3 composed with the third map p_F.

By definition $s = (\ell_1, \ldots, \ell_6) \in VSP(F, 6)$ if and only if the image by p_F of the 6-dimensional vector space (in $S^3W^\vee$) spanned by $e_{\ell_i}^3$ is of rank 3. Thus we get a map of $VSP(F, 6)$ into the Grassmannian $G(3, V)$, by $(\ell_1, \ldots, \ell_6) \mapsto p_F(\langle \ell_1, \ldots, \ell_6 \rangle)$.

Remark 3.5. The image of $VSP(F, 6)$ in the Plücker embedding of the Grassmannian is the anti-canonical model V_{22} of this Fano threefold, it is isomorphic to the variety of six-secant planes to the projected Veronese surface $\mathscr{V}'_{2,9}$.

Now it is reasonable to talk about conics on V_{22}. Denote by $F^\flat$ the dual quartic of $\mathsf{P}W$ of equation $\alpha_F^{-1}(e_\ell^2) \cdot e_\ell^2 = 0$, in other words we have

$$F^\flat = \{\ell \in \mathsf{P}W^\vee \mid \ell \in \mathscr{C}_\ell\}$$

(the quartic $F^\flat$ reduces to a double conic when $n = 5$), and denote by H_F the sextic of $\mathsf{P}W^\vee$ given by

$$H_F = \{\ell \in \mathsf{P}W^\vee \mid \operatorname{rank}(\alpha_F^{-1}(e_\ell^2)) \leqslant 1\}.$$

Let $\ell \in \mathsf{P}W^\vee \backslash H_F$, then the anti-polar conic $\mathscr{C}_\ell$ is smooth and we can write the abstract rational curve $\overline{\mathscr{C}_\ell}$ as $\mathsf{P}^1 = \mathsf{P}S_1$ with $\dim S_1 = 2$, i.e. we put $S_1 = \mathrm{H}^0(\mathcal{O}_{\overline{\mathscr{C}_\ell}}(1))$. Put $S_n := S^n S_1$, then $\mathsf{P}S_n$ is identified with the divisors of degree n on $\mathsf{P}S_1$ and we have

LEMMA 3.6. *The set of divisors of degree* 5 *on the anti-polar conic* $\mathscr{C}_\ell$ *given by* $\{\ell' + \mathscr{C}_\ell \cap \mathscr{C}_{\ell'}, \ell' \in \mathscr{C}_\ell\}$ *is a projective line in* $\mathsf{P}S_5$. *This* g_5^1 *admits a base point if* $\ell \in F^\flat$.

PROOF. Let D be such a divisor. By Corollary 3.4 the divisor D is completely determined by any one of its (sub)-divisor of degree 1, so the variety of such divisors is a curve of first degree in $\mathsf{P}S_5$.

COROLLARY 3.7. *The points of* $VSP(F, 6)$ *which contain a given line* ℓ *describe a conic* $\mathcal{C}_\ell$ *on the anti-canonical model* V_{22}. *The two conics* $\mathcal{C}_\ell$ *and* $\mathscr{C}_\ell$ *have the same rank.*

PROOF. Let ℓ be a point outside H_F. Then the image of $\mathscr{C}_\ell$ by ν_3 is a rational normal sextic projected by the map p_F to a smooth sextic inside $\mathsf{P}_\ell^4 := \mathsf{P}(p_F(\mathrm{H}^0(\mathcal{O}_{\mathscr{C}_\ell}(6))))$ – we have an injection $\mathrm{H}^0(\mathcal{O}_{\mathscr{C}_\ell}(6)) \subset S^3W^\vee$ and it is a simple matter to check $\mathrm{H}^0(\mathcal{O}_{\mathscr{C}_\ell}(6)) \cap \ker(p_F)$ is of dimension 2, moreover identifying $\ker(p_F)$ and W we have $\mathsf{P}(\mathrm{H}^0(\mathcal{O}_{\mathscr{C}_\ell}(6)) \cap W) = \ell \subset \mathsf{P}W$. Now P_ℓ^4 also contains the image of $\nu_3(\ell)$ and projecting from this latter point the sextic becomes:

(i) a rational sextic on a quadric of a P^3 generically;

(ii) a rational quintic on a quadric of a P^3 if $\ell \in F^\flat$.

These curves are obviously on a quadric, since a six-secant plane to $\mathscr{V}'_{2,9}$ passing through the point $p_F(\nu_3(\ell))$ will be mapped to a five-secant (resp. four-secant) line to this rational curve.

Now if $\ell \in H_F$, $\mathscr{C}_\ell$ breaks in two lines, say ℓ_1 and ℓ_2. We get two systems of six-secant planes to $\mathscr{V}'_{2,9}$ containing $p_F(\nu_3(\ell))$, one of these intersects $p_F(\nu_3(\ell_1))$ in two fixed points and intersects the twisted cubic $p_F(\nu_3(\ell_2))$

along a pencil of divisors of degree 3. In particular, such collection is mapped to a line by κ.

COROLLARY 3.8. *If p is not on H_F the threefold $p_1 p_2^{-1}(\mathcal{C}_p)$ is a quadric cone Γ_p of rank 4 in P^4_p. If $p \in H_F$ the cone Γ_p splits in two P_3's.*

REMARK 3.9. We already get a first interpretation in terms of abelian surfaces. Putting $W = W_3^\vee$ and choosing the unique $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-invariant quartic $\mathscr{K}_4$ of $\mathsf{P}W = \check{\mathsf{P}}_2^+$ for F we get $V = W_7$, $\mathscr{V}'_{2,9} = \kappa(\mathsf{P}_2^+)$, $F^\flat = \mathscr{K}'_4$. Now if $p \in \mathsf{P}_2^+$, the proper transform of the quadric cone Γ_p by κ^{-1} is by [10] birational to a Calabi Yau threefold, and by the preceding corollary contains – when $\mathscr{C}_p$ is smooth – *two* distinct pencils of special surfaces: the one induced by the six-secant planes, parameterized by $\mathcal{C}_p$ and corresponding generically to abelian surfaces, and another one induced by the second ruling (parameterized by $\mathscr{C}_p$) of planes of the cone. We think that these last ones are the same as the ones evoked in [10, remark 5.7].

