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SELF-IMPROVING PROPERTIES OF GENERALIZED POINCARÉ TYPE INEQUALITIES THROUGH REARRANGEMENTS

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Abstract

We prove, within the context of spaces of homogeneous type, L^p and exponential type self-improving properties for measurable functions satisfying the following Poincaré type inequality:

$$\inf_\alpha \big((f-\alpha)\chi_B\big)^*_\mu\big(\lambda\mu(B)\big) \le c_\lambda a(B).$$

Here, f^*_μ denotes the non-increasing rearrangement of f, and a is a functional acting on balls B, satisfying appropriate geometric conditions.

Our main result improves the work in [11], [12] as well as [2], [3] and [14]. Our method avoids completely the "good-λ" inequality technique and any kind of representation formula.

1. Introduction

The main purpose of this paper is both to improve the main result in [11], [12] and to provide a proof that avoids the use the classical good-λ inequality of Burkholder and Gundy (cf. [17]).

Let (X, d, μ) be a space of homogeneous type, and $a : \mathscr{B} \to [0, \infty)$ be a functional defined on the family $\mathscr{B}$ of all balls in X. Recall that d denotes a quasi-metric on X and that μ is a measure that is doubling with respect to d.

In [11], it is proved that if the functional a satisfies the *weighted D_r condition* defined below and if f is any given locally integrable function satisfying

$$\frac{1}{\mu(B)}\int_B |f - f_B|\,d\mu \le a(B) \tag{1}$$

for all balls B, then one can deduce higher L^p-integrability of f.

Definition 1.1. Let $0 < r < \infty$ and w be a weight. We say that a satisfies the *weighted D_r condition* if there exists a finite constant c such that for each

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ball B and any family $\{B_i\}$ of pairwise disjoint sub-balls of B,

$$\sum_i a(B_i)^r w(B_i) \leq c^r a(B)^r w(B). \tag{2}$$

In the main results of this paper we will be considering weights that belong to the $A_\infty(\mu)$ class of Muckenhoupt, see Section 3 below. The model example is given by the fractional average

$$a(B) = r(B)^\alpha \left(\frac{\nu(B)}{\mu(B)}\right)^{1/p} \tag{3}$$

where $\alpha \geq 0$, $p > 0$ and ν is a nonnegative measure. If D denotes the doubling order of μ (cf. (7)) and $\alpha p < D$, this functional satisfies the D_r condition with $r = Dp/(D - \alpha p)$.

The precise L^p self-improving phenomenon is the following result.

THEOREM 1.2 ([11]). *Let $\delta > 0$ and suppose that the functional a satisfies the weighted D_r condition* (2) *for some $0 < r < \infty$ and some $w \in A_\infty(\mu)$. Suppose that f is a locally integrable function satisfying* (1)*, then there exists a constant c such that for any ball $B \subset \mathsf{X}$*

$$\|f - f_B\|_{L^{r,\infty}(B,w)} \leq ca(\widehat{B}),$$

where $\widehat{B} = (1 + \delta)\kappa B$.

Here we denote by f_B the integral μ-average of f over B and by $\|\cdot\|_{L^{r,\infty}(B,w)}$ the normalized weak L^r norm. Also κ denotes the quasi-metric constant of d. Observe that the interesting δ's are those that are small, in fact we will assume that $0 < \delta < 1$.

Theorem 1.2 yields a different proof of the classical Poincaré-Sobolev inequalities which avoids completely the use of any representation formula. See [2], [11] for details and also [4] for a more complete background and for the references therein.

A natural question arises: Is it possible to relax the L^1 norm in (1) to derive such self-improving property? In this paper, we study this issue and weaken the hypothesis to obtain such self-improving property. More precisely, we will show that the L^1 norm can be replaced by any L^q quasi-norm with $0 < q < 1$. Even further, we will show that the left hand side of (1) can be replaced by a weaker expression which is defined in terms of non-increasing rearrangements.

Our main result is the following.

THEOREM 1.3. *Let $\delta > 0$ and suppose that the functional a satisfies the weighted D_r condition* (2) *for some $0 < r < \infty$ and some $w \in A_\infty(\mu)$. Suppose that f is a measurable function such that for any ball $B \subset \mathsf{X}$*

$$\inf_\alpha \big((f-\alpha)\chi_B\big)^*_\mu \big(\lambda\mu(B)\big) \le c_\lambda a(B), \qquad 0 < \lambda < 1. \tag{4}$$

then there exists a constant c such that for any ball $B \subset \mathsf{X}$

$$\|f - f_B\|_{L^{r,\infty}(B,w)} \le c a(\widehat{B}), \tag{5}$$

where $\widehat{B} = (1+\delta)\kappa B$.

Recall that the non-increasing rearrangement f^*_μ of a measurable function f is defined by

$$f^*_\mu(t) = \inf\big\{\lambda > 0 : \mu_f(\lambda) < t\big\}, \qquad 0 < t \le \mu(\mathsf{X}),$$

where $\mu_f(\lambda) = \mu\big\{x \in \mathsf{X} : |f(x)| > \lambda\big\}$, $\lambda > 0$, is the distribution function of f.

Chebyshev's inequality yields for any $\delta > 0$,

$$(f\chi_B)^*_\mu(\lambda\mu(B)) \le \left(\frac{1}{\lambda\mu(B)} \int_B |f|^\delta \, d\mu\right)^{1/\delta}.$$

Hence we have the following.

