MATH. SCAND. 97 (2005), 235–265

MAXIMAL OPERATORS FOR THE HOLOMORPHIC LAGUERRE SEMIGROUP

EMANUELA SASSO

Abstract

For each p in $[1, \infty)$ let $\mathbf{E}_p$ denote the closure of the region of holomorphy of the Laguerre semigroup $\{M_t^\alpha : t > 0\}$ on L^p with respect to the Laguerre measure μ_α. We prove weak type and strong type estimates for the maximal operator $f \mapsto \sup\{|M_z^\alpha f| : z \in \mathbf{E}_p\}$. In particular, we give a new proof for the weak type 1 estimate for the maximal operator associated to M_t^α. Our starting point is the well-known relationship between the Laguerre semigroup of half-integer parameter and the Ornstein-Uhlenbeck semigroup.

1. Introduction

The purpose of this paper is to analyse a class of maximal operators associated to the holomorphic Laguerre semigroup on the half-line R_+. We shall be working with the Laguerre probability measure μ_α of type α, with $\alpha \geq 0$, whose density is $\mu_\alpha(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}$ with respect to the Lebesgue measure. The Laguerre semigroup is the symmetric diffusion semigroup $\{M_t^\alpha : t \geq 0\}$ on $(\mathsf{R}_+, \mu_\alpha)$, whose infinitesimal generator is the differential operator

$$L_\alpha u(x) = -x \frac{\partial^2}{\partial x^2} u(x) - (1 + \alpha - x) \frac{\partial}{\partial x} u(x),$$

i.e. $u(x, t) = e^{-tL_\alpha} f(x)$ satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u = -L_\alpha u, \\ u(x, 0) = f(x). \end{cases}$$

By virtue of Stein's maximal theorem [12] the operator

$$M^\alpha f(x) = \sup_{t \geq 0} |M_t^\alpha f(x)|$$

is bounded on $L^p(\mu_\alpha)$, for $1 < p \leq \infty$. B. Muckenhoupt proved that M^α is of weak type $(1, 1)$ in the one-dimensional case [9]. This result has been

Received February 13, 2004; in revised form September 27, 2004.

extended by Dinger to the Laguerre semigroup, of order $\alpha = (\alpha_1, \dots, \alpha_d)$ on R^d_+, with $d > 1$, defined as the tensor product of d one-dimensional Laguerre semigroups of order α_i, with $i = 1, \dots, d$ [3].

The Laguerre semigroup can be extended to complex values of the parameter t. This is achieved by analytic continuation of the $L^2(\mu_\alpha)$-function $t \mapsto M_t^\alpha f$ to the half-plane $\{\Re z \geq 0\}$. Since M_t^α has an integral kernel $m_{\alpha,t}(x, y)$ with respect to the Laguerre measure, it is equivalent to substitute t by a complex variable in the expression of the kernel (for more details, see Section 2). The resulting operator M_z^α is well defined as a bounded operator on $L^2(\mu_\alpha)$ for all z, with $\Re z \geq 0$ and $\alpha \geq 0$. For any p, with $1 \leq p \leq \infty$, define

$$\mathbf{E}_p = \{x + iy : |\sin y/2| \leq \tan \phi_p \sinh x/2\},$$

where $\phi_p = \arccos |1-2/p|$. The above represents the region of holomorphy of M_z^α acting on $L^p(\mu_\alpha)$, i.e. M_z^α is bounded on $L^p(\mu_\alpha)$ if and only if z belongs to $\mathbf{E}_p$ (see [10]). The map $z \mapsto M_z^\alpha$ from $\mathbf{E}_p$ to the space of the bounded operators on $L^p(\mu_\alpha)$ is continuous in the strong operator topology and its restriction to the interior of $\mathbf{E}_p$ is analytic. Therefore the maximal operator

$$\mathcal{M}_{\alpha,p} f(x) = \sup_{z \in \mathbf{E}_p} |M_z^\alpha f(x)| \tag{1}$$

is well defined. The aim of this paper is to investigate the boundedness properties in $L^q(\mu_\alpha)$ of this maximal operator, whenever $\alpha \geq 0$. It is well known that weak type estimates for $\mathcal{M}_{\alpha,p}$ are a key tool to investigate the almost everywhere convergence of $M_z^\alpha f$ to $M_{z_0}^\alpha f$ as z tends to z_0, for f in $L^q(\mu_\alpha)$.

We remark that $\mathcal{M}_{\alpha,1}$ is the maximal operator for the heat kernel $m_{\alpha,t}$, which is known to be of weak type 1 and of strong type p for each $p > 1$ by the aforementioned results of Stein and Muckenhoupt. For $1 < p < 2$ we shall prove that the operator $\mathcal{M}_{\alpha,p}$ is of strong type q for each q in (p, p') and of weak type p if $p < \frac{2\alpha+2}{\alpha+3/2}$. On the other hand for $p > \frac{2\alpha+2}{\alpha+3/2}$ it is not of weak type p. Moreover $\mathcal{M}_{\alpha,2}$ is not of weak type 2.