3.3. *Boundary for the Klein quartic*

The boundary of $V_{22} = VSP(F, 6)$, as the set of nonreduced length six schemes apolar to F, is easily deducible from what follows. But for the general plane quartic F one needs to introduce a covariant of F, and this would be beyond the subject of this paper. So in this section, we focus on the surface $\Delta_F = \{s \in VSP(F, 6) \mid \lambda_s \neq (1^6)\}$ when the quartic F admits $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$ as its group of automorphisms.

We start by choosing a faithful embedding $1 \longrightarrow \mathrm{SL}(2, \mathsf{F}_7) \longrightarrow \mathrm{SL}(3, \mathsf{C})$, so the vector space W of our preceding section becomes a $\mathrm{SL}(2, \mathsf{F}_7)$-module (necessarily irreducible), say $W \simeq W_3^\vee$ and the decomposition $S^4 W_3 = \mathsf{C} \oplus W_6 \oplus W_8$ allows us to consider the *unique* $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-invariant quartic $\mathscr{K}_4$ of $\check{\mathsf{P}}_2^+ = \mathsf{P}W_3$. Such a quartic is called a Klein quartic and becomes the quartic F of our preceding section. All the quartic covariants of F are equal to F (when non zero) and the Klein quartic $\mathscr{K}'_4 \subset \mathsf{P}_2^+$ is (by unicity) the quartic $F^\flat$ of the last section.

We will need the classical

LEMMA 3.10. *There is a unique* $\mathrm{SL}(2, \mathsf{F}_7)$*-invariant even theta characteristic ϑ on the genus* 3 *curve $\mathscr{K}_4$ (resp. $\mathscr{K}'_4$).*

PROOF. The existence follows directly by the existence of a $\mathrm{SL}(2, \mathsf{F}_7)$-invariant injection $W_3 \longrightarrow S^2 U_4$ so that one can illustrate $\mathscr{K}_4$ as the Jacobian of a net of quadrics (in $\mathsf{P}U_4^\vee$). It is well known that such Jacobian is endowed with an even theta characteristic (cf. [1]). Reciprocally, such a theta characteristic on a curve of genus 3 comes with a net of quadrics and $\mathrm{Hom}_{\mathrm{SL}(2,\mathsf{F}_7)}(W_3, S^2 U) \neq 0$

if and only if the four dimensional vector space U equals $U_4^\vee$ as $\mathrm{SL}(2,\mathsf{F}_7)$-module.

We have

PROPOSITION 3.11. *Let $p \in \mathscr{K}_4'$ and $(x_1,x_2,x_3) \in \mathscr{K}_4' \times \mathscr{K}_4' \times \mathscr{K}_4'$ such that $\mathrm{h}^0(\vartheta + p - x_i) = 1$, then the anti-polar conic $\mathscr{C}_{x_i}$ of x_i with respect to $\mathscr{K}_4$ contains x_j.*

PROOF. Let us leave the plane P_2^+ and take a look at the configuration in $\mathsf{P}_5^+ = \mathsf{P}S^2W_3 = \mathsf{P}W_6 = \check{\mathsf{P}}_5^+$. The image of x_i by the Veronese embedding ν_2 lies on the quadric $Q_{\mathscr{K}_4'}$. On the other hand, noticing that $\mathrm{Hom}_{\mathrm{SL}_2\,\mathsf{F}_7}(\mathsf{C}, S^2S^2W_3) = \mathsf{C}$ this quadric can be interpreted

- as the inverse of the quadric $Q_{\mathscr{K}_4}$;
- as the Plücker embedding of the Grassmannian of lines of P_3^- using the $\mathrm{SL}(2,\mathsf{F}_7)$-invariant identification $S^2W_3 \simeq \Lambda^2 U_4$.

Let us denote by $\mathscr{K}_6'$ the Jacobian of the net of quadrics given by $W_3 \longrightarrow S^2U_4$ and remember that this curve is (by unicity) canonically isomorphic to $\mathscr{K}_4'$ itself. So $\nu_2(x_i)$ is a line in P_3^- (still denoted by $\nu_2(x_i)$) and this one turns out to be a trisecant line to the sextic $\mathscr{K}_6$ containing the image of p by the identification $\mathscr{K}_4' = \mathscr{K}_6'$. This interpretation of the $(3,3)$ correspondence on $\mathscr{K}_4' = \mathscr{K}_6'$ induced by the even theta characteristic as the incidence correspondence between $\mathscr{K}_6'$ and its trisecant lines is due to Clebsch. Now the three lines $\nu_2(x_i)$ are concurrent in p and then the three points $\nu_2(x_i)$ of P_5^+ span a projective plane contained in the inverse of the quadric $Q_{\mathscr{K}_4}$. In particular, $\alpha_{\mathscr{K}_4}^{-1}(\nu_2(x_i))\cdot\nu_2(x_j) = 0$ which is precisely what we need to claim that $x_j \in \mathscr{C}_{x_i}$.

Notice that using the same geometric interpretation we get immediately

COROLLARY 3.12. *If $p \in \mathscr{K}_4'$ then the anti-polar conic $\mathscr{C}_a$ intersects the hessian triangle T_p (i.e. the hessian of the polar cubic of p with respect to $\mathscr{K}_4'$) in points of the quartic $\mathscr{K}_4'$ (and $\mathscr{C}_p \cap \mathscr{K}_4' - 2p = T_p \cap \mathscr{K}_4' - 2x_1 - 2x_2 - 2x_3$)).*

PROPOSITION 3.13. *Let $s \in \Delta := \Delta_{\mathscr{K}_4}$, then there exists at least one point p in the support of ζ_s such that $p \in \mathscr{K}_4'$ and the type of ζ_s is one of the following*

	$p \notin \mathscr{H}_6$	$p \in \mathscr{H}_6$
	2, 1, 1, 1, 1	2, 2, 1, 1
type of ζ_s	3, 1, 1, 1	4, 2
	2, 2, 2	$(2,2,2)_c$

where $\mathscr{H}_6$ is the Hessian of $\mathscr{K}_4'$.

The types of ζ_s and the corresponding stratification of Δ is illustrated in Figure 1 and Figure 2 in the appendix.