COROLLARY 1.4. *Let f, a and ω as in Theorem* 1.3. *Suppose that for some $\delta < 1$ and for any ball $B \subset \mathsf{X}$*

$$\inf_\alpha \left(\frac{1}{\mu(B)} \int_B |f-\alpha|^\delta d\mu\right)^{1/\delta} \le a(B)$$

Then there is a geometric constant c such that for any ball $B \subset \mathsf{X}$

$$\|f - f_B\|_{L^{r,\infty}(B,w)} \le c a(\widehat{B}).$$

A similar result holds if the L^δ quasi-norm is replaced the by the quasi-norm $\|f\|_{L^{1,\infty}}$ since by a well known real analysis result

$$\left(\frac{1}{\mu(B)} \int_B |f|^\delta \, d\mu\right)^{1/\delta} \le c_\delta \, \|f\|_{L^{1,\infty}(B,\mu)}.$$

Our proof of Theorem 1.3 is essentially based on the following two ingredients. The first one is a relation between rearrangements and oscillations of a function f. This technique goes back to [1], and it was further developed in [8], [9]. We also use some ideas from the works [6], [15]. The second ingredient is an appropriate covering lemma of Calderón-Zygmund type. In the context of the standard Euclidean space R^n with doubling measure such covering lemmas can be obtained simply by application of the usual Calderón-Zygmund lemma to characteristic functions (see, e.g., [1], [8]. In the case of R^n with non-doubling measure a covering lemma of such type has been recently proved in [13]. Our covering lemma, in the context of spaces of homogeneous type, is presented in Section 3 below.

The main result of this paper can be applied to improve the results of [12], [14]. In [12], an exponential self-improving property was established assuming (1) with a functional a satisfying the so-called T_p condition, stronger than the D_r condition. In Section 5, we will improve this result by relaxing the initial assumption (1) as in Theorem 1.3. In [14], a non-homogeneous variant of Theorem 1.2 was proved in the context of R^n with any (non-doubling) measure. We will show that this result can be also proved under similar relaxed assumptions.

We also obtain an analogue of the main results of [8], [9] concerning the so-called local sharp maximal function, in the context of spaces of homogeneous type.

2. Preliminaries

2.1. Space of Homogeneous type

A quasimetric d on a set X is a function $d : \mathsf{X} \times \mathsf{X} \to [0, \infty)$ which satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all x, y;

(iii) there exists a finite constant $\kappa \geq 1$ such that

$$d(x, y) \leq \kappa(d(x, z) + d(z, y))$$

for all $x, y, z \in \mathsf{X}$.

Given $x \in \mathsf{X}$ and $r > 0$, let $B(x, r) = \{y \in \mathsf{X} : d(x, y) < r\}$ be the ball with center x and radius r. If $B = B(x, r)$ is a ball, we denote its radius r by $r(B)$ and its center x by x_B. If ν is a measure and E is a measurable set, $\nu(E)$ denotes the ν-measure of E.

DEFINITION 2.1. A space of homogeneous type (X, d, μ) is a set X together with a quasimetric d and a nonnegative Borel measure μ on X such that the doubling condition

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \tag{6}$$

holds for all $x \in \mathsf{X}$ and $r > 0$.

The balls $B(x, r)$ are not necessarily open, but by a theorem of Macias and Segovia [10], there is a continuous quasimetric d' which is equivalent to d (i.e., there are positive constants c_1 and c_2 such that $c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y)$ for all $x, y \in \mathsf{X}$) for which every ball is open. We always assume that the quasimetric d is continuous and that balls are open.

If C is the smallest constant for which (6) holds, then the number $D = \log C$ is called the doubling order of μ. By iterating (6), we have

$$\frac{\mu(B)}{\mu(\tilde{B})} \leq C_\mu \left(\frac{r(B)}{r(\tilde{B})}\right)^D \qquad \text{for all balls} \quad \tilde{B} \subset B. \tag{7}$$

A weight ω is a nonnegative measurable function integrable on any ball. Also if E is any μ-measurable set we let $\omega(E) = \int_E \omega(x) d\mu(x)$.

A relevant class of weights is given by the $A_\infty(\mu)$ class of Muckenhoupt: if there are positive constants c, ε such that

$$\omega(E) \leq c \left(\frac{\mu(E)}{\mu(B)}\right)^\varepsilon \omega(B) \tag{8}$$

for every ball B and every measurable set $E \subset B$.

2.2. Rearrangements

We mention here some simple properties of rearrangements. It follows easily from the definition that

$$\mu_f(f_\mu^*(t)) \leq t \qquad \text{and} \qquad \mu\{x \in \mathsf{X} : |f(x)| \geq f_\mu^*(t)\} \geq t.$$

Next, for any $0 < \lambda < 1$,

$$(f + g)_\mu^*(t) \leq f_\mu^*(\lambda t) + g_\mu^*((1 - \lambda)t). \tag{9}$$

For any measurable set R with finite positive measure we have,

$$\inf_R |f| \leq (f \chi_R)_\mu^*(\mu(R)). \tag{10}$$

Finally we recall that the (weighted normalized) weak L^r norm is defined by

$$\|g\|_{L^{r,\infty}(B,\omega)} = \sup_{\lambda > 0} \lambda \left(\frac{\omega(\{x \in B : |g(x)| > \lambda\})}{\omega(B)}\right)^{1/r}$$

or, equivalently,

$$\|g\|_{L^{r,\infty}(B,\omega)} = \sup_{0 < t \leq \omega(B)} (g \chi_B)_\omega^*(t) \left(\frac{t}{\omega(B)}\right)^{1/r}.$$

We will need the following two propositions about local rearrangements.