We follow the same strategy adopted by [6] to study the maximal operators associated to the holomorphic Ornstein-Uhlenbeck semigroup. By the periodicity properties of the semigroup M_z^α, we may restrict the parameter z to the region $\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leq \Im z \leq \pi\}$. To obtain the positive results we decompose the operator into a "local" part, whose kernel is supported in a sort of neighbourhood of the diagonal, and in a "global" part. For the negative result, we provide counterexamples by analysing the behaviour of $m_{\alpha,z}$ on the boundary of $\mathbf{E}_p$. To be more specific, a critical point is $z_p = |\log(p-1)| + i\pi$. Therefore it is natural to investigate the smaller maximal operator defined by taking (1) only over the set obtained deleting from $\mathbf{E}_p$ a small neighbourhood of the point z_p. Observe that at this point the operator $M^\alpha_{z_p}$, with $\alpha \in \mathsf{N}/2-1$, may be reduced to the Fourier transform from $L^p(\mathsf{R}^n, dx)$ to $Ł^{p'}(\mathsf{R}^n, dx)$ acting on radial functions.

The paper is organized as follows. In Section 2 we recall some basic properties of the Laguerre semigroup and we decompose the maximal operators in "local" and "global" parts. In Section 3 we estimate the local part while Section 4 is devoted to showing the positive results regarding the global parts of the maximal operators. Finally the negative results will be proved in Section 5.

2. Preliminaries and statement of results

The Laguerre semigroup on R_+ is the family of integral operators $\{M^\alpha_t : t \geq 0\}$ defined by the following kernel expressed in terms of the standard Bessel function J_α.

$$
\begin{aligned}
(2)\quad m_{\alpha,t}(x,y) = \Gamma(\alpha+1)(1-e^{-t})^{-1} \exp\left(-\frac{e^{-t}(x+y)}{1-e^{-t}}\right) \\
(-xye^{-t})^{-\alpha/2} J_\alpha\left(2\frac{(-xye^{-t})^{1/2}}{1-e^{-t}}\right),
\end{aligned}
$$

with respect to the Laguerre measure μ_α (see, for instance, [3]). Since this kernel admits an analytic continuation on the half-plane $\Re z \geq 0$, it is easy to see that the Laguerre semigroup has analytic continuation to a family of operators $\{M^\alpha_z : \Re z \geq 0\}$ from $\mathscr{D}(\mathsf{R}_+)$ to $\mathscr{D}'(\mathsf{R}_+)$ such that

$$
(3)\qquad M^\alpha_{z+2i\pi} f(x) = M^\alpha_z f(x), \qquad M^\alpha_{\bar z} f(x) = \overline{M^\alpha_z \overline{f(x)}}.
$$

By [10] the operator M^α_z extends to a bounded operator on $L^p(\mu_\alpha)$, for $1 \leq p \leq \infty$ and $\alpha \geq 0$, if and only if z belongs to the set $\mathbf{E}_p$, defined in the previous section. The set $\mathbf{E}_p$ is a closed $2\pi i$-periodic subset of the right half-plane. Moreover, if $1/p + 1/p' = 1$, then $\mathbf{E}_p = \mathbf{E}_{p'}$ and $\mathbf{E}_p \subset \mathbf{E}_q$, for each $1 < p < q < 2$. In particular, the end-point cases are $\mathbf{E}_1 = \{x + ik\pi : x \geq 0, k \in \mathsf{Z}\}$ and $\mathbf{E}_2 = \{z : \Re z \geq 0\}$.

Our purpose is to investigate the boundedness of the maximal operator $\mathscr{M}_{\alpha,p}$, defined in (1), on $L^q(\mu_\alpha)$, for $1 \leq q \leq \infty$. It turns out that we may restrict the parameter z to the "fundamental domain" $\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leq \Im z \leq \pi\}$. Indeed, by (3) and the properties of the region $\mathbf{E}_p$, it is easy to see that the maximal operator

$$
\mathscr{M}^*_{\alpha,p} f(x) = \sup_{z\in\mathbf{F}_p} |M^\alpha_z f(x)|,
$$

and $\mathscr{M}_{\alpha,p}$ are simultaneously of weak or strong type.

