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A MONGE-AMPÈRE NORM FOR DELTA-PLURISUBHARMONIC FUNCTIONS

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Abstract

We consider differences of plurisubharmonic functions in the energy class $\mathcal{F}$ as a linear space, and equip this space with a norm, depending on the generalized complex Monge-Ampère operator, turning the linear space into a Banach space $\delta\mathcal{F}$. Fundamental topological questions for this space is studied, and we prove that $\delta\mathcal{F}$ is not separable. Moreover we investigate the dual space. The study is concluded with comparison between $\delta\mathcal{F}$ and the space of delta-plurisubharmonic functions, with norm depending on the total variation of the Laplace mass, studied by the first author in an earlier paper [7].

1. Introduction and notations

Convex-, subharmonic-, and plurisubharmonic functions are all convex cones in some larger linear space. Given any such cone, K say, we can investigate the space of differences from this cone δK. Such studies are often motivated by algebraic completion of the cone, and differences of convex functions were considered by F. Riesz in as early as 1911.

δ-convex functions, or *d.c.* functions as they sometimes are denoted, were studied by Kiselman [15], and Cegrell [8], and have been given attention in many areas ranging from nonsmooth optimization to super-reflexive Banach spaces [13].

δ-subharmonic where first given a systematic treatise in [3]. δ-plurisubharmonic functions were studied by Cegrell [7], and Kiselman [15], where the topology was defined by neighbourhood basis of the form $(U \cap \mathcal{PSH}) - (U \cap \mathcal{PSH})$, U a neighbourhood of the origin in L^1_{loc}.

In this paper we study a subset of δ-plurisubharmonic functions. Let Ω be a hyperconvex domain in $\mathbf{C}^n$, then $\mathcal{F}(\Omega)$ is a convex cones in the linear space $L^1_{loc}(\Omega)$. Let $\delta\mathcal{F}(\Omega)$ denote the set of functions $u \in L^1_{loc}(\Omega)$ that can be written as $u = u_1 - u_2$, where $u_i \in \mathcal{F}(\Omega)$.

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We will define a norm, depending on the Monge-Ampère operator, for functions in this class and discuss some of the topological questions that this norm raises.

For convenience we will denote the class of negative plurisubharmonic functions on a domain Ω by $\mathcal{PSH}^-(\Omega)$, and as in [9] we will denote the class of bounded plurisubharmonic functions with boundary value zero and finite total Monge-Ampère mass by $\mathcal{E}_0(\Omega)$.

For the notation of the so called *energy class* $\mathcal{F}(\Omega)$ on a hyperconvex domain Ω we refer to the paper [10]. As for now we remind the reader that the generalized complex Monge-Ampère operator is well defined in $\mathcal{F}(\Omega)$, and functions from $\mathcal{F}(\Omega)$ has finite total Monge-Ampère mass, but that the so called "comparison principle" do not hold in general, even if it is true that if $u \geq v$ on Ω then $\int_\Omega (dd^c u)^n \leq \int_\Omega (dd^c v)^n$.

In almost all results in this paper the domain Ω does not matter much, except for the results in Section 5, and therefore we will often suppress the reference to Ω from the notation.

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2. Definition of the norm

DEFINITION 2.1. Let Ω be a hyperconvex set in $\mathbb{C}^n$. Assume that $u \in \delta\mathcal{F}(\Omega)$, then we define the norm of u to be:

$$\|u\| = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{F}}} \left[\left(\int_\Omega (dd^c(u_1 + u_2))^n \right)^{\frac{1}{n}} \right].$$

Note that for functions $u \in \mathcal{F}$ we have $\|u\|^n = \int (dd^c u)^n$. To see this choose $u_2 = 0$ in the infimum of the definition and hence $\|u\|^n \leq \int (dd^c u)^n$. For an inequality in the other direction let $u_1, u_2 \in \mathcal{F}$ be any representation of $u = u_1 - u_2$. Since $u_2 \leq 0$ we have $u \geq u_1 - u_2 + 2u_2 = u_1 + u_2$, and thus $\int (dd^c u)^n \leq \int (dd^c(u_1 + u_2))^n$ thus $\int (dd^c u)^n \leq \|u\|^n$.

The following Lemma will be used repeatedly.

LEMMA 2.2. *Suppose $u, v \in \mathcal{F}$, $h \in \mathcal{E}_0$, and that p, q are positive natural numbers such that $p + q = n$. Then*

$$\int -h(dd^c u)^p \wedge (dd^c v)^q \leq \left(\int -h(dd^c u)^n \right)^{\frac{p}{n}} \left(\int -h(dd^c v)^n \right)^{\frac{q}{n}}$$

PROOF. Cf. [10].

The following inequality is very useful when working with the Monge-Ampère operator, and will be essential for our work.

THEOREM 2.3 (Błocki's inequality, [4]). *Let* Ω *be an open subset of* $\mathbb{C}^n$*, and let* $h, u, v_1, \ldots, v_2 \in \mathcal{PSH} \cap \mathcal{C}(\Omega)$*. Furthermore, suppose* $u \leq h$*, and* $u = h$ *close to* $\partial\Omega$*, and that* $-1 \leq v_j \leq 0$ *for* $1 \leq j \leq n$*. Then*

$$\int_\Omega (h-u)^n dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq n! \int_\Omega (-v_n)(dd^c u)^n.$$

LEMMA 2.4. *If* $\lambda \in \mathbb{R}$ *then* $\|\lambda u\| = |\lambda| \|u\|$.