PROPOSITION 2.2. *For any measurable function f, any weight ω, and each measurable set $R \subset \mathsf{X}$ with $0 < \omega(R) < \infty$, and for any $0 < \lambda < 1$:*

$$\big(f\chi_R\big)^*_\omega\big(\lambda\omega(R)\big) \le 2\inf_c\big((f-c)\chi_R\big)^*_\omega\big(\lambda\omega(R)\big) + \big(f\chi_R\big)^*_\omega\big((1-\lambda)\omega(R)\big).$$

PROOF. Essentially the same was proved in [8] in a less general context. The proof of this proposition follows the same lines, and we shall recall it only for the sake of completeness.

Using (9) and (10), for any constant c we get

$$\begin{aligned}|c| &\le \inf_{x\in R}\big(|f(x)-c|+|f(x)|\big) \le \big((|f-c|+|f|)\chi_R\big)^*_\omega\big(\omega(R)\big)\\ &\le \big((f-c)\chi_R\big)^*_\omega\big(\lambda\omega(R)\big) + \big(f\chi_R\big)^*_\omega\big((1-\lambda)\omega(R)\big).\end{aligned}$$

From this and from the estimate

$$\big(f\chi_R\big)^*_\omega\big(\lambda\omega(R)\big) \le \big((f-c)\chi_R\big)^*_\omega\big(\lambda\omega(R)\big) + |c|$$

we get immediately the required inequality.

PROPOSITION 2.3. *Let $\omega \in A_\infty(\mu)$. For any ball B and any measurable f, and for any $0 < \lambda < 1$:*

$$\big(f\chi_B\big)^*_\omega\big(\lambda\omega(B)\big) \le \big(f\chi_B\big)^*_\mu\Big(\Big(\frac{\lambda}{c}\Big)^{1/\varepsilon}\mu(B)\Big),$$

where c and ε are A_∞-constants of ω.

PROOF. It follows from the definitions of A_∞ and of the rearrangement that for any $\delta > 0$,

$$\begin{aligned}&\omega\Big\{x\in B : |f(x)| > \big(f\chi_B\big)^*_\mu\Big(\Big(\frac{\lambda}{c}\Big)^{1/\varepsilon}\mu(B)\Big)+\delta\Big\}\\ &\qquad\le c\left(\frac{\mu\big\{x\in B : |f(x)| > \big(f\chi_B\big)^*_\mu\big(\big(\frac{\lambda}{c}\big)^{1/\varepsilon}\mu(B)\big)+\delta\big\}}{\mu(B)}\right)^\varepsilon \omega(B) < \lambda\omega(B),\end{aligned}$$

which is equivalent to that

$$\big(f\chi_B\big)^*_\omega\big(\lambda\omega(B)\big) \le \big(f\chi_B\big)^*_\mu\Big(\Big(\frac{\lambda}{c}\Big)^{1/\varepsilon}\mu(B)\Big)+\delta.$$

Letting $\delta \to 0$ yields the required inequality.

3. A basic covering lemma

In this section we prove a covering lemma of Calderón-Zygmund type that will be a key object in the proof of Theorem 1.3. To do this we define for each ball an appropriate basis. We adapt ideas from [11].

For a given ball B and for $x \in B$ we define the basis

$$\mathscr{B}_B(x) = \{B' : x \in B', x_{B'} \in B \text{ and } r(B') \leq \delta r(B)\}.$$

It is easy to see that any ball $B' \in \mathscr{B}_B(x)$ is contained in $\widehat{B} = \kappa(1+\delta)B$.

Lemma 3.1. *Let ω be a doubling weight. There are positive constants $c_1, c_2 \leq 1$ and $c_3 \geq 1$ such that for any $0 < \lambda < 1$, any ball B and any set $E \subset B$ with $\omega(E) \leq c_1\lambda\omega(B)$, there is a countable family of pairwise disjoint balls $\{B_i\}$ from $\cup_{x\in E}\mathscr{B}_B(x)$ such that*

(i) $\omega(B_i \cap B) \geq c_2\omega(B_i)$;

(ii) $\omega(B_i \cap E) \geq \lambda\omega(B_i \cap B)$;

(iii) $\omega(E) \leq c_3\lambda \sum_i \omega(B_i \cap B)$.

Proof. Since ω is doubling we have as in (7) that

$$\frac{\omega(B)}{\omega(\widetilde{B})} \leq c_4 \left(\frac{r(B)}{r(\widetilde{B})}\right)^{c_5} \tag{11}$$

for every pair $\widetilde{B}$, B of balls such that $\widetilde{B} \subset B$. The constants c_4 and c_5 depend on the doubling condition of ω and on κ.