Let $\tilde{\mu}_\alpha$ be the Borel measure on R_+ with density $\tilde{\mu}_\alpha(x) = 2\frac{x^{2\alpha+1}e^{-x^2}}{\Gamma(\alpha+1)}$, with respect to the Lebesgue measure, and consider the map Φ defined on test functions by

$$\Phi f(x) = f(x^2). \tag{4}$$

The map Φ is an isometry between $L^q(\mu_\alpha)$ and $L^q(\tilde{\mu}_\alpha)$ and between $L^{q,\infty}(\mu_\alpha)$ and $L^{q,\infty}(\tilde{\mu}_\alpha)$. Define $\widetilde{M}_z^\alpha = \Phi M_z^\alpha \Phi^{-1}$. It is quite straightforward to see that $\widetilde{m}_{\alpha,z}(x,y) = m_{\alpha,z}(x^2,y^2)$ is the integral kernel of $\widetilde{M}_z^\alpha$. Clearly we may reduce the problem to the study of the boundedness of $\widetilde{\mathscr{M}}^*_{\alpha,p}$ on $L^q(\tilde{\mu}_\alpha)$, defined by

$$\widetilde{\mathscr{M}}^*_{\alpha,p} f(x) = \sup_{z\in\mathbf{F}_p} |\widetilde{M}_z^\alpha f(x)|.$$

More generally, we shall consider the family of maximal operators $\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma}$ defined as follows. Let z_p denote the point on the boundary of $\mathbf{F}_p$ with imaginary part π. For each σ, with $0 \le \sigma < |z_p|$, let $\mathbf{F}_{p,\sigma} = \{z \in \mathbf{F}_p : |z - z_p| \ge \sigma\}$. Define

$$\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma} f(x) = \sup_{z\in\mathbf{F}_{p,\sigma}} |\widetilde{M}_z^\alpha f(x)|.$$

We are now ready to state our results. Since $\mathbf{E}_p = \mathbf{E}_{p'}$, we only need study the boundedness of $\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma}$ for $1 \le p \le 2$.

THEOREM 2.1. *For $\alpha \ge 0$, the following hold:*

(1) *The operator $\widetilde{\mathscr{M}}^*_{\alpha,1}$ is of weak type 1 and of strong type q for every q in $(1,\infty]$;*
(2) *Let $1 < p < 2$. The operator $\widetilde{\mathscr{M}}^*_{\alpha,p}$ is of strong type q whenever $p < q < p'$;*
(3) *Let $1 < p < 2$. The operator $\widetilde{\mathscr{M}}^*_{\alpha,p}$ is of weak type p, when $p < \frac{2\alpha+2}{\alpha+3/2}$, but it is not of weak type p, when $p > \frac{2\alpha+2}{\alpha+3/2}$;*
(4) *If $1 < p < 2$ and $0 < \sigma < |z_p|$, the operator $\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma}$ is of weak type p and p', but not of strong type p;*
(5) *If $\alpha \notin \frac{2\mathsf{N}-1}{2}$, the operator $\widetilde{\mathscr{M}}^*_{\alpha,2,\sigma}$, with $0 \le \sigma < \pi$, is not of weak type 2.*

REMARK 2.2. We shall see that the results of statements (1) and (2) can be extended to the d-dimensional case (see Remark 4.6). Moreover, every negative result holds also in higher dimension. Indeed, by restricting the operators to functions which depend only on one variable in R^d_+, one sees that it suffices to consider the one-dimensional case $d = 1$. In particular, by (3), $\widetilde{\mathscr{M}}^*_{\alpha,p}$ cannot be of weak type p whenever $p > \min_{i=1,\dots,d}\left(\frac{2\alpha_i+2}{\alpha_i+3/2}\right)$, with $\alpha = (\alpha_1,\dots,\alpha_d)$.

REMARK 2.3. Recall that the holomorphic Ornstein-Uhlenbeck semigroup acting on square integrable functions on R^n, with respect to the Gaussian measure $\mathrm{d}\gamma(x) = \pi^{-n/2} e^{-|x|^2}\,\mathrm{d}x$, is the family of operators $\mathscr{H}_z$, with $\Re z \geq 0$, defined by

$$\mathscr{H}_z f(x) = \int_{\mathsf{R}^n} h_z(x, y) f(y)\,\mathrm{d}\gamma(y), \qquad \text{if} \quad z \neq i\pi\mathsf{Z},$$

$$\mathscr{H}_{ik\pi} f(x) = f((-1)^k x), \qquad \text{for} \quad k \in \mathsf{Z}.$$

Here

$$h_z(x, y) = (1 - e^{-2z})^{-n/2} \exp\left(-\frac{|e^{-z}x - y|^2}{1 - e^{-2z}}\right) e^{|y|^2}$$

is the Mehler kernel with respect to the Gaussian measure γ. It is well known that the region of holomorphy of the Ornstein-Uhlenbeck semigroup on $L^p(\gamma)$ is $2\mathbf{E}_p$ (see [6]). The maximal operators

$$\mathscr{H}_p^* f(x) = \sup_{z \in 2\mathbf{E}_p} |\mathscr{H}_z f(x)|,$$

$$\mathscr{H}_{p,\sigma}^* f(x) = \sup_{z \in 2\mathbf{F}_{p,\sigma}} |\mathscr{H}_z f(x)|$$

have been investigated in [6] and [11]. In particular, in [6] the authors have proved the analogue of our Theorem 2.1 for the operators $\mathscr{H}_p^*$ and $\mathscr{H}_{p,\sigma}^*$. Note however that, while the operator $\mathscr{M}_{\alpha,p}^*$ is of weak type p for $p < \frac{2\alpha+2}{\alpha+3/2}$, the operator $\mathscr{H}_p^*$ is never of weak type p, for all $p > 1$. Since when $\alpha = n/2 - 1$ and $n > 1$, the Laguerre operator can be interpreted as $\mathscr{H}_z$ acting on the radial functions on R^n (see [7]), we obtain that for $p < \frac{2n}{n+1}$, the maximal operator $\mathscr{H}_p^*$, restricted to the space of radial functions, is of weak type p.