PROOF. Let $\lambda \geq 0$. From the definition, we have

$$\begin{aligned}
\|u\|^n &= \inf_{u_1 - u_2 = u} \int_\Omega (dd^c(u_1+u_2))^n \\
&= \inf_{u_1 - u_2 = u} \int_\Omega \left(dd^c\left(\frac{\lambda}{\lambda}(u_1+u_2)\right)\right)^n \\
&= \inf_{u_1 - u_2 = u} \int_\Omega \lambda^{-n}(dd^c(\lambda u_1 + \lambda u_2))^n \\
&= \lambda^{-n} \inf_{\tilde{u}_1 - \tilde{u}_2 = \lambda u} \int_\Omega (dd^c(u_1+u_2))^n = \lambda^{-n}\|\lambda u\|^n.
\end{aligned}$$

Hence $\lambda\|u\| = \|\lambda u\|$.

If $\lambda < 0$ we have $\lambda u = -\lambda(-u)$, and the same line of reasoning as above applies.

LEMMA 2.5. *Suppose* Ω *is a hyperconvex domain in* $\mathbb{C}^n$ *and that* $u, v \in \mathcal{F}(\Omega)$*, then*

$$\int_\Omega (dd^c(u+v))^n \leq \left[\left(\int_\Omega (dd^c u)^n\right)^{\frac{1}{n}} + \left(\int_\Omega (dd^c v)^n\right)^{\frac{1}{n}}\right]^n$$

PROOF. Take $h \in \mathcal{E}_0$ and let us consider the left hand side in the inequality above.

$$\begin{aligned}
\int_\Omega -h(dd^c(u+v))^n &= \sum_{j=0}^n \binom{n}{j} \int_\Omega -h(dd^c u)^{n-j} \wedge (dd^c v)^j \\
&\leq \sum_{j=0}^n \binom{n}{j} \left(\int_\Omega -h(dd^c u)^n\right)^{\frac{n-j}{n}} \left(\int_\Omega -h(dd^c v)^n\right)^{\frac{j}{n}} \\
&= \left[\left(\int_\Omega -h(dd^c u)^n\right)^{\frac{1}{n}} + \left(\int_\Omega -h(dd^c v)^n\right)^{\frac{1}{n}}\right]^n
\end{aligned}$$

where the inequality comes from the "Hölder-inequality" in Lemma 2.2. Fix $w \in \Omega$, and take $h = \max(k \cdot g_\Omega, -1)$, where $g_\Omega(z, w)$ is the pluricomplex Green function with pole at w, then $h \in \mathcal{E}_0$ and $h \searrow -1$ on Ω and the Lemma follows.

Now we are in a position to prove the triangle-inequality for $\delta\mathcal{F}$.

COROLLARY 2.6. *Suppose* Ω *is a hyperconvex domain in* $\mathbb{C}^n$ *and that* $u, v \in \delta\mathcal{F}(\Omega)$, *then*

$$\|u + v\| \leq \|u\| + \|v\|. \tag{1}$$

PROOF. Take $\epsilon > 0$, then there is $u_i, v_i \in \mathcal{F}$ such that

$$\left(\int_\Omega (dd^c(u_1 + u_2))^n\right)^{1/n} < \|u\| + \epsilon$$

and

$$\left(\int_\Omega (dd^c(v_1 + v_2))^n\right)^{1/n} < \|v\| + \epsilon.$$

According to Lemma 2.5 we have

$$\begin{aligned}\|u\| + \|v\| - 2\epsilon &> \left(\int_\Omega (dd^c(u_1 + u_2))^n\right)^{1/n} + \left(\int_\Omega (dd^c(v_1 + v_2))^n\right)^{1/n} \\ &\geq \left[\int_\Omega (dd^c(u_1 + u_2 + v_1 + v_2))^n\right]^{1/n},\end{aligned}$$

and furthermore, since $u_1 + v_1 - (u_2 + v_2) = u - v$, $u_1 + v_1$ and $u_2 + v_2$ are two of the functions in the set we take infimum over we have

$$\left[\int_\Omega (dd^c(u_1 + u_2 + v_1 + v_2))^n\right]^{1/n} \geq \|u + v\|.$$

Hence $\|u + v\| \leq \|u\| + \|v\|$.

LEMMA 2.7. *If* $\|u\| = 0$, *then* $u = 0$.

PROOF. Take $\epsilon > 0$. Since

$$\|u\| = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{F}}} \left[\left(\int_\Omega (dd^c(u_1 + u_2))^n\right)^{\frac{1}{n}}\right].$$

there is $\tilde{u}_i \in \mathcal{F}$ such that $\int_\Omega (dd^c(\tilde{u}_1 + \tilde{u}_2))^n < \epsilon$.

Take a sequence $\{v_j\} \subset \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$, such that $v_j \searrow \tilde{u}_1 + \tilde{u}_2$ as $j \to \infty$. Let $t > 0$ and define $h_t = \max\{v_j, -t\}$. According to Błocki's inequality (Theorem 2.3) we have

$$n!\epsilon > n! \int_\Omega (dd^c v_j)^n > \int_\Omega (h_t - v_j)^n \, dV,$$

hence

$$n!\epsilon > \|h_t - v_j\|_{L^n} \operatorname{vol}(\Omega).$$

Letting $t \searrow 0$ we get

$$\frac{n!\epsilon}{\operatorname{vol}(\Omega)} > \|v_j\|_{L^n},$$

independent of j. Thus $\|u_1 + u_2\|_{L^n} < C\epsilon$, and letting $\epsilon \to 0$ we get $\|u\|_{L^n} = 0$, so $u = 0$, except for a set of measure zero, but since $u \in \delta\mathcal{F}$ we have $u \equiv 0$.