We will be using the following important geometric fact (see, e.g., [5, p. 2]):

Let B and B' two balls such that $x_{B'} \in B$ and $r(B') \leq r(B)$. Then, then there is a geometric constant c_2 such that

$$\omega(B \cap B') \geq c_2\, \omega(B'). \tag{12}$$

Set now

$$c_1 = \frac{c_2}{c_4}\left(\frac{\delta}{1+\delta}\frac{1}{\kappa^2(4\kappa+1)}\right)^{c_5}.$$

Let $E \subset B$ be any set with $\omega(E) \leq c_1\lambda\omega(B)$. Suppose that for some $B' \in \mathscr{B}_B$ we have

$$\frac{\omega(B' \cap E)}{\omega(B' \cap B)} > \lambda. \tag{13}$$

Observe that by (12) $\omega(B' \cap B) \geq c_2\, \omega(B')$ since $B' \in \mathscr{B}_B$ recalling that we assume that $\delta < 1$. Hence, this combined with (13) yields

$$\frac{c_2}{c_1} < \frac{\omega(\widehat{B})}{\omega(B')} \leq c_4 \left(\frac{\kappa(1+\delta) r(B)}{r(B')} \right)^{c_5},$$

and therefore,

$$r(B') < \frac{\delta}{\kappa(4\kappa+1)} r(B). \tag{14}$$

We now define

$$\Omega_\lambda = \left\{ x \in B : \sup_{B' \in \mathscr{B}_B(x)} \frac{\omega(B' \cap E)}{\omega(B' \cap B)} > \lambda \right\},$$

and for $x \in \Omega_\lambda$ we let

$$r(x) = \sup \left\{ r(B') : B' \in \mathscr{B}_B(x) \text{ and } \frac{\omega(B' \cap E)}{\omega(B' \cap B)} > \lambda \right\},$$

and note that by (14).

$$r(x) < \frac{\delta}{\kappa(4\kappa+1)} r(B).$$

Denote by E' the set of ω-density points of E (recall that ω is doubling). Then, by Lebesgue differentiation theorem $E = E'$ ω-a.e. and $E' \subset A$. For any $x \in E'$ we can find a ball $B_x \in \mathscr{B}_B$ such that

$$\frac{\omega(B_x \cap E)}{\omega(B_x \cap B)} > \lambda \qquad \text{and} \qquad \frac{r(x)}{2} < r(B_x) \leq r(x).$$

Thus the ball B_x satisfies both (i) and (ii).

Now, if we denote $B^* = \kappa(4\kappa+1)B$ we have $B_x^* \in \mathscr{B}_B$ by (14) and

$$\omega(B_x^* \cap E) \leq \lambda \omega(B_x^* \cap B).$$

Applying the Vitali-type covering lemma (cf. [5, p. 14]) to the family $\{B_x\}_{x \in E'}$, we get a countable subfamily of pairwise disjoint balls $\{B_i\}$ such that $E' \subset \cup_i B_i^*$. Every B_i clearly satisfies item (i) and (ii) but further,

$$\begin{aligned} \omega(E) &\leq \sum_i \omega(B_i^* \cap E) \leq \lambda \sum_i \omega(B_i^* \cap B) \\ &\leq \lambda \sum_i \omega(B_i^*) \leq c_{\kappa,\omega} \lambda \sum_i \omega(B_i) \leq \frac{c_{\kappa,\omega}}{c_2} \lambda \sum_i \omega(B_i \cap B), \end{aligned}$$

and thus we have verified item (iii) of the lemma, which completes the proof.

4. Proof of Theorem 1.3

Adapting ideas from [8], [9] we prove in this section the main theorem of this paper.

PROOF OF THEOREM 1.3. The goal is to prove the following estimate: there is a constant c' such that for any ball B and for any $0 < t < \omega(B)$

$$(f\chi_B)^*_\omega(t) \le c'\, a(\widehat{B})\left(\frac{\omega(\widehat{B})}{t}\right)^{\frac{1}{r}} + (f\chi_B)^*_\mu(\lambda\mu(B)). \tag{15}$$

First we will prove this estimate for small values of t.

The key result in the proof is the following estimate for the rearrangement of f: there exists a constant λ', $0 < \lambda' < 1$ such that for any $0 < t < \frac{c_1\lambda'}{2}\omega(B)$

$$(f\chi_B)^*_\omega(t) \le c\, a(\widehat{B})\left(\frac{\omega(\widehat{B})}{t}\right)^{\frac{1}{r}} + (f\chi_B)^*_\omega(2t). \tag{16}$$

Throughout the proof $c_3 \ge 1$, $c_2, c_1 \le 1$ are the geometric constants from Lemma 3.1 and c and ε are the constants from the A_∞-condition (8) of ω.

For any t we consider the set $E = E_t$

$$E = \{x \in B : |f(x)| \ge (f\chi_B)^*_\omega(t)\}.$$

We can assume that $(f\chi_B)^*_\omega(t) > (f\chi_B)^*_\omega(2t)$, since otherwise (16) is trivial. Observe that $t \le \omega(E) \le 2t$.

Set $\lambda' = \frac{1}{4c_3}$ and $\lambda = \left(\frac{c_2\lambda'}{c}\right)^{1/\varepsilon}$ and let $0 < t < \frac{c_1\lambda'}{2}\omega(B)$. Then we have that

$$\omega(E) \le c_1\lambda'\omega(B),$$

and we can apply Lemma 3.1 to the set E and number λ'. Hence, we get a countable family of pairwise disjoint balls $\{B_i\}$ satisfying (i)–(iii) of Lemma 3.1. Thus combining item (ii) and Proposition 2.2 we have,

$$\begin{aligned}
(f\chi_B)^*_\omega(t) \le \inf_{x\in E}|f(x)| &\le \inf_i \inf_{x\in E\cap B_i}|f(x)| \\
&\le \inf_i (f\chi_{E\cap B_i})^*_\omega\big(\omega(E\cap B_i)\big) \\
&\le \inf_i (f\chi_{B\cap B_i})^*_\omega\big(\lambda'\omega(B\cap B_i)\big) \\
&\le \inf_i \Big[2\inf_\alpha \big((f-\alpha)\chi_{B\cap B_i}\big)^*_\omega\big(\lambda'\omega(B\cap B_i)\big) \\
&\qquad + \big(f\chi_{B\cap B_i}\big)^*_\omega\big((1-\lambda')\omega(B\cap B_i)\big)\Big].
\end{aligned}$$