PROOF. From the preceding section, a point s of $VSP(\mathscr{K}_4, 6)$ is in Δ if and only if the support of ζ_s intersects the quartic curve $\mathscr{K}_4'$. So let $p \in \mathscr{K}_4'$, by Corollary 3.4 the only thing to understand is the type of ζ_s when the point p moves along the conic $\mathscr{C}_p$. We have the alternative: the conic $\mathscr{C}_p$ is smooth (case i) or $p \in \mathscr{H}_6 := H_{\mathscr{K}_4'}$ (case ii).

(i) Denote (once again) by S_n the $(n+1)$-dimensional vector space $\mathrm{H}^0(\mathcal{O}_{C_a}(n))$, we have $S_n = S^n S_1$. As $p \in \mathscr{K}_4'$, the $(1,5)$ correspondence between the two (isomorphic) rational curves $\mathcal{C}_p$ and $\mathscr{C}_p$ has a base point, namely the point p itself on $\mathscr{C}_p$ and then reduces to a $(1,4)$ correspondence. The induced pencil of divisors of degree 4 in $\mathsf{P}S_4$ intersects the variety of non reduced divisors in six points (as any generic pencil in $\mathsf{P}S_4$) and the expected types of ζ_s are hence $(2,1,1,1,1)$ generically, $(3,1,1,1)$ once and $(2,2,1,1)$ six times (each corresponding to a point of $\mathscr{C}_p \cap \mathscr{K}_4' - \{p\}$). But by the preceding proposition, if $p' \in \mathscr{C}_p \cap \mathscr{K}_4'$ and $p \neq p'$, then the two conics $\mathscr{C}_p$ and $\mathscr{C}_{p'}$ intersect in $p + p' + 2p''$ with $p'' \in \mathscr{K}_4'$ hence the six expected subschemes ζ_s of type $(2,2,1,1)$ on $\mathcal{C}_p$ become three ζ_s of type $(2,2,2)$ for the particular Klein quartic. Notice that in such a case, the scheme ζ_s has a length decomposition $2\cdot(p+p'+p'')$ and there exists a point $q \in \mathscr{K}_4'$ so that $\mathrm{h}^0(\vartheta + q - x) = 1$ whenever $x \in \{p, p', p''\}$. Let us denote by q_x the intersection of $\mathscr{C}_x$ with the line $\overline{xq}$, then

$$p^3 \cdot q_p + p'^3 \cdot q_{p'} + p''^3 \cdot q_{p''} = 0$$

is an equation of $\mathscr{K}_4$.

(ii) Suppose now the point p is one of 24 points of intersection of the quartic $\mathscr{K}_4'$ and its Hessian $\mathscr{H}_6$. Such points come 3 by 3 and the group μ_3 acts on each triplet (so there is an order p_1, p_2, p_3 on such triplet). Put $p = p_1$. The conic $\mathscr{C}_{p_1}$ is no longer smooth and decomposes in two lines, say $\ell = p_1 p_2$ and $\ell' = p_2 p_3$. Each generic point q of the line ℓ gives us a point $2p_1 + q + q' + 2p_3 = \mathscr{C}_{p_1} \cap \mathscr{C}_q + p_1 + q$ of Δ (hence of type $(2,2,1,1)$) with $q' \in \ell$ defined such that the degree 4 divisor $p_2 + p_1 + q + q'$ on the line ℓ is harmonic. One can even provide the corresponding equation of the quartic $\mathscr{K}_4$:

$$\epsilon(\beta x + \alpha z)^4 - \epsilon(\beta x - \alpha z)^4 - 2\alpha\beta\{(x + \epsilon(\beta^2 z - \alpha^2 y))^4 - x^4\} + 2\alpha^3\beta((y + \epsilon z)^4 - y^4) = 0$$

with coefficients in $\mathsf{C}[\epsilon]/\epsilon^2$. We can forget points of Δ arising from a point of ℓ', for such points can be constructed as the preceding ones by starting with the point p_2 instead of the point p_1. The possible degeneracies follow easily: when $(\alpha : \beta)$ tends to $(1 : 0)$ we get back to the well known $(2, 2, 2)_c$ case, the last equation becomes

$$(z+\epsilon x)^4-z^4+(x+\epsilon\, y)^4-x^4+(y+\epsilon z)^4-y^4 = 4\epsilon(z^3x+x^3y+y^3z) = 0.$$

The last possible degeneration arises when $(\alpha : \beta)$ tends to $(1 : 0)$ in which case we get $(4, 2)$ as partition of 6.

4. Degenerated abelian surfaces

Now that we have seen the boundary Δ of V_{22} in Hilb$(6, \mathsf{P}_2^+)$ we will show that points on Δ naturally correspond to generalized G_7-embedded abelian surfaces that are singular. We shall call them 'degenerated abelian surfaces'. The method we are going to employ is very naïve: given $s \in \Delta$, find a surface A_s in P^6 which intersects P_2^+ along s and check that this surface is sent to a projective plane by the map κ, i.e. that $\mathrm{h}^0(\mathscr{I}_{A_s}(7))^{G_7} = 5$. The so-called translation scrolls are natural candidates for 'degenerated abelian surfaces'. Given a G_7-invariant elliptic normal curve E of degree 7 with an origin in P_2^+ and a point $\sigma \in E$, let $l_{x,\sigma}$ be the bisecant line through the points x and $x+\sigma$ on E. Then the union of bisecant lines $(E, \sigma) = \cup_{x\in E} l_{x,\sigma}$ form a surface that is called a *translation scroll*. We will show that the general point on Δ corresponds to a translation scroll, while further degenerations are formed by reducible surfaces. Finally we prove Theorem 1.1 by showing that any point outside the boundary Δ corresponds to a smooth abelian surface. In this section, we work up to the action of $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$.

4.1. Translation scrolls

We will need the

PROPOSITION 4.1. *Every translation scroll of an elliptic normal curve of degree 7 by a 2-torsion point is a smooth elliptic scroll of degree 7 and contains 3 elliptic normal curves of degree 7.*

PROOF. Cf. [4, Proposition 1.1] or [15].