A remark on other energy classes

Since other type of energy-classes, for instance $\mathcal{E}_p(\Omega)$ also are convex cones we can form the linear spaces $\delta\mathcal{E}_p$. It is natural to try to generalize our norm to a norm for these spaces. Consider a hyperconvex domain $\Omega \subset \mathbb{C}^2$, and the energy class $\mathcal{E}_1(\Omega)$. Since $\int_\Omega (dd^c u)^2$ is not finite in general we have to replace it with $\int_\Omega -u(dd^c u)^2$. Thus the natural generalization of the norm would be to take $u \in \delta\mathcal{E}_1$, and set

$$q(u) = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_1}} \left\{ \left(\int_\Omega -(u_1 + u_2)\big(dd^c(u_1 + u_2)\big)^2 \right)^{\frac{1}{3}} \right\}.$$

Unfortunately q is not a norm, since it does not satisfy the triangle inequality. Using the energy estimate in [11], and repeating the calculations in Lemma 2.5 we only get $q(u + v) \leq e^{2/3}(q(u) + q(v))$.

3. On the Topology of $\delta\mathcal{F}$

THEOREM 3.1. *$(\delta\mathcal{F}, \|\cdot\|)$ is a Banach space.*

PROOF. Lemmata 2.4 and 2.7, and Corollary 2.6 shows that $(\delta\mathcal{F}, \|\cdot\|)$ is a normed vector space. It remains to show completeness.

Suppose (u_n) is a Cauchy sequence in $\delta\mathcal{F}$. For each integer k there is an integer n_k such that $\|u_n - u_m\| < 2^{-k}$ for $n, m > n_k$. We choose the n_k's such that $n_{k+1} > n_k$.

We have $u_{n_k} = u_{n_1} + (u_{n_2} - u_{n_1}) + \cdots + (u_{n_k} - u_{n_{(k-1)}})$. Since $u_{n_j} \in \delta\mathcal{F}$ for $j = 1, \ldots, k$ we can write $u_{n_j} - u_{n_{j-1}} = \phi_j^1 - \phi_j^2$, for $\phi_j^1, \phi_j^2 \in \mathcal{F}$, where the ϕ_j^1 and ϕ_j^2 are chosen such that

$$\|u_{n_j}-u_{n_{j-1}}\| = \inf\left(\int (dd^c(\varphi^1+\varphi^2))^n\right)^{1/n} \geq \left(\int (dd^c(\phi_j^1+\phi_j^2))^n\right)^{1/n} - 2^{-j-1}.$$

Then we have

$$\begin{aligned} u_{n_k} &= u_{n_1} + (\phi_2^1 - \phi_2^2) + \cdots + (\phi_k^1 - \phi_k^2) \\ &= u_{n_1} + (\phi_2^1 + \cdots + \phi_k^1) - (\phi_2^2 + \cdots + \phi_k^2) \end{aligned}$$

and since $\sum_{j=2}^k \phi_j^1 \in \mathcal{PSH}^-(\Omega)$ is a decreasing sequence and

$$\begin{aligned} &\left(\int \left(dd^c\left(\sum_{j=2}^k \phi_j^1\right)\right)^n\right)^{1/n} \\ &\quad\leq \left(\int \left(dd^c\left(\sum_{j=2}^k \phi_j^1 + \phi_j^2\right)\right)^n\right)^{1/n} \leq \sum_{j=2}^k \left(\int \left(dd^c\left(\phi_j^1 + \phi_j^2\right)\right)^n\right)^{1/n} \\ &\quad\leq \sum_{j=2}^k \left(\|u_{n_j} - u_{n_{j-1}}\| + 2^{-j-1}\right)^{1/n} = \sum_{j=2}^k \left(2^{-j} + 2^{-j-1}\right)^{1/n} < \frac{1}{\sqrt[n]{2}-1}. \end{aligned}$$

Thus $\sum_{j=2}^k \phi_j^1$ is an decreasing sequence of plurisubharmonic functions with bounded total mass, and in the same way $\sum_{j=2}^k \phi_j^2$ is. Therefore u_{n_k} is convergent to some $u \in \delta\mathcal{F}$, and since (u_n) is a Cauchy sequence $u_n \to u$.

LEMMA 3.2. *$\mathcal{F}$ is closed in the topology of $\delta\mathcal{F}$.*

PROOF. Take any Cauchy-sequence (u_m) in $\mathcal{F}$. Choose a suitable sparse subsequence (u'_m), then $u_p = u_0 + u_1 - u_0 + \cdots + u_p - u_{p-1}$, and by the exact same reasoning as in the proof of completeness for $\delta\mathcal{F}$, we get that $u_p \to u \in \mathcal{F}$.

PROPOSITION 3.3. *The continuous functions are not dense in $\delta\mathcal{F}$. Furthermore $\delta\mathcal{F}$ is* not *separable.*

PROOF. Let us denote the Lelong number of u at x with $\nu(u, x)$. The Lelong number at the origin is of course a linear functional on all of $\delta\mathcal{F}$, furthermore $\nu(\cdot, 0)$ is a continuos linear functional on $\delta\mathcal{F}$, by Theorem 4.3 or directly by the estimate:

$$(2\pi\nu(u, x))^n \leq (dd^c u)^n(\{x\}),$$

for functions $u \in \mathcal{F}$ (see e.g. [10]).

For all functions $u \in \mathcal{PSH} \cap \mathcal{C}$ we have $\nu(u, 0) = 0$, thus $\log |z|$ can not be approximated by continuos functions in our topology.