Applying item (i) together with Proposition 2.3 we obtain for each i

$$\begin{aligned}\inf_\alpha\big((f-\alpha)\chi_{B\cap B_i}\big)^*_\omega\big(\lambda'\omega(B\cap B_i)\big) &\le \inf_\alpha\big((f-\alpha)\chi_{B_i}\big)^*_\omega\big(c_2\lambda'\omega(B_i)\big)\\ &\le \inf_\alpha\big((f-\alpha)\chi_{B_i}\big)^*_\mu\big(\lambda\mu(B_i)\big)\\ &\le c_\lambda a(B_i),\end{aligned}$$

where in the last step we have used the basic hypothesis (4) to the ball B_i. Hence,

$$(f\chi_B)^*_\omega(t) \le \inf_i\Big(2c_\lambda a(B_i) + \big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big)\Big).$$

Since the balls B_i are disjoint and contained in $\widehat{B}$ we have by the D_r condition (2)

$$\sum_i a(B_i)^r w(B_i) \le d^r a(\widehat{B})^r w(\widehat{B}).$$

We split the family of balls as follows: $i \in I$ if

$$a(B_i) \le 10^{1/r} d a(\widehat{B}) \left(\frac{w(\widehat{B})}{t}\right)^{\frac{1}{r}} \tag{17}$$

and $i \in II$ if it satisfies the opposite inequality. The proof below shows that the family I is not empty. We claim that

$$\sum_{i\in I} w(B_i\cap B) \ge (4-1/10)t. \tag{18}$$

Indeed, by the D_r condition

$$\sum_{i\in II} w(B_i) \le t/10,$$

and hence (18) follows from the item (iii) of Lemma 3.1 and our choice of λ'.

Therefore we have

$$(f\chi_B)^*_\omega(t) \le 2c_\lambda 10^{1/r} d a(\widehat{B}) \left(\frac{w(\widehat{B})}{t}\right)^{\frac{1}{r}} + \inf_{i\in I}\big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big).$$

Set now for $i \in I$

$$E_i = \big\{x \in B_i\cap B : |f(x)| \ge \big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big)\big\}.$$

By (18),

$$\omega(\cup_{i\in I} E_i) = \sum_{i\in I} \omega(E_i) \geq (1-\lambda') \sum_{i\in I} \omega(B_i \cap B) \geq \big(1-\tfrac{1}{4}\big)\big(4-\tfrac{1}{10}\big)t > 2t.$$

Thus,

$$\begin{aligned}\inf_{i\in I}\big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big) &\leq \inf_{i\in I}\inf_{x\in E_i} |f(x)| \\ &= \inf_{x\in\cup_{i\in I}E_i} |f(x)| \leq (f\chi_B)^*_\omega\big(\omega(\cup_{i\in I}E_i)\big) \leq (f\chi_B)^*_\omega(2t),\end{aligned}$$

and hence (16) is proved for the values $0 < t < \frac{c_1\lambda'}{2}\omega(B)$.

Now, for one of these values of t, there is $k = 1, \ldots$, such that

$$\frac{c_1\lambda'}{2^{k+1}}\omega(B) \leq t < \frac{c_1\lambda'}{2^k}\omega(B).$$

Since $c_1 \approx 2^{-m}$, for some $m = 1, 2, \ldots$, iterating (16) yields

$$\begin{aligned}(f\chi_B)^*_\omega(t) &\leq c a(\widehat{B}) \left(\frac{\omega(\widehat{B})}{t}\right)^{\frac{1}{r}} \sum_{j=1}^{k+m} (2^{-j})^{\frac{1}{r}} + (f\chi_B)^*_\omega(\lambda'\omega(B)) \\ &\leq c'\, a(\widehat{B}) \left(\frac{\omega(\widehat{B})}{t}\right)^{\frac{1}{r}} + (f\chi_B)^*_\omega(\lambda'\omega(B)). \qquad (19)\end{aligned}$$

Next, we observe that (19) trivially holds for $t \geq \lambda'\omega(B)$, and hence this formula holds for any $0 < t < \omega(B)$.

Applying Proposition 2.3 to the second term on the right-hand side of (19), we obtain (15).