Let us start with $s \in \Delta$ with $\lambda \in \{(2, 1, 1, 1, 1), (3, 1, 1, 1)\}$. Only one point of s has a multiple structure, say $p \in \mathscr{K}_4'$ and the support of s consists in four *distinct* points on the conic $\mathscr{C}_p$. The stabilizer of $(s)_{\mathrm{red}}$ under $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C}) \simeq \mathrm{Aut}(\mathscr{C}_p)$ is in general isomorphic to Z_2^2 (if it is bigger consider the subgroup

$$\{\mathrm{Id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$$

in $\mathrm{Stab}_{\mathsf{P}\,\mathrm{SL}(2,\mathsf{C})}((s)_{\mathrm{red}}) \subset \mathfrak{S}_4$). Consider the double cover E_s of $\mathscr{C}_p$ ramified at the four points $\mathsf{Z}_2^2 \cdot p$. It is a smooth elliptic curve and we choose p as origin on E_s. Note that, by Proposition 3.11, E_s depends only on p so it will be denoted by E_p. Now the linear system $|7 \cdot p|$ is a H_7-module and we can embed E_p in P^6 in 168 distinct ways, one of them sends $p \in E_p$ to $p \in \mathsf{P}_2^+$. Next $p \in s$ is double along a line that will intersect $\mathscr{C}_p$ in a further point, say p_s (of course $p = p_s$ if $\lambda = (3, 1, 1, 1)$). Denote by σ_s one of the inverse images of p_s by the $2:1$ map $E_p \longrightarrow \mathscr{C}_p$.

PROPOSITION 4.2. *The translation scroll* (E_p, σ_s) *intersects* P_2^+ *along* s *and is mapped by* κ *to a plane.*

PROOF. First the bisecant variety of E_p intersects P_2^+ along $\mathscr{C}_p$: Indeed such variety intersects P_2^+ along a conic (E_p being of degree 7 and invariant under a symmetry which preserves P_2^+). Next by the previous proposition such a conic must contain the three pairs of points of $\mathscr{K}_4'^2$ such as (q_1, q_2) where $\{p, q_1, q_2\}$ are associated to the same point of $\mathscr{K}_4'$ under the $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-invariant $(3, 3)$ correspondence on $\mathscr{K}_4'$. By Proposition 3.11, this conic is nothing but $\mathscr{C}_p$.

Next as $\pm\sigma$ moves along $\mathscr{C}_p$, the set $(E_p, \pm\sigma) \cap \mathscr{C}_p - 2p$ describes a *pencil* of degree 4 divisors on $\mathscr{C}_p$, we need to identify this pencil with our pencil of degree 4 divisors on $\mathscr{C}_p$ (proof of Proposition 3.13, item (i). But both pencils contain the three divisors of type $(2, 2)$ such as $2q_1 + 2q_2$ so they are equal.

The point now is to prove that (E_p, σ_s) is mapped to a plane under κ, i.e. that $\mathrm{h}^0(\mathcal{O}_{(E_p,\sigma_s)}(7))^{G_7} = 3$ or rather $\mathrm{h}^0(\mathcal{O}_{(E_p,\sigma_s)}(7))^{G_7} \leqslant 3$ by the previous paragraphs. The scroll (E_p, σ_s) is a P^1-bundle over E_p (in two ways, these correspond to the choices σ_s and $-\sigma_s$ to define the scroll) so we have a map

$$(E_p, \sigma_s) \longrightarrow E_p$$

and if R denotes a generic fiber we get a sequence

$$\mathrm{H}^0(\mathcal{O}_{(E_p,\sigma_s)}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathcal{O}_R(7))^{G_7} \longrightarrow 0$$

which turns out to be exact: if a G_7-invariant septimic S contains the line R, then its intersection with (E_p, σ_s) contains $E_p \cup G_7 \cdot R$ which is of degree $7 + 49 \times 2 \times 1$. Now our scroll is of degree 14 and by Bezout's theorem we conclude $(E_p, \sigma_s) \subset S$. Using a semi-continuity argument we just need to find one fiber R such that $\mathrm{h}^0(\mathcal{O}_R(7))^{G_7} \leq 3$. Choose one of the forty-nine P_2^+ and pick up one of the four lines of (E_p, σ_s) which intersects it. Then the restriction

$$\mathrm{H}^0(\mathcal{O}_{\mathsf{P}V_0}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathcal{O}_R(7))^{G_7}$$

is of rank 3 at most which is precisely what we needed.

Of course to get the $(2, 2, 2)$ cases it is natural to make σ_s tend to a 2-torsion point. Then the translation scroll (E_p, σ_s) tends to a *smooth* scroll of degree 7 and everything is lost in virtue of the following remark.

REMARK 4.3. Any translation scroll (E_p, σ_s) where σ_s is a non-trivial 2-torsion point of E_p is contained in all our G_7-invariant septimic hypersurfaces. Indeed such a scroll intersects any of the forty-nine P_3^-'s along a line and contains the curve E_p itself, once again Bezout's theorem together with the inequality $49 \times 1 + 7 > 7 \times 7$ allow us to conclude.

However we have

PROPOSITION 4.4. *If* $\lambda_s = (2, 2, 2)$*, then there exist an elliptic curve* E_p*, a two torsion point* σ_s *on* E_p *and a double structure on the translation scroll* (E_p, σ_s) *intersecting* P_2^+ *along* s *and mapped to a plane by* κ*.*

PROOF. Let $\lambda_s = (2, 2, 2)$. By Proposition 4.1 one and only one smooth translation scroll X intersects P_2^+ along $(s)_{\mathrm{red}}$ so we just need to find a double structure $\tilde{X}$ on X such that $\mathrm{h}^0(\mathscr{I}_{\tilde{X}}(7))^{G_7} = 5$. Now X contains two (in fact three by 4.1) elliptic curves E_p and $E_{p'}$ and by definition is contained in the two corresponding bisecant varieties S_{E_p} and $S_{E_{p'}}$. But these two varieties are the proper transforms by κ^{-1} of the two quadric cones Γ_p and $\Gamma_{p'}$ (cf. Corollary 3.8). These two cones intersect along the six-secant plane to $\kappa(\mathsf{P}_2^+)$ corresponding to s so we are done! The double structure is then easy to understand: one considers the double structures on $X \backslash E_p$ (resp. $X \backslash E_{p'}$) defined by the embedding $X \longrightarrow S_{E_p}$ (resp. $X \longrightarrow S_{E_{p'}}$) and such structures coincide on $X \backslash (E_p \cup E_{p'})$.