For the second statement of the proposition, let us assume that $\delta\mathcal{F}$ is indeed separable. Let $\{u_i\}$ be a dense subset of $\delta\mathcal{F}$. It is well known that the set where the Lelong number is positive for a given function u, is of Lebesgue measure zero. Thus the union of the sets where the Lelong number is positive for functions from $\{u_i\}$ is also of Lebesgue-measure zero. Take any point x not in this union, i.e. $\nu(u_i, x) = 0$ for all u_i's, and then we see that $v(z) = \int_\Omega \log |z| \, \delta_x$ cannot be approximated from functions in $\{u_i\}$.

A vector space L over R with an order structure defined by a binary relation "$\leq$" being reflexive, transitive and anti-symmetric is called an *ordered vector space* over R if the relation satisfies:

(1) translation-invariance, $x \leq y \Longrightarrow x + z \leq y + z$ for all $x, y, z \in L$

(2) $x \leq y \Longrightarrow \lambda x \leq \lambda y$ for all $x, y \in L$ and $\lambda > 0$.

Clearly every vector space of real-valued functions f on a parameter set X is an ordered vector space under the natural order $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

If L is a topological vector space, and an ordered vector space, we say that if is an *ordered topological vector space* if the positive cone $C = \{x \mid x \geq 0\}$ is closed on L. In particular $\delta\mathcal{F}$ is an ordered topological vector space since $\{u \in \delta\mathcal{F} \mid u \geq 0\}$ is closed on the topology of $\delta\mathcal{F}$.

A comprehensive treatise of ordered topological vector spaces is found in the book of Schaefer and Wolff [18].

It is natural to ask wether $\delta\mathcal{F}$ has even more ordered structure.

Remember that a *vector lattice* is an ordered vector space L over R such that $\sup(x, y)$ and $\inf(x, y)$ exist for every pair $(x, y) \in L \times L$. For a vector lattice L set $|x| = \sup(x, -x)$. Of course $\delta\mathcal{F}$ is a vector lattice since $\sup(u, v) = \max(u, v)$ exist and the same for infimum.

Given a topological vector space L over R, with a vector lattice structure, a set $X \subset L$ is called solid if $x \in X$ and $|x| \leq |y|$ imply that $y \in X$.

We call L *locally solid* if it has a 0-neighbourhood base of solid sets, i.e. the norm is compatible with the lattice structure.

Unfortunately $\delta\mathcal{F}$ is not locally solid. It suffices to show that the the unit ball $\mathsf{B} \subset L$ is not *solid*, (see e.g. [18] or [17]), and this is showed in the example below.

EXAMPLE 3.4. Consider the function $f(\zeta) = \max(\log |\zeta|, -1) - \max(\log |\zeta|, -1/2)$ in the unit-disc D in C^1. We have $\| \log |\zeta| \| = \pi$, and $|f| \leq |\log |\zeta||$.

Since $\max(\log|\zeta|, -1) = p_\mu$, and $\max(\log|\zeta|, -1/2) = p_\nu$, where μ and ν are the Lebesgue measure on the circles $\{|\zeta| = e^{-1}\}$ and $\{|\zeta| = e^{-1/2}\}$, and therefore have disjunct support we can calculate that $\|f\| = \pi + \pi$. Thus $\delta\mathcal{F}(\mathsf{D})$ is not locally solid. In particular: $\delta\mathcal{F}$ is not a so-called Banach lattice. (A Banach lattice is a locally solid Banach space.)

4. The dual space

Let us denote the topological dual of $\delta\mathcal{F}$ by $(\delta\mathcal{F})'$.

It is natural to ask which elements of the dual can be given by Borel measures.

THEOREM 4.1. *Take $\psi \in \mathcal{F}$. Suppose $\Psi \in (\delta\mathcal{F})'$ is given by*

$$\Psi(u) = \int dd^c u \wedge (dd^c\psi)^{n-1},$$

then $\|\Psi\| = \|\psi\|^{n-1}$, and if $\psi \neq 0$ there is no Borel measure on Ω such that $\Psi(u) = \int u\, d\mu$.

PROOF. Let $u \in \mathcal{F}$. According to Lemma 2.2 we have

$$\Psi(u) = \int dd^c u \wedge (dd^c\psi)^{n-1} \leq \left(\int (dd^c u)^n\right)^{\frac{1}{n}} \left(\int (dd^c\psi)^n\right)^{\frac{n-1}{n}}.$$

Thus $\Psi(u) \leq \|u\| \cdot \|\psi\|^{n-1}$. Take $f \in \delta\mathcal{F}$ and choose any $u, v \in \mathcal{F}$ such that $f = u - v$, then

$$\begin{aligned}|\Psi(f)| &= |\Psi(u - v)| \leq |\Psi(u)| + |\Psi(v)| = \Psi(u) + \Psi(v)\\ &= \Psi(u + v) \leq \left(\int (dd^c(u + v))^n\right)^{\frac{1}{n}} \cdot \|\psi\|^{n-1},\end{aligned}$$

and we get that

$$|\Psi(f)| \leq \inf_{u-v=f;\, u,v\in\mathcal{F}} \left(\int (dd^c(u + v))^n\right)^{1/n} \cdot \|\psi\|^{n-1} = \|f\| \cdot \|\psi\|^{n-1}$$

On the other hand, take $u = \|\psi\|^{-1}\psi$. Then $\|u\| = 1$ and

$$\Psi(u) = \int dd^c(\|\psi\|^{-1}\psi) \wedge (dd^c\psi)^{n-1} = \|\psi\|^{n-1}.$$