To investigate $\widetilde{\mathscr{M}}_{\alpha,p}^*$, we essentially adopt the same techniques of [6]. We split the operator in a "local" part and the remaining or "global" part. To describe this decomposition, first we must write the kernel of $\widetilde{M}_z^\alpha$ as an average over $[-1, 1]$ of a family of kernels $\widetilde{m}_{\alpha,z}(\cdot, \cdot, s)$ depending on an additional variables s in $[-1, 1]$. Indeed, by using the integral form of Bessel functions, for $\alpha \geq 0$ (see, for instance, [4, p. 15]), and by considering the action of the isometry Φ, we may write the kernel of $\widetilde{M}_z^\alpha$ as

$$\widetilde{m}_{\alpha,z}(x, y) = \int_{-1}^{1} \widetilde{m}_{\alpha,z}(x, y, s)\,\mathrm{d}s,$$

where

(5)

$$\widetilde{m}_{\alpha,z}(x, y, s) = (1 - e^{-z})^{-\alpha-1} e^{\frac{1}{2}\frac{1}{e^{z/2}+1}(x^2+y^2+2xys) - \frac{1}{2}\frac{1}{e^{z/2}-1}(x^2+y^2-2xys)} \Pi_\alpha(s)$$

and $\Pi_\alpha(s) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\sqrt{\pi}}(1-s^2)^{\alpha-1/2}$. It will be useful to observe that $\int_{[-1,1,]} \Pi_\alpha(s)\,\mathrm{d}s = 1$. To split the operator we introduce two sets: the "local" region

$$L = \Big\{(x,y,s) \in \mathsf{R}_+ \times \mathsf{R}_+ \times [-1,1] : \\ (x^2+y^2-2xys) \le \min\left(1, \frac{1}{x^2+y^2+2xys}\right)\Big\},$$

and its complement G, the "global" region. This choice is suggested by the description of the corresponding local region in polar coordinates in the Ornstein-Uhlenbeck case [6]. Observe that the diagonal $\{(x,x,1)\}$, is contained in L, i.e. L is a neighbourhood of the diagonal in $\mathsf{R}_+ \times \mathsf{R}_+ \times \{1\}$. The local and global parts of the operator $\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma}$ are defined by

$$\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,\sigma} f(x) = \sup_{z\in\mathbf{F}_{p,\sigma}} \left| \int_{\mathsf{R}_+} \int_{-1}^1 \widetilde{m}_{\alpha,z}(x,y,s)\chi_L(x,y,s)\,\mathrm{d}s f(y)\,\mathrm{d}\tilde{\mu}_\alpha(y) \right|,$$

$$\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma} f(x) = \sup_{z\in\mathbf{F}_{p,\sigma}} \left| \int_{\mathsf{R}_+} \int_{-1}^1 \widetilde{m}_{\alpha,z}(x,y,s)\chi_G(x,y,s)\,\mathrm{d}s f(y)\,\mathrm{d}\tilde{\mu}_\alpha(y) \right|.$$

Clearly

$$\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma} f(x) \le \widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,\sigma} f(x) + \widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma} f(x),$$

we shall study the operators $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,\sigma}$ and $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$ separately. First, however, it is convenient to simplify the expression of $\widetilde{m}_{\alpha,z}(x,y)$ by means of the change of variable

$$z = \tau(\zeta) = 2\log\frac{1+\zeta}{1-\zeta}.$$

The same change of variable, without the factor 2 in front of logarithm, was introduced in [6], to which we refer the reader for the properties of the map τ. Here we only recall that τ is a biholomorphic transformation of a neighbourhood of 0, which maps the ray $\mathsf{R}_+e^{i\phi_p}$ onto $\partial\mathbf{E}_p \cap \{z \in \mathsf{C} : 0 \le \Im z < 2\pi\}$. Therefore the heat kernel becomes

$$\widetilde{m}_{\alpha,\tau(\zeta)}(x,y) = \Gamma(\alpha+1)\frac{(1+\zeta)^{2+2\alpha}}{(4\zeta)^{1+\alpha}} \exp\left(-(x^2+y^2)\frac{(1-\zeta)^2}{4\zeta}\right) \\ \left(i\frac{xy}{4}\left(\frac{1}{\zeta}-\zeta\right)\right)^{-\alpha} J_\alpha\left(i\frac{xy}{2}\left(\frac{1}{\zeta}-\zeta\right)\right).$$