To finish the proof we observe that if α is any number, $f - \alpha$ also satisfies the initial assumption (4) and hence (15) holds for $f - \alpha$ as well:

$$((f-\alpha)\chi_B)^*_\omega(t) \leq c'\, a(\widehat{B}) \left(\frac{\omega(\widehat{B})}{t}\right)^{\frac{1}{r}} + ((f-\alpha)\chi_B)^*_\mu(\lambda\mu(B)).$$

Recalling that

$$\|g\|_{L^{r,\infty}(B,\omega)} = \sup_{0<t\leq\omega(B)} (g\chi_B)^*_\omega(t)\left(\frac{t}{\omega(B)}\right)^{1/r},$$

we have by multiplying $\left(\frac{t}{\omega(B)}\right)^{\frac{1}{r}}$ and taking the supremum over $0 < t < \omega(B)$ that

$$\begin{aligned}\|f-\alpha\|_{L^{r,\infty}(B,\omega)} &\le c'\,a(\widehat{B})\left(\frac{\omega(\widehat{B})}{\omega(B)}\right)^{\frac{1}{r}} + ((f-\alpha)\chi_B)^*_\mu(\lambda\mu(B))\\ &\le c''\,a(\widehat{B}) + ((f-\alpha)\chi_B)^*_\mu(\lambda\mu(B))\end{aligned}$$

(we have used that ω is doubling). Taking the infimum over all α and using again (4) combined with the D_r condition we have

$$\begin{aligned}\inf_\alpha \|f-\alpha\|_{L^{r,\infty}(B,\omega)} &\le c''\,a(\widehat{B}) + \inf_\alpha\big((f-\alpha)\chi_B)^*_\mu(\lambda\mu(B)\big)\\ &\le c''\,a(\widehat{B}) + c_\lambda a(B)\\ &\le c'''\,a(\widehat{B}).\end{aligned}$$

The proof is now complete.

5. Exponential self-improving properties

By imposing a stronger condition on the functional a it has been proved in [12] that it is possible to go beyond the L^p self-improving property to deduce higher integrability. In this section we improve the main result from that paper.

We will be assuming the following geometrical condition which is different but within the "spirit" of the D_r condition.

DEFINITION 5.1. Let $1 < p < \infty$. We say that the functional a satisfies the T_p condition if there exists a finite constant c such that for each ball $B \subset \mathsf{X}$

$$\sum_j a(B_j)^p \le c^p a(B)^p \tag{20}$$

whenever $\{B_j\}$ is a family of pairwise disjoint sub-balls of B.

We remark that the T_p condition is stronger than the D_r condition, for any r, since in particular the functional a is increasing, class T_p is increasing, namely if $B' \subset B$ then $a(B') \le ca(B)$. The main examples are given by the expression

$$a(B) = \left(\int_B g^p\right)^{\frac{1}{p}},$$

where $g \in L^p_{\mathrm{loc}}(\mathsf{X})$ and, more generally,

$$a(B) = \nu(B)^{\frac{1}{p}}$$

where ν is a locally finite measure.

We will assume that X satisfies that the annuli are not empty, namely

$$\text{if } x \in \mathsf{X} \text{ and } 0 < r < R < \infty \text{ then } B_R(x) \setminus B_r(x) \neq \emptyset. \tag{21}$$

Also, this condition can be replaced by assuming that X is connected as done in [12].

THEOREM 5.2 ([12]). *Suppose that the functional a satisfies the T_p condition for some $1 < p < \infty$, and that w is a doubling measure on X that is absolutely continuous with respect to μ. Also, let $\delta > 0$ be given.*

If f is a locally integrable function for which there exist constants $\tau \geq 1$ such that for all balls B

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \leq a(\tau B), \tag{22}$$

then there exists a constant C independent of f such that

$$\|f - f_B\|_{\exp L^{p'}(B,w)} \leq C a(\tau \widehat{B}). \tag{23}$$

for any ball B.

Using this result we now state the following consequence of Theorem 1.3.

COROLLARY 5.3. *Let $\omega \in A_\infty(\mu)$. Let f be a measurable function and suppose that a satisfies the T_p condition with respect to ω. Suppose that for any ball $B \subset \mathsf{X}$*

$$\inf_\alpha \big((f - \alpha)\chi_B\big)^*_\mu \big(\lambda \mu(B)\big) \leq c_\lambda a(B), \qquad 0 < \lambda < 1.$$

Then, there is a geometric constant c such that for any ball $B \subset \mathsf{X}$

$$\|f - f_B\|_{\exp L^{p'}(B,w)} \leq c a(\widehat{\widehat{B}}). \tag{24}$$

The proof is the following. Since the T_p condition implies the D_r condition, Theorem 1.3 applied to the underlying measure μ yields

$$\inf_\alpha \|f - \alpha\|_{L^{r,\infty}(B,\mu)} \leq c a(\widehat{B})$$

for any ball B. On the other hand, if we choose $r > 1$ we have that

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \leq 2 \inf_\alpha \frac{1}{\mu(B)} \int_B |f - \alpha|\, d\mu \leq c_r \inf_\alpha \|f - \alpha\|_{L^{r,\infty}(B,\mu)} .$$

From this and from the previous estimate we get that (22) holds with $\tau = (1+\delta)\kappa$, which, in view of (23), yields (24).

6. Local sharp maximal function

Essentially the same ideas used in the proof of Theorem 1.3 can be applied to obtain a rearrangement inequality for the Local Sharp Maximal Function. This is a maximal type operator that was introduced by Strömberg in [15] following ideas of John's paper [6]. Given a measurable function f, the local sharp maximal function $M^{\#}_{\lambda,\mathscr{B}_B} f$ by

(25)
$$M^{\#}_{\lambda,\mathscr{B}_B} f(x) = \sup_{B' \in \mathscr{B}_B(x)} \inf_c \big((f-c)\chi_{B'}\big)^*_\mu\big(\lambda\mu(B')\big) \quad (x \in B, 0 < \lambda < 1),$$

where $\mathscr{B}_B(x)$ is the basis defined in Section 3.