Thus

$$\|\Psi\| = \sup_{\|f\|=1} |\Psi(f)| = \|\psi\|^{n-1}.$$

To see that Ψ is not given by a Borel measure, take $u, v \in \mathcal{F}$ such that $u = v$ near $\partial\Omega$. Then

$$\int_\Omega dd^c u \wedge (dd^c \psi)^{n-1} = \int_\Omega dd^c v \wedge (dd^c \psi)^{n-1}, \tag{2}$$

by "Stokes' theorem", and if $\Psi(u) = \int_\Omega u\, d\mu$ then $\int_\Omega (v - u)\, d\mu = 0$. Since $\mathcal{C}_0^\infty \subset \delta\mathcal{E}_0$ (see Lemma 3.1, [10]) it follows that $d\mu$ has its support on the boundary of Ω. But then $\Psi(u) = 0$ for all $u \in \mathcal{E}_0$. Take a sequence $\{u_j\} \subset \mathcal{E}_0$ such that $u_j \searrow \psi$, and by continuity we get $\int_\Omega (dd^c \psi)^n = 0$, thus $\psi = 0$.

EXAMPLE 4.2. Suppose $q > 1$. Let $g \in L^q(\Omega)$. For any $u \in \mathcal{F}(\Omega)$, define $T(u) = \int ug\, dV$, then $T \in (\delta\mathcal{F})'$.

PROOF. From [12] we have for every $u \in \mathcal{F}$ with $\int (dd^c u)^n \leq 1$ there is a constant A, depending only on Ω such that $\int e^{-u}\, dV \leq A$. Thus $u \in L^p, \forall p$.

THEOREM 4.3. *If T is a linear functional on $\delta\mathcal{F}$ such that $T(x) \geq 0$, for all $x \in \mathcal{F}$, then T is continuous.*

PROOF. Take a bounded sequence $\{f_k\} \subset \delta\mathcal{F}$, such that $\|f_k\| < M$. By construction there is $x_k, y_k \in \mathcal{F}$ such that $f_k = x_k - y_k$, and $\|x_k + y_k\| < M+1$. We have $\|x_k\| = \|f_k + y_k\| \leq \|f_k\| + \|y_k\| \leq M + \|y_k\| \leq M + \|x_k + y_k\| \leq 2M + 1$, where the second to last inequality follows from that $y_k \geq x_k + y_k$, thus $\int (dd^c y_k)^n \leq \int (dd^c (x_k + y_k))^n$,

If T is bounded on all bounded sequences $\{x_k\} \subset \mathcal{F}$ then $|T(f_k)| = |T(x_k) - T(y_k)| \leq |T(x_k)| + |T(y_k)|$, and $T(f_k)$ would be bounded as well.

Suppose T is *not* continuous. Then there has to be a bounded sequence $\{f_k\} \subset \delta\mathcal{F}$ such that $\{T(f_k)\}$ is not bounded. Thus there has to be a bounded sequence $\{x_k\}$ in $\mathcal{F}$ such that $T(x_k) > k > 0$.

Now define $\phi = \sum_{k=1}^\infty k^{-2} x_k$. Since $\mathcal{F}$ is a convex cone and $\{x_k\}$ is bounded $\phi \in \mathcal{F}$. Note that $T(\phi) = T\big(\sum_1^p x_k\big) + T\big(\sum_{p+1}^\infty x_k\big) \geq T\big(\sum_1^p x_k\big)$, since $T \geq 0$ on $\mathcal{F}$. But then $T(\phi) \geq \sum_1^p k^{-1}$, for all positive numbers p, i.e. $T = +\infty$, and we have a contradiction.

Let us recall the notion of dual cones.

DEFINITION 4.4. If C is a cone in the topological vector space L, the *dual cone* C' of C is defined to be the set

$$C' = \{T \in L \mid T(u) \geq 0 \text{ if } u \in C\}.$$

THEOREM 4.5. $(\delta\mathcal{F})' = \mathcal{F}' - \mathcal{F}' = \delta\mathcal{F}'$.

PROOF. This follows more or less immediately from [16], (see also Lemma 1 p. 218 [18]), since one can show that $\mathcal{F}$ is a so called normal cone, but to avoid giving the rather abstract definitions of normal cones, we give a self contained proof.

Take $T \in (\delta\mathscr{F})'$ and define $p : \mathscr{F} \to \mathsf{R}_+$ by $p(u) := \sup\{T(v) \mid u \leq v \leq 0\}$. By the linearity of T, $p(\lambda u) = \lambda p(u)$, for $\lambda \geq 0$, and since $\{\phi \mid u + v \leq \phi \leq 0\} \supset \{\phi \mid u \leq \phi \leq 0\} + \{\phi \mid v \leq \phi \leq 0\}$, $p(u + v) \geq p(u) + p(v)$ also. Thus the set $V = \{(t, u) \mid 0 \leq t \leq p(u)\} \subset \mathsf{R} \times \delta\mathscr{F}$ is a convex cone.

Clearly $\mathsf{R} \times \delta F$ is a normable space. Take a sequence $\{u_k\} \subset \mathscr{F}$ such that $\|u_k\| \to 0$, as $k \to \infty$. If $\varphi \in \mathscr{F}$ and $u_k \leq \varphi \leq 0$ then $\int (dd^c\varphi)^n \leq \int (dd^c u_k)^n$, and hence $\|\varphi\| \leq \|u_k\|$, thus $p(u_k) \to 0$, as $k \to \infty$ by the continuity of T. We conclude that $(1, 0) \notin \bar{V}$.