For $\alpha \ge 0$, we may also write

$$\widetilde{m}_{\alpha,\tau(\zeta)}(x,y) = \int_{-1}^1 \widetilde{m}_{\alpha,\tau(\zeta)}(x,y,s)\,\mathrm{d}s,$$

where

(6)
$$\widetilde{m}_{\alpha,\tau(\zeta)}(x,y,s) = \frac{(1+\zeta)^{2+2\alpha}}{(4\zeta)^{1+\alpha}} e^{\frac{x^2+y^2}{2}} e^{-\left(\frac{\zeta}{4}(x^2+y^2+2xys)+\frac{1}{4\zeta}(x^2+y^2-2xys)\right)} \Pi_\alpha(s).$$

REMARK 2.4. We prove now some identities which will be useful in the sequel. Assume that f, g are in $L^1(\mathsf{R}_+, m_\alpha)$, where m_α is the measure with density

$$m_\alpha(y) = e^{y^2} \tilde{\mu}_\alpha(y). \tag{7}$$

Observe that the measure m_α is simply proportional to a power of y times Lebesgue measure.

We define the *generalized translation* as

$$\tau_y^\alpha f(x) = \int_{[-1,1]} f\left(\sqrt{x^2+y^2-2xys}\right)\Pi_\alpha(s)\,\mathrm{d}s,$$

and the *generalized convolution* of f and g as

$$f\#_\alpha g(x) = \int_{\mathsf{R}_+}\int_{[-1,1]} f\left(\sqrt{x^2+y^2-2xys}\right)g(y)\Pi_\alpha(s)\,\mathrm{d}s\,\mathrm{d}m_\alpha(y). \tag{8}$$

For $\alpha = \frac{n}{2} - 1$, these correspond to the average over the sphere of a radial funcion and to the convolution of radial functions in R^n, respectively. It is well known that generalized translations and generalized convolution share many of the properties of ordinary translations and convolution in R^n [8]. In particular

(i) the function $y \mapsto \tau_y^\alpha f$ is continuous in $L^p(m_\alpha)$;

(ii) $f\#_\alpha g = g\#_\alpha f$.

Namely, by the change of variable $s = \cos\theta$, we have

(9)
$$f\#_\alpha g(x) = \int_{\mathsf{R}_+}\int_{[0,\pi]} f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)g(y)(\sin\theta)^{2\alpha}\mathrm{d}s\,\mathrm{d}m_\alpha(y).$$

For each $\bar{x}$ in R^2, let x denote the absolute value of $\bar{x}$. If $\bar{x}$ and $\bar{y}$ are in R^2, let θ be the angle between the nonzero vectors $\overline{x}$ and $\overline{y}$. Interpreting (9) as an integral on R^2 in spherical coordinates, we obtain that

$$f\#_\alpha g(x) = \int_{\mathsf{R}^2} f(|\bar{x}-\bar{y}|)g(|\bar{y}|)\left(1-\left(\frac{\overline{x}\cdot\overline{y}}{|\overline{x}||\overline{y}|}\right)^2\right)^\alpha |\overline{y}|^{2\alpha} 2\frac{\mathrm{d}\overline{y}}{\Gamma(\alpha+1/2)\sqrt{\pi}}.$$

Now by the further change of variables, $\overline{y} - \overline{x} = \overline{w}$, we have

$$\begin{aligned} f\#_\alpha g(x) &= \int_{\mathsf{R}^2} f(|\bar{w}|)g(|\bar{x} - \bar{w}|)\left(1 - \left(\frac{\overline{x}\cdot\overline{w}}{|\overline{x}||\overline{w}|}\right)^2\right)^\alpha |\overline{w}|^{2\alpha} 2\frac{\mathrm{d}\overline{w}}{\Gamma(\alpha+1/2)\sqrt{\pi}} \\ &= g\#_\alpha f(x). \end{aligned}$$

Namely, by the sine theorem

$$\left(1 - \left(\frac{\overline{x}\cdot\overline{y}}{|\overline{x}||\overline{y}|}\right)^2\right)|\overline{y}|^2 = \left(1 - \left(\frac{\overline{x}\cdot\overline{w}}{|\overline{x}||\overline{w}|}\right)^2\right)|\overline{w}|^2.$$

This concludes the proof of item (ii).