THEOREM 6.1. *Let $\omega \in A_\infty(\mu)$. There are constants $\lambda, \lambda_0 < 1$ such that for any measurable function f and any ball $B \subset \mathsf{X}$,*

(26)
$$(f\chi_B)^*_\omega(t) \le 2\big((M^{\#}_{\lambda,\mathscr{B}_B} f)\chi_B\big)^*_\omega(t/2) + (f\chi_B)^*_\omega(2t) \qquad (0 < t \le \lambda_0\omega(B)).$$

REMARK 6.2. Variants of this theorem was proved in [8], [9]. On the other hand the Theorem is new in the context of spaces of homogeneous type.

PROOF. We are going to outline the proof since it follows along the line of the proof Theorem 1.3. Following the notation there we set $\lambda' = \frac{1}{4c_3}$ and $\lambda = \big(\frac{c_2\lambda'}{c}\big)^{1/\varepsilon}$ and we choose $\lambda_0 = \frac{c_1\lambda'}{2}$.

Fix $t \le \lambda_0\omega(B)$. Set

$$E = \{x \in B : |f(x)| \ge (f\chi_B)^*_\omega(t)\}$$

and

$$\Omega = \big\{x \in B : M^{\#}_{\lambda,\mathscr{B}_B} f(x) > \big((M^{\#}_{\lambda,\mathscr{B}_B} f)\chi_B\big)^*_\omega(t/2)\big\}.$$

We easily obtain that $t/2 \le \omega(E\setminus\Omega) \le 2t \le c_1\lambda'\omega(B)$. Applying Lemma 3.1 to the set $E \setminus \Omega$ and number λ', we get a countable family of pairwise disjoint balls $\{B_i\}$ satisfying (i)–(iii) of that lemma. By item (ii) and Proposition 2.2,

$$(f\chi_B)^*_\omega(t) \le \inf_i \Big(2\inf_c \big((f-c)\chi_{B_i\cap B}\big)^*_\omega\big(\lambda'\omega(B_i\cap B)\big) + \big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big)\Big).$$

Next, using the fact that the balls B_i belong to $\cup_{x\in E\setminus\Omega}\mathscr{B}_B(x)$ and applying item (i) and Proposition 2.3, we obtain for each i

$$\inf_c \big((f-c)\chi_{B_i\cap B}\big)^*_\omega\big(\lambda'\omega(B_i\cap B)\big) \le \big((M^{\#}_{\lambda,\mathscr{B}_B} f)\chi_B\big)^*_\omega(t/2).$$

Therefore,

$$(f\chi_B)^*_\omega(t) \le 2\big((M^\#_{\lambda,\mathscr{B}_B} f)\chi_B\big)^*_\omega(t/2) + \inf_i \big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big).$$

Now, exactly as in the prove of Theorem 1.3 one can show that

$$\inf_i \big(f\chi_{B_i\cap B}\big)^*_\omega\big((1-\lambda')\omega(B_i\cap B)\big) \le (f\chi_B)^*_\omega(2t),$$

which completes the proof.

The standard iteration of (26) (e.g. [8, p. 282]) yields the following.

COROLLARY 6.3. *Let $\omega \in A_\infty(\mu)$. Then for any measurable function f and each ball $B \subset \mathsf{X}$,*

(27)

$$\inf_c \big((f-c)\chi_B\big)^*_\omega(t) \le \frac{4}{\log 2}\int_{t/4}^{\omega(B)} \big((M^\#_{\lambda,\mathscr{B}_B} f)\chi_B\big)^*_\omega(\tau)\frac{d\tau}{\tau} \quad (0 < t \le \omega(B)),$$

where λ is the constant from Theorem 6.1.

It follows immediately from (27) the following.

COROLLARY 6.4 (The John-Strömberg inequality). *Let $\omega \in A_\infty(\mu)$. Then for any ball $B \subset \mathsf{X}$ and any measurable f,*

$$\inf_c \big((f-c)\chi_B\big)^*_\omega(t) \le \frac{4\|M^\#_{\lambda,\mathscr{B}_B}\|_\infty}{\log 2}\log\frac{4\omega(B)}{t} \qquad (0 < t \le \omega(B)). \tag{28}$$

The John-Strömberg inequality was proved in the paper by [6], [15] in the usual euclidean context with the Lebesgue measure. In the context of spaces homogeneous type this inequality is implicitly contained in [16] but our proof is different.

Let BMO(B) be the space of all locally integrable f for which

$$\|f\|_{\mathrm{BMO}(B)} \equiv \sup_{B'\in\mathscr{B}_B}\frac{1}{\mu(B')}\int_{B'}|f-f_{B'}|\,d\mu < \infty.$$

By Chebyshev's inequality,

$$\|M^\#_{\lambda,\mathscr{B}_B}\|_\infty \le \frac{1}{\lambda}\|f\|_{\mathrm{BMO}(B)},$$

and therefore (28) implies the following.

COROLLARY 6.5 (The John-Nirenberg inequality). *Let $\omega \in A_\infty(\mu)$. Then for any ball $B \subset \mathsf{X}$ and any locally integrable f,*

$$\inf_c \big((f-c)\chi_B\big)^*_\omega(t) \le \frac{6\|f\|_{\mathrm{BMO}(B)}}{\lambda} \log \frac{4\omega(B)}{t} \qquad (0 < t \le \omega(B)). \tag{29}$$

The John-Nirenberg inequality was originally proved in [7] in the classical situation. In the context of spaces homogeneous type this inequality is contained in [16]. See also [12].