4.2. Union of seven quadrics

We still have to consider the missing cases, namely schemes s such that $\lambda_s \in \{(2, 2, 1, 1), (2, 2, 2)_c, (4, 2)\}$. These are degenerations of the preceding ones. A degenerated elliptic curve is nothing but a heptagon, and such curves come in triplets (E_0, E_1, E_2) (cf. Figure 3 of the appendix) with $E_i = \bigcup_{k=0}^{6} \overline{e_k e_{k+1+i}}$ and $\{e_x\}_{x \in \mathsf{Z}_7}$ is an orbit of minimal cardinality under the action of G_7.

The Heisenberg action on each curve E_i reduces to an action of Z_7. Let us denote by P_I the projective space spanned by the points $\{e_i\}_{i \in I}$.

For $i \in \{0, 1, 2\}$ put $B_i = \bigcup_{k=0}^{6} \mathsf{P}_{I_i^k}$ with $I_i^k = \{k + i + 1, k - i - 1, k + 3i + 3, k - 3i - 3\}$. Then the bisecant variety of E_i is $B_j + B_k$ with $\{i, j, k\} = \{0, 1, 2\}$.

Finally let us choose one of the forty nine P_2^+ and suppose E_i intersects it in p_{i+1}. We are then ready to check the remaining cases:

Let $s \in \Delta$ such that $\lambda_s = (2, 2, 1, 1)$ and $s = 2 \cdot p_1 + 2 \cdot p_3 + q + q'$ with q and q' on the line $\overline{p_1 p_2}$ (cf. Figure 1 of the appendix). So we have $s \in \mathcal{C}_{p_1} \cap \mathcal{C}_{p_3}$.

The corresponding degenerated abelian surface A_s needs to be on the bisecant varieties of E_0 and E_2 so we have $A_s \subset B_1$. As B_1 is the union of seven P^3's the surface A_s is the union of seven quadrics.

When s moves along $\mathcal{C}_{p_1} \cap \mathcal{C}_{p_3}$ we get the two other kinds of degenerations:

- if $\lambda_s = (2, 2, 2)_c$, then the surface A_s degenerates in the union of the fourteen planes $B_0 \cap B_1 \cup B_1 \cap B_2 \cup B_2 \cap B_0$;
- if $\lambda_s = (4, 2)$, then the surface A_s degenerates in the union of the seven planes $\bigcup_{k=0}^{6} \mathsf{P}_{\{k,k+3,k-3\}}$ double along B_1.

4.3. The smooth case

In order to complete the proof of Theorem 1.1, we need to show that a generalized G_7-embedded abelian surface A is smooth and abelian provided the type of ζ_A is $(1, 1, 1, 1, 1, 1)$. Now, by the Enriques-Kodaira classification of surfaces (see [3, chapter VI]) complex tori are entirely characterized by their numerical invariants. So any generalized G_7-embedded abelian surface is an abelian surface provided it is smooth. Our strategy is to consider A as a divisor on a Calabi Yau threefold as in remark 3.9.

First we treat the case of surfaces A singular along a curve:

LEMMA 4.5. *A generalized G_7-embedded abelian surface A, singular along a curve, intersects 'the' plane P_2^+ in a non reduced scheme.*

PROOF. Let A be a singular generalized G_7-embedded abelian surface. The proposition is obviously true if A is singular in codimension 0 that is to say if A carries a double structure. For such surface, the intersection of its reduced structure (of degree 7) with any P_2^+ cannot be six distinct points so $\zeta_A = A \cap \mathsf{P}_2^+$, which is of length six, cannot be reduced.

By assumption the singular locus of A contains a curve C. We can also assume C is G_7-invariant (if not we replace C by its orbit under G_7).

If C has degree 7, it is necessarily elliptic and, being G_7 invariant, intersects P_2^+ in a point of the Klein curve $\mathscr{K}_4'$ so we are done. Indeed the rationality of C (if irreducible) is totally excluded (such a curve admits either a unique four-secant plane, a unique trisecant line or a (unique) double point, this would be a contradiction with the irreducibility of V_0 as H_7-module), but C can still split in the union of seven lines. We want to prove that C in this case is a heptagon (that is to say elliptic). The stabilizer of one of the lines under the action of H_7 is isomorphic to Z_7 so we get, on each line $\ell \subset C$, two fixed points under the action of $\mathrm{Stab}_{H_7}(\ell) \simeq \mathsf{Z}_7$ and then an orbit of fourteen points on C. Noticing all the components have the same stabilizer (the only group morphism from Z_7 to the symmetric group $\mathfrak{S}_6$ is constant) and considering the symmetries of

G_7 it is easy to prove that these fourteen points coincide two by two, implying C is a heptagon.

If the singular locus C has degree 14, then the reduced structure of it has degree 7 or 14. Only the latter is a problem. The normalization of the surface would have sectional genus (-6) so it would consist of at least seven components. They have the same degree i.e. 1 or 2, so their number must be 7 and their degree must be 2. In particular, either the surface A is contained in one orbit of seven P_3's under G_7 or the reduced structure of A consists of seven planes. In both cases, it is a simple matter to conclude for the only orbits of seven P_3's under G_7 are listed in the subsection 4.2 (consider for instance their possible intersections with the forty-nine P_3^+) so the surface A appears already in the subsection 4.2 and the proposition is true for such surfaces.

The last possible case is when C has degree 21, but then the surface A has 14 components (its normalization would have sectional genus (-13)). Therefore C splits and, as $\gcd(49, 21) = 7$, contains three G_7-invariant curves of degree 7 so we are back to the first case.

End of the proof of Theorem 1.1.

Let A be a generalized G_7-embedded abelian surface, preimage of a six-secant plane of $\kappa(\mathsf{P}_2^+)$ by κ, i.e. $\zeta_A = (1, 1, 1, 1, 1, 1)$. Since $A \cap \mathscr{K}_4' = \emptyset$ we may divide in two cases; A intersects $\mathscr{H}_6$ but not $\mathscr{K}_4'$ in P_2^+, and A intersects neither $\mathscr{H}_6$ nor $\mathscr{K}_4'$.