Moreover, by choosing $g = 1$, $f\#_\alpha g$ is well defined for a.e. $x \in \mathsf{R}_+$ and

$$\int_{\mathsf{R}_+}\int_{[-1,1]} f\left(\sqrt{x^2+y^2-2xys}\right)\Pi_\alpha(s)\,\mathrm{d}s\,\mathrm{d}m_\alpha(y) = \int_{\mathsf{R}_+} f(y)\,\mathrm{d}m_\alpha(y).$$

3. Results for the "local" part

In this section we shall prove that $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,\sigma}$ is of weak type 1 and of strong type q, for each q in $(1,\infty]$ and $\sigma \geq 0$. Since $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,\sigma} \leq \widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p,0} = \widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$, it is enough to consider the latter operator. In the following, we will use the measure m_α defined in (7). Moreover, for each $\alpha \geq 0$ and $t > 0$, let $k_{\alpha,t}(x,y)$ be the function

$$(x,y) \mapsto \int_{-1}^{1} k_{\alpha,t}(x,y,s)\Pi_\alpha(s)\,\mathrm{d}s,$$

where $k_{\alpha,t}(x,y,s) = (4t)^{-\alpha-1}\exp\left(-\frac{x^2+y^2-2xys}{4t}\right)$.

LEMMA 3.1. *Suppose that* $\alpha \geq 0$. *Let* $\{T^\alpha_t : t > 0\}$ *be the family of integral operators, defined by*

$$T^\alpha_t f(x) = \int_{\mathsf{R}_+} k_{\alpha,t}(x,y)f(y)\,\mathrm{d}m_\alpha(y),$$

on $C^\infty_c(\mathsf{R}_+)$. *Then* $\{T^\alpha_t : t > 0\}$ *is a diffusion semigroup on* (R_+, m_α). *Moreover the maximal operator*

$$T^* f(x) = \sup_{t>0}|T^\alpha_t f(x)|$$

is of weak type $(1,1)$.

PROOF. Let $H^1(\mathsf{R}_+, m_\alpha)$ denote the space of all functions f, such that both f and its distributional derivative f' are in $L^2(m_\alpha)$. Let Q_α be the quadratic form, defined for f in $H^1(\mathsf{R}_+, m_\alpha)$ by

$$Q_\alpha(f) = \int_0^\infty \left| \frac{d}{dx} f \right|^2 (x)\, \mathrm{d}m_\alpha(x).$$

The form Q_α with dense domain $H^1(\mathsf{R}_+, m_\alpha)$ is closed, so Q_α is the form of a self adjoint operator $-\Delta_\alpha \geq 0$, which on $C_c^2(\mathsf{R}_+)$ coincides with the differential operator

$$-\frac{d^2}{dx^2} - \frac{(2\alpha+1)}{x}\frac{d}{dx}.$$

We claim that $-\Delta_\alpha$ is the infinitesimal generator of $\{T_t^\alpha : t > 0\}$. Indeed, by Remark 2.4 we have that

(10)
$$\int_{\mathsf{R}_+} k_{\alpha,t}(x, y)\, \mathrm{d}m_\alpha(y) = (4t)^{-\alpha-1} \int_{\mathsf{R}_+} \int_{[-1,1]} \exp\left(-\frac{w^2}{4t}\right) \Pi_\alpha(s)\, \mathrm{d}s\, \mathrm{d}m_\alpha(w)$$
$$= \int_{\mathsf{R}_+} e^{-w^2}\, \mathrm{d}m_\alpha(w) = 1,$$

and for each $\delta > 0$ it is quite simple to prove that

$$\lim_{t\to 0^+} \int_\delta^\infty k_{\alpha,t}(x, y)\, \mathrm{d}m_\alpha(y) = 0. \tag{11}$$

By (10) and (11), it is easy to prove that for every f

$$\lim_{t\to 0^+} \int_{\mathsf{R}_+} k_{\alpha,t}(x, y) f(y)\, \mathrm{d}m_\alpha(y) = f(x)$$

in $L^2(m_\alpha)$. So the claim is proved once we verify that $u_t(x) = T_t^\alpha f(x)$ solves the Cauchy problem

$$\begin{cases} \partial_t u_t = -\Delta_\alpha u_t \\ u_0 = f. \end{cases}$$

We have that $\partial_t k_{\alpha,t}(x, y) = -\Delta_{\alpha x} k_{\alpha,t}(x, y)$. Indeed, by a straightforward calculation, we get that

$$\int_{[-1,1]} [\partial_t k_{\alpha,t}(x, y, s) + \Delta_{\alpha x} k_{\alpha,t}(x, y, s)] \Pi_\alpha(s)\, \mathrm{d}s = 0.$$

Now by Lebesgue dominated convergence theorem we can differentiate under the integral sign. Hence $\partial_t u_t = -\Delta_\alpha u_t$. This proves the claim.