Since $\delta\mathscr{F}$ is locally convex there is a closed real hyperplane $H = \{t, u) \mid h(t, x) = -1\}$, separating $\bar{V}$ and $(1, 0)$ where we can choose h such that $h \geq 0$ on V and $h(1, 0) = -1$. Since $(\mathsf{R} \times \delta\mathscr{F})'$ is algebraically isomorphic with $(\mathsf{R} \oplus \delta\mathscr{F})'$, (see Theorem 4.3 p. 137, [18]) we have $h(t, u) = \alpha t + g(u)$. Now $h(1, 0) = \alpha = -1$.

Since $(0, u) \in V$, for all $u \in \mathscr{F}$, and $g \in (\delta\mathscr{F})'$, we have $g(u) \geq 0$ on $\mathscr{F}$ according to our choice of H. V was chosen such that $(p(u), u) \in V$, hence $h(p(u), u) = -p(u) + g(u) \geq 0$, and we get $T(u) \leq p(u) \leq g(u)$. To sum up: $T = g - (g - T)$, where $g - T \geq 0$. Note that by Theorem 4.3, linear operators that are positive on $\mathscr{F}$ are continuous.

We can extend the definition of the Monge-Ampère operator to the whole of $\delta\mathscr{F}$. Suppose $u \in \delta\mathscr{F}$, then $u = u_1 - u_2$, for some $u_1, u_2 \in \mathscr{F}$, and we can define $(dd^c u)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (dd^c u_1)^{n-j} \wedge (dd^c u_2)^j$. To see that this definition is independent of the choice of the functions from $\mathscr{F}$, suppose $u = u_1 - u_2 = v_1 - v_2$, and that $h \in \mathscr{E}_0$. Then

$$\begin{aligned}
&\int h\, dd^c(u_1 - u_2) \wedge \cdots \wedge dd^c(u_1 - u_2) \\
&\quad = \int (u_1 - u_2)\, dd^c h \wedge dd^c(u_1 - u_2) \wedge \cdots \wedge dd^c(u_1 - u_2) \\
&\quad = \int (v_1 - v_2)\, dd^c h \wedge dd^c(u_1 - u_2) \wedge \cdots \wedge dd^c(u_1 - u_2) \\
&\quad = \int h\, dd^c(v_1 - v_2) \wedge dd^c(u_1 - u_2) \wedge \cdots \wedge dd^c(u_1 - u_2),
\end{aligned}$$

and continuing iteratively we have

$$\int h\, dd^c(u_1 - u_2) \wedge \cdots \wedge dd^c(u_1 - u_2) = \int h\, dd^c(v_1 - v_2) \wedge \cdots \wedge dd^c(v_1 - v_2).$$

COROLLARY 4.6. *The following functionals are all continuous on* $\delta\mathcal{F}$*:*

- *The total mass of the Monge-Ampère measure.*
- *Demailly's generalized Lelong numbers* $\nu(dd^c u, \varphi)$ *for the current* $dd^c u$ *with weight* φ.

For a definition of $\nu(T, \varphi)$—the Lelong number of the current T with weight φ see [14].

5. Comparison with delta-subharmonic functions

Let us turn our attention to the class of delta-subharmonic functions in domains in $\mathbb{C}^n$.

If we have a generating family of seminorms on a Fréchet space X and if K is a closed convex cone in X we can turn K into a Fréchet space with topology defined by the seminorms

$$\|f\|_j = \inf\{|g|_j + |h|_j \;;\; f = g - h, \text{ for } g \text{ and } h \text{ in } K\}, \qquad j \in \mathbb{N},$$

where $|\cdot|_j$ are a generating family of seminorms on X.

DEFINITION 5.1. *The set* δm. Let $m(\Omega)$ be the set of positive measures that can be written as $\mu = \Delta\varphi$, for some $\varphi \in \mathcal{PSH}(\Omega)$. We denote the space of differences from this cone by $\delta m(\Omega)$ as usual.

Since any open domain $\Omega \subset \mathbb{C}^n$ is para-compact it suffices to define a seminorm for any compact $K \Subset \Omega$ and generate the topology from these seminorms.

Using the topology on $\delta\mathcal{PSH}$, the delta-plurisubharmonic functions, defined in the introduction we have a continuity property of the Laplace operator.

THEOREM 5.2. *Assume that* Ω *is pseudoconvex then* $\delta m(\Omega)$ *is a Fréchet space with seminorms defined by*

$$\|\mu\|_K = \inf\left(\int_K \mu_1 + \mu_2 \mid \mu = \mu_1 - \mu_2, \mu_1, \mu_2 \in m(\Omega)\right), \qquad K \Subset \Omega.$$

Furthermore the Laplace operator $\Delta : \delta\mathcal{PSH}(\Omega) \mapsto \delta m(\Omega)$ *is continuos.*

PROOF. Cf. [7]. (By assuming that Ω is pseudoconvex we don't have to deal with some homotopy intricacies.)

DEFINITION 5.3. *The set* δM. Let us denote the set of all positive real Borel measures on Ω by $M(\Omega)$, and the signed real Borel measures as $\delta M(\Omega)$. Then the *total variation* of a measure $\mu \in \delta M(\Omega)$ is by Jordan's decomposition theorem given as

$$|\mu| = \inf\left(\int_\Omega \mu_1 + \mu_2 \mid \mu = \mu_1 - \mu_2, \mu_1, \mu_2 \in M(\Omega)\right).$$

We will view $\delta M(\Omega)$ as a Banach space with norm defined by the equation above.

Let Δ denote the Laplacian as a map from $\delta\mathcal{F}$ to δM. Clearly Δ is a linear map. Continuity of the map, however, turns out to be more subtle.