If A intersects $\mathscr{H}_6$ in P_2^+, then A is a *smooth* plane curve fibration and has a trisecant line in P_2^+: see construction in [11].

So we are left with the case that $A \cap \mathscr{H}_6 = A \cap \mathscr{K}_4' = \emptyset$. Thus, A is neither a translation scroll nor a plane curve fibration.

By the previous lemma, A is irreducible with isolated singularities. First we compute some invariants of A.

LEMMA 4.6. $\omega_A = \mathcal{O}_A$ *and* $\chi(\mathcal{O}_A) = 0$.

PROOF. Let us choose $a \in \mathsf{P}_2^+ \cap A$ and consider (identifying P_2^+ with $\mathsf{P}W_3$) the Calabi Yau threefold Y_a preimage of the cone Γ_a by κ. We know that $Y_a = \bigcup_{t \in \mathcal{C}_a} A_t$. Y_a has a quadratic singularity at a, and the general surface A_t is smooth at a, so after a small resolution of Y_a at a, the surface A_t will be Cartier there.

We want to prove that the surface A is Cartier as a divisor on Y_a except in the preimage $\kappa^{-1}(\kappa(a))$. Since A is the pullback of a plane on Γ_a by κ, it could fail to be Cartier only in $\kappa^{-1}(\kappa(a))$ and in the restriction of the base locus of κ to A. In fact, it could fail to be Cartier only on the intersection $B_a = \bigcap_{t \in \mathcal{C}_a} A_t \subset Y_a$.

Consider the restriction of κ to a smooth abelian surface A_t in Y_a. The restriction of the base locus of κ contains already the intersection of A_t with the 49 P_3^-'s, i.e. 49×10 points so the degree of κ restricted to A_t is at most $7 \times 7 \times 14 - 49 \times 10 = 196$. On the other hand, the G_7-orbits of degree 98 form a Kummer surface (cf. Proposition 2.6). Since $\kappa(A_t)$ is a plane, the degree is at least 196. Therefore B_a contains no points outside $\kappa^{-1}(\kappa(a)))$, and A is Cartier on Y_a outside $\kappa^{-1}(\kappa(a)))$.

After a small resolution Y'_a of Y_a along $\kappa^{-1}(\kappa(a)))$, the strict transform of A is Cartier everywhere on Y'_a. We may even assume that the small resolution restricted to A is an isomorphism.

Now, Y'_a is still Calabi-Yau and the surfaces A_t form a pencil without base points on Y'_a, so we have that $\omega_A = \mathcal{O}_A$. Indeed, the general A_t is smooth (in particular A_t is a smooth elliptic fibration for $t \in \mathcal{H}_6$) with normal bundle $\mathcal{O}_A(A) = \mathcal{O}_A$ so $\omega_A = \mathcal{O}_A(K_{Y_a}) = \mathcal{O}_A$. Furthermore $\chi(\mathcal{O}_A) = \chi(\mathcal{O}_{A_t}) = 0$ as claimed.

Finally, we show that A is smooth. By assumption, A has only isolated singularities. The length of any G_7-orbit is a multiple of 7, so A has at least 7 singular points. Furthermore, A has trivial canonical sheaf and $\chi(\mathcal{O}_A) = 0$. Let $\tilde{A} \to A$ be a minimal desingularization of A. Then there are no (-1)-curves in the exceptional locus. On the other hand the canonical divisor K on $\tilde{A}$ is supported on the exceptional locus. Let H be the pullback of the hyperplane divisor on A. Then $H \cdot K = 0$, so $K^2 \leq 0$, with equality only if K is trivial. Furthermore any effective pluricanonical divisor is supported on the exceptional locus. In fact we get that $h^0(mK) = h^0(K) = p_g$, for $m > 0$. If $p_g = 1$, and $K^2 < 0$, then $\tilde{A}$ is necessarily a nonminimal surface, i.e. it contains (-1)-curves. But any such curve is contained in K, so by assumption it is contained in the exceptional locus of the desingularization map. This contradicts the minimality of the desingularization. If K is trivial, then the singularities are rational double points, while $\tilde{A}$ is an abelian surface. This is again a absurd. If $p_g = 0$, then $\tilde{A}$ has no effective pluricanonical divisors, and $K^2 < 0$. So $\tilde{A}$ is birational to a ruled surface. Let F be a general member of its ruling. Then $K \cdot F = -2$. On the other hand the image in A of the support of K is invariant under G_7, so $K \cdot F$ must be divisible by 7. This is a contradiction that completes the proof.

We add a characterization of the intersection of $A \cap \mathsf{P}_3^-$:

REMARK 4.7. Choose coordinates $(y_i)_{i\in\{0,\ldots,6\}}$ in V_0 together with one of the forty-nine P_3^-'s of equations $y_4 = y_3$, $y_5 = y_2$, $y_6 = y_1$. Using the N_7-invariant isomorphism $\Lambda^2 V_4 = W_3^\vee \otimes V_0$ let us introduce for $x = (x_1 : x_2 : x_3) \in \mathsf{P}W_3$

and $y = (y_0 : - : y_4) \in \mathsf{P}_3^- \subset \mathsf{P}V_0$ the matrix

$$M_y(x) = \begin{pmatrix} x_2y_2 & -x_3y_1 - x_1y_3 & x_3y_0 & -x_2y_1 - x_1y_2 \\ -x_3y_3 & -x_3y_2 + x_2y_3 & -x_2y_1 + x_1y_2 & x_1y_0 \\ -x_1y_1 & x_2y_0 & x_3y_1 & x_3y_2 + x_2y_3 \end{pmatrix}$$

It is the restriction to P_3^- of a skew-symmetric Moore matrix $M(x, y)$ (see remarks 2.5, 3.9 and [10]). This matrix $M(x, y)$ defines the Calabi Yau threefold which is the strict transform of the cone Γ_x by κ^{-1}. For general $y \in \mathsf{P}_3^-$, $M_y(x)$ defines six points in $\mathsf{P}W_3$, meaning there is *one* abelian surface containing y and six Calabi Yau threefolds of the preceding type which contain this surface. But $M_y(x)$ may degenerate for special points $y \in \mathsf{P}_3^-$ (for such cases we get more than six points in $\mathsf{P}W_3$):

rank of $\Lambda^3 M_y(x)$	y in	x in	abelian varieties passing through y
3	$\mathscr{K}_6$	$\mathscr{H}_6$	have a trisecant line
2	C_{18}	$\mathscr{K}_4'$	translation scrolls
1	Z	$\mathscr{K}_4' \cap \mathscr{H}_6$	reducible

where $\mathscr{K}_6$ is the unique $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-invariant curve of degree 6 and genus 3 in P_3^-, C_{18} is a $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$-invariant curve of degree 18 and genus 35 in P_3^- (analogue of the Bring curve in the $(1, 5)$ case) and Z is the minimal orbit (of cardinality eight) under the action of $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$ on P_3^-.