We observe that, by (10) T_t^α can be extended to a contraction on $L^1(m_\alpha)$ and, by duality, to a contraction on $L^\infty(m_\alpha)$. By interpolation we get that T_t^α is a diffusion semigroup. Notice that $t \mapsto T_t^\alpha f$ is a continuous function on R_+ for each $f \in L^2(m_\alpha)$. Thus, since the supremum over $t > 0$ coincides with the supremum over all rational $t > 0$, $T^* f$ is a measurable function. By the Littlewood-Paley-Stein theory, the maximal operator T^* is bounded on $L^p(m_\alpha)$, for $1 < p < \infty$. It remains now to prove the weak type (1, 1) boundedness. Let $\mathscr{E}_t$ be the ergodic means of $\{T_t^\alpha : t > 0\}$, i.e.

$$\mathscr{E}_t f = \frac{1}{t}\int_0^t T_\sigma^\alpha f \,\mathrm{d}\sigma,$$

on $L^p(m_\alpha) \cap L^2(m_\alpha)$ and with $t > 0$. The associated ergodic maximal operator is defined by

$$\mathscr{E}^* f = \sup_{t>0} |\mathscr{E}_t f|,$$

on $L^p(m_\alpha) \cap L^2(m_\alpha)$. The Hopf-Dunford-Schwartz ergodic maximal theorem asserts that

$$m_\alpha\{x \in \mathsf{R}_+ : \mathscr{E}^* f(x) > \lambda\} \le \frac{2}{\lambda}\|f\|_1, \qquad \forall \lambda > 0, \quad \forall f \in L^1(m_\alpha). \tag{12}$$

For $f \ge 0$, since T_t^α is positivity preserving, by Fubini's theorem we have that

$$\begin{aligned}
\mathscr{E}_{2t} f(x) &\ge \frac{1}{2t}\int_t^{2t} T_\sigma^\alpha f \,\mathrm{d}\sigma \\
&= \frac{1}{2t}\int_t^{2t}\int_{\mathsf{R}_+} k_{\alpha,\sigma}(x,y) f(y)\,\mathrm{d}m_\alpha(y)\,\mathrm{d}\sigma \\
&= \int_{\mathsf{R}_+}\frac{1}{2t}\int_t^{2t} k_{\alpha,\sigma}(x,y) f(y)\,\mathrm{d}\sigma\,\mathrm{d}m_\alpha(y) \\
&\ge \int_{\mathsf{R}_+}\frac{1}{2t}\int_t^{2t} (4\sigma)^{-\alpha-1}\,\mathrm{d}\sigma \int_{[-1,1]} e^{-\frac{x^2+y^2-2xys}{4t}}\,\Pi_\alpha(s)\,\mathrm{d}s f(y)\,\mathrm{d}m_\alpha(y) \\
&= C_\alpha T_t^\alpha f,
\end{aligned}$$

for some positive constant C_α. So, for any f in $L^1(m_\alpha)$,

$$T^* f \le \sup_{t>0} T_t^\alpha |f| \le C_\alpha \sup_{t>0} \mathscr{E}_t |f|. \tag{13}$$

Now the weak type (1, 1) estimate for T^* follows from (12) and (13) and this concludes the proof.

REMARK 3.2. By the definition of generalized convolution, $T_t^\alpha f = f\#_\alpha k$, with $k(x) = (4t)^{-\alpha-1}e^{-\frac{x^2}{4t}}$. So when $\alpha = \frac{n}{2} - 1$, with $n \in \mathsf{N}$ and $n > 1$, the operator T_t^α corresponds to the heat semigroup acting on radial functions of R^n.

LEMMA 3.3. *For each $p \in [1, 2)$, there exists a constant C such that for every t in $(0, 1]$ and (x, y, s) in L*

$$\sup_{|\phi|\le\phi_p} |\widetilde{m}_{\alpha,\tau(te^{i\phi})}(x, y, s)| \le Ct^{-\alpha-1}e^{y^2} \exp\left(-\frac{\cos\phi_p}{4t}(x^2 + y^2 - 2xys)\right)\Pi_\alpha(s).$$

PROOF. By (6) we have

$$|\widetilde{m}_{\alpha,\tau(te^{i\phi})}(x, y, s)| \le Ct^{-\alpha-1}e^{\frac{x^2+y^2}{2}} \exp\left(-\frac{\cos\phi}{4t}(x^2 + y^2 - 2xys)\right)\Pi_\alpha(s).$$

Since, if $(x, y, s) \in L$, then

$$\begin{aligned} x^2 - y^2 &\le \sqrt{(x^2 - y^2)^2 + 4(1 - s^2)x^2y^2} \\ &\le \sqrt{(x^2 + y^2 + 2xys)(x^2 + y^2 - 2xys)} \\ &\le 1, \end{aligned}$$

then we may majorise $e^{\frac{x^2+y^2}{2}}$ by Ce^{y^2} and the result follows.