THEOREM 5.4. *Suppose Ω is a strict pseudoconvex domain with $\mathcal{C}^\infty$-smooth boundary, then the map $\Delta : \delta\mathcal{F} \to \delta M$ is continuous.*

PROOF. According to [6] the solution $\varphi \in \mathcal{PSH}(\Omega)$ to the Dirichlet problem:

$$\begin{cases} (dd^c\varphi)^n = 1 & \text{on } \Omega \\ \varphi = -\|z\|^2 & \text{on } \partial\Omega \end{cases}$$

satisfy $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$. Thus it follows that $\|z\|^2 + \varphi \in \mathcal{E}_0(\Omega)$.

Direct calculation gives that

$$dd^c u \wedge (dd^c\|z\|^2)^{n-1} = 4^{n-1}(n-1)!\Delta u.$$

Thus we have that

$$\begin{aligned} 4^{n-1}(n-1)!\int_\Omega \Delta u &= \int_\Omega dd^c u \wedge (dd^c\|z\|^2)^{n-1} \\ &\le \int_\Omega dd^c u \wedge (dd^c(\|z\|^2+\varphi))^{n-1} \\ &\le \left(\int_\Omega (dd^c u)^n\right)^{1/n} \left(\int_\Omega (dd^c(\|z\|^2+\varphi))^n\right)^{(n-1)/n} \\ &\le C\cdot\left(\int_\Omega (dd^c u)^n\right)^{1/n} \end{aligned}$$

for some positive constant C. The second inequality above follows from Lemma 2.2.

Take $u \in \delta\mathcal{F}$ and any $\epsilon > 0$, then there is a choice of u_1, u_2 such that $u = u_1 - u_2$ where $\int (dd^c(u_1+u_2))^n < \|u\|^n + \epsilon$. According to the calculation above we have

$$\int_\Omega \Delta u_1 + \Delta u_2 = \int_\Omega \Delta(u_1+u_2) \le C'\cdot\left(\int_\Omega (dd^c(u_1+u_2))^n\right)^{1/n} < C'\|u\| + \epsilon$$

for some constant C', not depending on ϵ. Let $\epsilon \to 0$ and the theorem follows.

Unfortunately, continuity of Δ does not hold in general, in particular not where the boundary of the domain is “flat”, as can be seen from the following example.

EXAMPLE 5.5. Let $u_k = \max(k \log |z_1|, (1/k) \log |z_2|)$. Then there is a constant c, not depending on k, such that

$$\int_{\mathsf{D}^2} \Delta u_k \geq c \cdot k,$$

but

$$\int_{\mathsf{D}^2} (dd^c(u_k))^2 = (2\pi)^{-2}.$$

PROOF. Take $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathsf{D})$, where D is the unit disc. Then

$$\begin{aligned}
\int_{\mathsf{D}^2} \chi_1 \chi_2 \Delta u_k &= \int_{\mathsf{D}^2} u_k(z_1, z_2) \Delta(\chi_1(z_1)\chi_2(z_2)) \\
&= \int_{\mathsf{D}^2} u_k(z_1, z_2)\big(\chi_2(z_2)\Delta_1\chi_1(z_1) + \chi_1(z_1)\Delta_2\chi_2(z_2)\big) \\
&= \int_{\mathsf{D}} \chi_2 \int_{\mathsf{D}} u_k \Delta_1 \chi_1 + \int_{\mathsf{D}} \chi_1 \int_{\mathsf{D}} u_k \Delta_2 \chi_2 \geq \int_{\mathsf{D}} \chi_2 \int_{\mathsf{D}} u_k \Delta_1 \chi_1 \\
&= \int_{\mathsf{D}} \chi_2 \int_{\mathsf{D}} \chi_1 \Delta_1 u_k.
\end{aligned}$$

Take χ_2 such that $\chi_2 \equiv 1$ on $\mathsf{D}(1/2)$. For z_2 fixed with $|z_2| < 1/2$ we know that $\Delta_1 \max(k \log |z_1|, k^{-1} \log |z_2|)$ is k times the (normalized) Lebesgue measure on the circle $\{z_1 \in \mathsf{C}\, ;\ |z_1|^{k^2} = |z_2|\}$. Choose χ_1, depending on k, such that $\chi_1 \equiv 1$ at least where $|z_1| \leq \left(\frac{1}{2}\right)^{1/k^2}$. After making all these choices we have

$$\int_{\mathsf{D}} \chi_2 \int_{\mathsf{D}} \chi_1 \Delta_1 u_k = \int_{\mathsf{D}} \chi_2 \frac{k}{2\pi}\, dz_2 \wedge d\bar{z}_2 > c \cdot k,$$

for some constant c, independent of k.

It is well known that $\int_{\mathsf{D}^2} (dd^c(u_k))^2 = (2\pi)^{-2}$.

REMARK 5.6. Let Ω be a hyperconvex domain and take a sequence $\{u_k\}$ in $\mathscr{F}(\Omega)$ such that $\int \Delta u_k$ diverges. Exhaust Ω with smooth, strict pseudoconvex domains from inside, then Theorem 5.4 implies that the Laplace mass of the u_k has to be pushed out towards the boundary.

Let $U(0, f)$ be the *Perron-Bremermann function* of f, i.e. the largest locally bounded plurisubharmonic function that has boundary values at most f. (See e.g. [9])

In his Doctoral Thesis, Åhag [1] generalized the notion $\mathscr{F}_p(f)$ of energy classes with "boundary data" f, from [9], and introduced $\mathscr{F}(f, \Omega)$. Assume that

$$\lim_{\Omega \ni z \to \zeta} U(0, f) = f(\zeta)$$

for every $\zeta \in \partial\Omega$, then we define the $\mathcal{F}(f, \Omega)$ to be set of plurisubharmonic functions on Ω such that there is a $\varphi \in \mathcal{F}$ such that $U(0, f) \geq u \geq \varphi + U(0, f)$.