4.4. Some questions

Let A be a smooth abelian surface embedded in $\mathsf{P}V_0$. As already noticed in Proposition 2.6 there is a factorization

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A).$$

What is the ramification locus of the last map? Of course, the map to the right needs to define a $K3$ surface so it is certainly a sextic curve (of genus 4), but it doesn't explain how to recover it (using representation theory for instance). A good understanding of this point should enable one to get a reconstruction method, as in the $(1, 5)$ case, of the abelian surface $A^\vee$. Note that this ramification locus admits the six points $\kappa(A) \cap \kappa(\mathsf{P}_2^+)$ as double points. Next the surface A intersects P_3^- in ten points, so projecting A from P_3^- we get maps

$$A^\flat \xrightarrow{2:1} K^\flat_A \xrightarrow{2:1} \mathsf{P}_2^+.$$

The last map is ramified along a sextic curve with the six points $A \cap \mathsf{P}_2^+$ as double points. Hence there are two maps $K^\flat_A \xrightarrow{2:1} \mathsf{P}_2^+$ and $K^\flat_A \xrightarrow{2:1} \kappa(A^\vee)$.

Is it possible to find an (in fact 168) identification(s) between P_2^+ and $\kappa(A^\vee)$ such that the two maps coincide? The answer is positive in the $(1, 5)$ case and allows one to identify the moduli space of $(1, 5)$ polarized abelian surfaces (without level structure) up to duality. Note that one can easily show that the two sets of six points $A \cap \mathsf{P}_2^+$ and $\kappa(A) \cap \kappa(\mathsf{P}_2^+)$ are associated in Coble's sense. This phenomenon is in fact true for any Fano threefold V_{22}: given a six-secant plane to $\mathscr{V}'_{2,9}$ (Veronese surface isomorphic to $\mathsf{P}W$), it intersects it along six points associated to the corresponding set in $\mathsf{P}W$.

It is possible to show that the Fano threefold $VSP(\mathscr{K}_4, 6)$ is 'stable' by association of points i.e. that there exists a (dual) Klein curve C in the plane $\kappa(A)$ such that $\kappa(A) \cap \kappa(\mathsf{P}_2^+)$ is a point of the corresponding variety $VSP(C, 6)$, isomorphic, up to $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$, to $VSP(\mathscr{K}_4, 6)$. It is therefore natural to ask whether one can find a quotient of $VSP(\mathscr{K}_4, 6)$ by $\mathsf{P}\,\mathrm{SL}(2, \mathsf{F}_7)$ and an involution defined by association that forms the moduli space of $(1, 7)$ polarized abelian surfaces (without level structure) up to duality?

5. Appendix

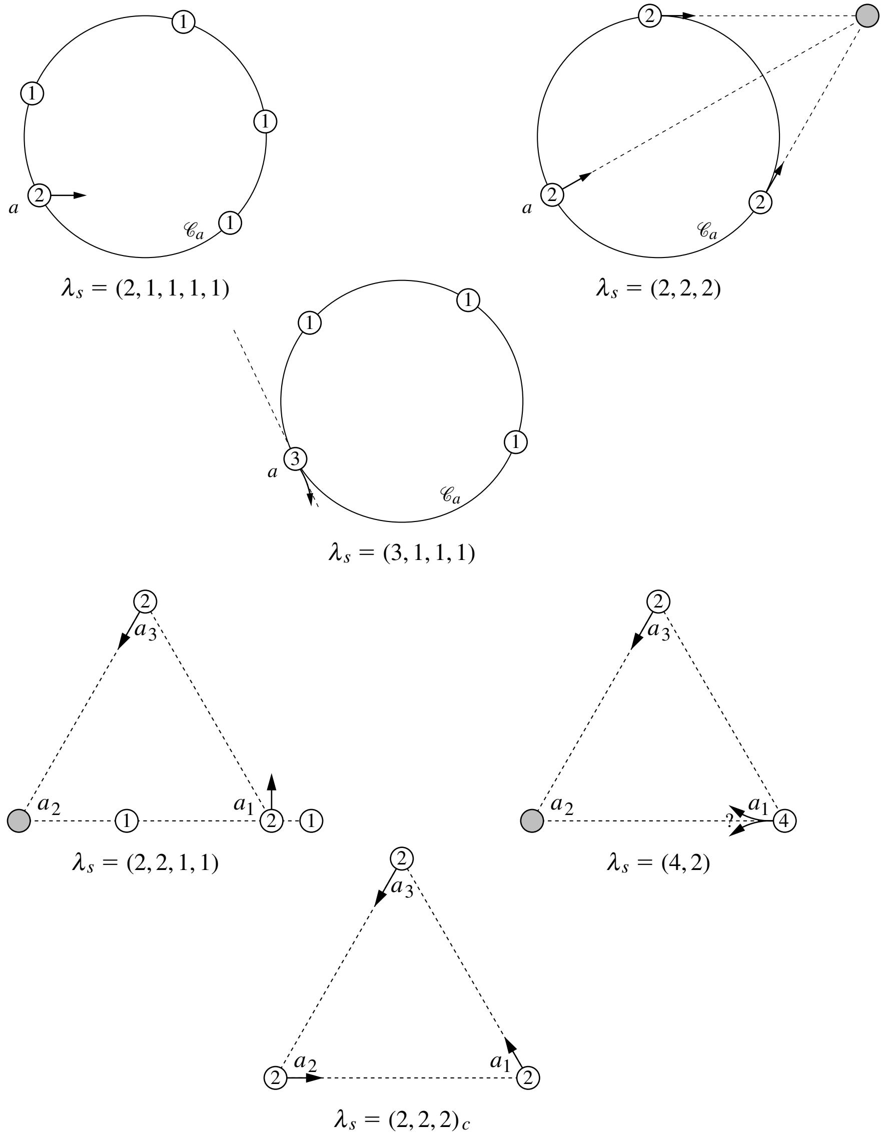

FIGURE 1. Possible configurations of ζ_s when $s \in \Delta$, where each arrow gives the (first) direction along which the point is doubled (oriented in a purely decorative way).