7. The non-homogeneous context

The technique used in the proof of Theorem 1.3 can be adapted to the non-homogeneous context. That is, consider for example, the usual Euclidean space R^n and let μ be any nonnegative Radon measure vanishing on any hyperplane parallel to the axes (cf. [13]). For a given cube Q and for each x in the interior of Q we define the basis (cf. [13], [14])

$$C_Q(x) = \{Q_x(r)\},$$

where $Q_x(r)$ is the unique cube with sidelength r, containing x, contained in Q and with center y closest to x. A main geometrical property of the cube $Q_x(r)$ is that the ratio of any two sidelengths of the unique rectangle R_x centered at x such that $R_x \cap Q = Q_x(r)$ is bounded by 2.

The following covering lemma, proved in [13], is essentially based on this property of the basis $\{Q_x(r)\}$.

LEMMA 7.1. *Let E be a subset of Q, and suppose that $\mu(E) \le \lambda\omega(Q)$, $0 < \lambda < 1$. Then there exists a sequence of cubes $\{Q_i\}$ contained in Q such that*

(i) $\mu(Q_i \cap E) = \lambda\mu(Q_i)$;

(ii) $\bigcup_i Q_i = \bigcup_{k=1}^{B_n} \bigcup_{i\in F_k} Q_i$, *where each of the family $\{Q_i\}_{i\in F_k}$ is formed by pairwise disjoint cubes;*

(iii) $E' \subset \cup_i Q_i$, *where E' is the set of μ-density points of E.*

Actually the proof of this lemma shows that one can choose the sequence $\{Q_i\}$ from $\cup_{x\in E} C_Q(x)$. Next, since any $A_\infty(\mu)$ weight ω (cf. [14]) vanishes also on hyperplanes parallel to the axes, we have that Lemma 7.1 holds also for ω. Further, from items (i)–(iii) of this lemma we obtain that there exists a subsequence of pairwise disjoint cubes $\{Q_i\}$ for which we have analogs of items (ii) and (iii) of Lemma 3.1:

$$\omega(E \cap Q_i) = \lambda\omega(Q_i)$$

and

$$\omega(E) \leq B_n \lambda \sum_i \omega(Q_i).$$

So, we see that the family $\mathcal{Q}$ of all cubes in R^n satisfies an analogue of Lemma 3.1 with respect to any $\omega \in A_\infty(\mu)$ with constants $c_1 = c_2 = 1$ and $c_3 = B_n$.

Next, we note that full analogs of Propositions 2.2 and 2.3 hold in this context and therefore, full analogs of Theorems 1.3 and 6.1 hold as well. We remark that this version of Theorem 1.3 yields an improvement of the main result from [14]. Finally we note that a similar version of Theorem 6.1 was proved in [9].

REFERENCES

1. Bennett, C., DeVore, R. A., and Sharpley, R., *Weak-L^∞ and* BMO, Ann. of Math. (2) 113 (1981), 601–611.
2. Franchi, B., Pérez, C., and Wheeden, R. L., *Self-Improving Properties of John-Nirenberg and Poincaré Inequalities on Spaces of Homogeneous Type*, J. Funct. Anal. 153 (1998), 108–146.
3. Hajłasz, P., and Koskela, P., *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris 320 (1995), 1211–1215.
4. Hajłasz, P., and Koskela, P., *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000), 688.
5. Genebashvili, I., Gogatishvili, A., Kokilashvili, V., and Krbec, M., *Weight theory for integral transforms on spaces of homogeneous type*, Pitman Monogr. Surveys Pure Appl. Math. 92 (1998).
6. John, F., *Quasi-isometric mappings*, in “Seminari 1962–1963 di Analisi, Algebra, Geometria E Topologia”, Rome (1965), 462–473.
7. John, F., and Nirenberg, L., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.
8. Lerner, A. K., *On weighted estimates of non-increasing rearrangements*, East J. Approx. 4 (1998), 277–290.
9. Lerner, A. K., *On the John-Strömberg characterization of* BMO *for nondoubling measures*, Real. Anal. Exchange 28 (2003), no. 2, 649–660.
10. Macias, R., Segovia, C., *Lipschitz functions on spaces of homogeneous type*, Adv. Math. 33 (1979), 257–270.
11. MacManus, P., and Pérez, C., *Generalized Poincaré inequalities: Sharp self-improving properties*, Internat. Math. Res. Notices 2 (1998), 101–116.
12. MacManus, P., and Pérez, C., *Trudinger’s inequality without derivatives*, Trans. Amer. Math. Soc. 354 (2002), 1997–2012.
13. Mateu, J., Mattila, P., Nicolau, A., and Orobitg, J., BMO *for nondoubling measures*, Duke Math. J. 102 (2000), no. 3, 533–565.
14. Orobitg, J., and Pérez, C., *A_p weights for nondoubling measures in R^n and applications*, Trans. Amer. Math. Soc. 354 (2002), no. 5, 2013–2033.
15. Strömberg, J.-O., *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), 512–544.
16. Strömberg, J.-O., and Torchinsky, A., *Weighted Hardy spaces*, Lecture Notes in Math. 1381 (1989).

17. Torchinsky, A., *Real-variable Methods in Harmonic Analysis*, Pure Appl. Math. Vol. 123, Orlando etc., Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers) (1986).

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