EXAMPLE 5.7. Let

$$u(z) = \sum_{k=1}^{\infty} \max(\log|z_1|, k^{-4}\log|z_2|).$$

Then $u \in \mathcal{F}(\mathsf{D}^2)$, but $\int_{\mathsf{D}^2} \Delta u = +\infty$. Furthermore take $f = |z_2|^2 - 1$, then $f \in \mathscr{C}^\infty(\overline{\mathsf{D}}^2)$ and $(dd^c f)^2 = 0$ but $(dd^c(u+f))^2$ is not bounded on D^2.

PROOF. Let $u_k = \max(\log|z_1|, k^{-4}\log|z_2|)$, then $\int_{\mathsf{D}^2}(dd^c(u_k))^2 = (2\pi k^2)^{-2}$.

By Lemma 2.5 we get

$$\int_{\mathsf{D}^2}\left(dd^c\left(\sum_{k=1}^{N} u_k\right)\right)^2 \leq \left(\sum_{k=1}^{N}\left(\int (dd^c u_k)^2\right)^{1/2}\right)^2 = \left(\sum_{k=1}^{N}\frac{1}{2\pi k^2}\right)^2 \leq \frac{\pi^2}{144},$$

thus $u \in \mathcal{F}$, and $u + f \in \mathcal{F}(f)$. But we have

$$\int dd^c u_k \wedge dd^c(|z_2|^2 - 1) = \int dd^c u_k \wedge (2i\, dz_2 \wedge d\bar{z}_2) = 16\int \Delta_1 u_k > c,$$

where the constant c is independent of k, by the inequality in Example 5.5 above. Thus

$$\int\left(dd^c\left(f + \sum_{k=1}^{N} u_k\right)\right)^2 \geq 2\int\sum_{k=1}^{N} dd^c u_k \wedge dd^c(|z_2|^2 - 1) \geq N,$$

and we get that the total mass of $u + f$ diverges.

To ensure that if $u \in \mathcal{F}(f)$ we have $\int_\Omega (dd^c u)^n < +\infty$ Åhag introduced the concept of *compliant functions* f. A continuous function $f : \partial\Omega \to \mathsf{R}$ is said to be compliant if the Perron-Bremermann function $U(0, f)$ satisfies $U(0, f) = f$ on the boundary and $\int (dd^c(U(0, f) + U(0, -f)))^n < +\infty$.

Åhag proved, using the smoothness result for the Monge-Ampère operator of Caffarelli-Kohn-Nirenberg-Spruck [6], that under the assumption that Ω is strict pseudoconvex and smooth, any smooth boundary function is compliant.

In relation to this Åhag [2] has posed the following problem:

PROBLEM. Suppose Ω is hyperconvex, $f \in \mathscr{C}^\infty$, and $f = U(0, f)$. If $u \in \mathcal{F}(\Omega)$, is $\int_\Omega (dd^c(u + f))^n < +\infty$?

According to Example 5.7 above the answer to this problem is no, not always. To see this it simply suffices to take f and u as in the example.

Since the dual of the space δM is well understood it would be nice to pull back $(\delta M)'$ to $(\delta F)'$. At the moment this does not seems feasible considering the example below.

EXAMPLE 5.8. Let B be the unit ball in C^2. The inverse Laplace-operator $\Delta^{-1} : \delta M(\mathsf{B}) \mapsto \delta\mathcal{F}(\mathsf{B})$ is *not* continuous. Let $u(z_1, z_2) = -(1 - |z_1|^2)^{1/2} + |z_2|$. Then $u \in \mathcal{PSH} \cap \mathcal{C}(\mathsf{B})$, and $u = 0$ on the boundary of the ball. Away from the z_1-axis we have that

$$4\,\partial\bar{\partial}u = \left(\frac{|z_1|^2}{(1-|z_1|^2)^{3/2}} + \frac{2}{(1-|z_1|^2)^{1/2}}\right)dz_1 \wedge d\bar{z}_1 + \frac{1}{|z_2|}dz_2 \wedge d\bar{z}_2.$$

Thus setting $r = |z_1|$ and $\rho = |z_2|$ we calculate

$$\begin{aligned}\int_{\mathsf{B}} \Delta u\, dV &= 4(2\pi)^2 \int_0^1 \int_0^{\sqrt{1-r^2}} \left(\frac{r^2}{(1-r^2)^{3/2}} + \frac{2}{(1-r^2)^{1/2}} + \frac{1}{\rho}\right) r\rho\, d\rho\, dr \\ &= 4\pi^2.\end{aligned}$$

In [5] Błocki pointed out that even though $(dd^c(-(1-|z_1|^2)^{1/2}))^2 = (dd^c|z_2|)^2 = 0$ we still have that $\int_{\mathsf{B}}(dd^c u)^n\, dV = +\infty$, since for any real number $0 < a < 1$, we have

$$\begin{aligned}\int_{\mathsf{B}} (dd^c u)^n\, dV &\geq \frac{1}{16}(2\pi)^2 \int_0^a \int_0^{\sqrt{1-r^2}} \frac{2-r^2}{(1-r^2)^{3/2}} r\, d\rho\, dr \\ &= \frac{\pi^2}{4}\int_0^a \frac{2r-r^3}{1-r^2}\, dr = \frac{\pi^2}{8}(a^2 - \log(1-a^2))\end{aligned}$$

which of course diverges as a tends to one.

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