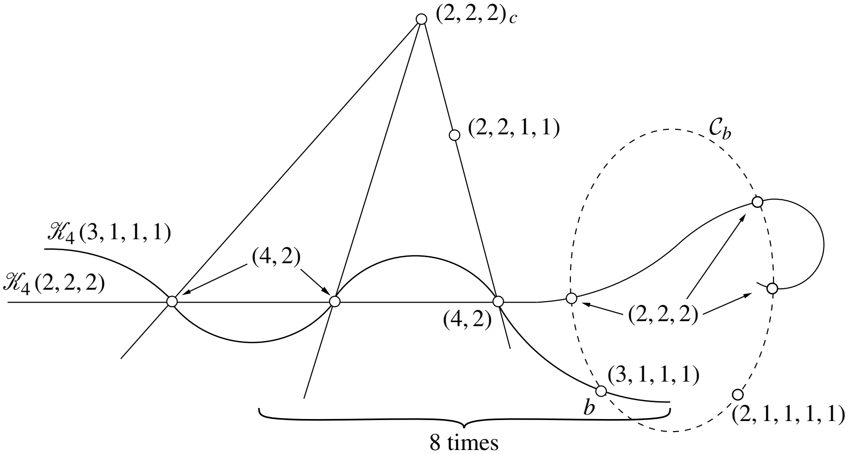

FIGURE 2. Stratification of the surface Δ.

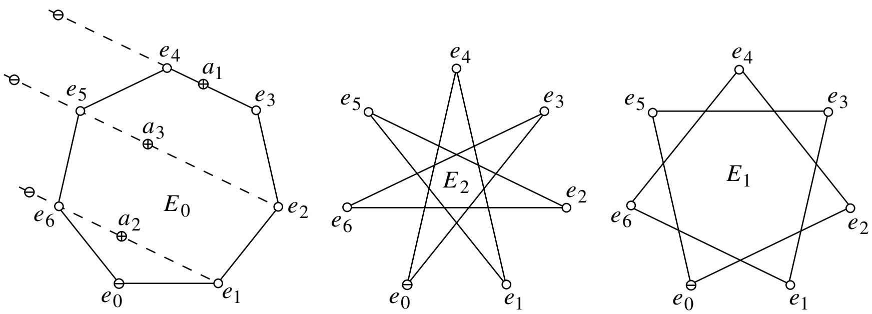

FIGURE 3. A triplet of degenerate elliptic curves.

REFERENCES

1. Barth, W., *Moduli of vector bundles on the projective plane*, Invent. Math. 42 (1977), 63–91.
2. Barth, W., Hulek, K., and Moore, R., *Degenerations of the Horrocks-Mumford surfaces*, Math. Ann. 277 (1987), 735–755.
3. Barth, W., Peters, C., and Van de Ven, A., *Compact Complex Surfaces*, Ergeb. Math. Grenzgeb. 4 (1984).
4. Ciliberto, C., and Hulek, K., *A remark on the geometry of elliptic scrolls and bielliptic surfaces*, Manuscripta Math. (1998), 231–224.
5. Dolgachev, I., *Invariant stable bundles over modular curves $X(p)$*, in *Recent progress in algebra (Taejon/Seoul, 1997)*, Contemp. Math. 224 (1999), 65–99.
6. Dolgachev, I., and Kanev, V., *Polar covariants of cubics and quartics*, Adv. Math. 98 (1993), 216–301.
7. Elliot, E. B., *An introduction to the Algebra of Quantics*, Chelsea reprint, 1964.

8. Fricke, R., *Lehrbuch der Algebra II*, Braunschweig, 1926.
9. Fulton, W., and Harris, J., *Representation Theory*, Grad. Texts in Math. 129 (1991).
10. Gross, M., and Popescu, S., *Calabi-Yau threefolds and moduli of abelian surfaces I*, Compositio Math. 127 (2001), 169–228.
11. Hulek, K., and Ranestad, K., *Abelian surfaces with two plane cubic fibrations and calabi-yau threefolds*, In Th. Peternell and F.-O. Schreyer, editors, *Complex Analysis and Algebraic Geometry*, 275–316, de Gruyter, 2000.
12. Lange, H., and Birkenhake, C., *Complex Abelian Varieties*, Springer-Verlag, 1992.
13. Manolache, N., and Schreyer, F.-O., *Moduli of* $(1, 7)$*-polarized abelian surfaces via syzygies*, Math. Nachr. 226 (2001), 177–203.
14. Marini, A., *On a family of* $(1, 7)$*-polarize abelian surfaces*, Math. Scand. 95 (2004), 181–225.
15. Miele, M., *Klassifikation der Durchschnitte Heisenberg-Invariante Systeme von Quadriken in* P^6, PhD thesis, Erlangen, 1993.
16. Mukai, S., *Fano 3-folds*, In *Vectors Bundles and Special Proj. Embeddings (Bergen 1989)*, London Math. Soc. Lecture Note Ser. 179 (1992), 255–263.
17. Schreyer, F.-O., *Geometry and algebra of prime Fano 3-folds of genus 12*, Compositio Math. 127 (2001), 297–319.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LILLE 1
F-59655 VILLENEUVE DIASCQ CEDEX
FRANCE
E-mail: Franck.Melliez@math.univ-lille1.fr

MATEMATISK INSTITUTT
UNIVERSITETET I OSLO
P.O.BOX 1053 BLINDERN
N-0316 OSLO
NORWAY
E-mail: ranestad@math.uio.no