THEOREM 3.4. *For each $p \in [1, 2)$, the operator $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ is of weak type 1 and of strong type q, whenever $1 < q \le \infty$.*

PROOF. For any fixed $f \ge 0$, Lemma 3.3 yields

$$\begin{aligned} \widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p} f(x) \le C \sup_{t>0} t^{-\alpha-1} \int_{\mathsf{R}_+}\int_{[-1,1]} \exp\left(-\frac{\cos\phi_p}{4t}(x^2 + y^2 - 2xys)\right) \\ \chi_L(x, y, s)\Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}m_\alpha(y). \end{aligned} \tag{14}$$

We claim that $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ is bounded on $L^\infty(\tilde{\mu}_\alpha)$. Namely, by Remark 2.4, we get

$$\begin{aligned} &|\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p} f(x)| \\ &\quad \le C\|f\|_\infty \sup_{t>0} t^{-\alpha-1} \int_{\mathsf{R}_+}\int_{[-1,1]} \exp\left(-\frac{\cos\phi_p}{4t}w^2\right)\Pi_\alpha(s)\, \mathrm{d}s\, \mathrm{d}m_\alpha(w). \end{aligned}$$

Since $\alpha \geq 0$, the integral

$$t^{-\alpha-1} \int_{\mathsf{R}_+} \int_{[-1,1]} \exp\left(-\frac{\cos\phi_p}{4t} w^2\right) \Pi_\alpha(s)\, \mathrm{d}s\, \mathrm{d}m_\alpha(w)$$

is finite and bounded by a constant independent of x and t. This concludes the proof of our claim.

It only remains to prove that $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ is also of weak type 1. By Lemma 3.3, our operator is bounded by

$$\mathscr{W} f(x) = \sup_{t>0} t^{-\alpha-1} \int_{\mathsf{R}_+} e^{-\cos\phi_p \frac{x^2+y^2-2xys}{4t}} \chi_L(x, y, s) \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}m_\alpha(y),$$

whose kernel is supported in the local region. Since, by Lemma 3.1, $\mathscr{W}$ is of weak type (1, 1) with respect to the measure m_α, $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ is of weak type (1, 1), with respect to the same measure. We consider the vector-valued operator S given by

$$Sf(x) = \left\{ \int_{\mathsf{R}_+} \int_{[-1,1]} \widetilde{m}_{\alpha,t}(x, y, s) \chi_L(x, y, s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \right\}_{t\in\mathsf{Q}}.$$

It is clear that $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p} f(x) = \|Sf(x)\|_{\ell_\infty}$. In order to prove that $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ maps $L^1(\tilde{\mu}_\alpha)$ into weak-$L^1(\tilde{\mu}_\alpha)$, it is enough to prove that S maps $L^1(\tilde{\mu}_\alpha)$ into weak-$L^1_{\ell^\infty}(\tilde{\mu}_\alpha)$. Now the conclusion follows by applying to S a vector-valued version of the arguments in Section 5.2 of [10] (see also [5, Section 5]).

4. Results for the "global" part

In this section we shall estimate the global maximal operators $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$, for $1 \leq p < 2$ and $\sigma \geq 0$. Our estimates are a consequence of the inequality

$$\begin{aligned}(15)\quad & \sup_{|\phi|\leq\phi_p} |\widetilde{m}_{\alpha,\tau(te^{i\phi})}(x, y, s)| \\ &\qquad \leq C t^{-\alpha-1} e^{\frac{x^2+y^2}{2} - \frac{\cos\phi_p}{4}(t(x^2+y^2+2xys)+\frac{1}{t}(x^2+y^2-2xys))} \Pi_\alpha(s),\end{aligned}$$

for all t in (0, 1] and all (x, y, s) in $\mathsf{R}_+ \times \mathsf{R}_+ \times [-1, 1]$, which follows easily from (6). First we give two different expressions for the right hand side of this inequality. Since $\cos\phi_p = \frac{2}{p} - 1$, we have that

$$\begin{aligned} & e^{\frac{x^2+y^2}{2} - \frac{\cos\phi_p}{4}\left(t(x^2+y^2+2xys)^2+\frac{1}{t}(x^2+y^2-2xys)\right)} \\ &\qquad = \exp\left(\frac{x^2}{p} + \frac{y^2}{p'} - \frac{\cos\phi_p}{4t} Q_t(x, y, s)\right) \\ &\qquad = \exp\left(\frac{x^2}{p'} + \frac{y^2}{p} - \frac{\cos\phi_p}{4t} Q_{-t}(x, y, s)\right), \end{aligned}$$

where $Q_\tau(x,y,s)$ is a quadratic form in x, y defined by

$$Q_\tau(x,y,s) = (1+\tau)^2x^2 - 2xys(1-\tau^2) + (1-\tau)^2y^2.$$

For each x in R_+, consider the section $G(x) = \{(y,s) : (x,y,s) \in G\}$ and for every fixed $\delta > 0$, define

$$J_\pm(x,t) = \int_{G(x)} \exp\left(-\frac{\delta}{t} Q_{\pm t}(x,y,s)\right) \Pi_\alpha(s)\,\mathrm{d}s\,\mathrm{d}m_\alpha(y).$$

LEMMA 4.1. *For each $\delta > 0$ and $(x,y,s) \in G$,*

(i) *there exists a constant C such that*

$$\sup_{0<t\le 1} t^{-\alpha-1} \exp\left(-\frac{\delta}{t} Q_t(x,y,s)\right) \le C\left[(1+x)^{2\alpha+2} \wedge (x^2(1-s^2))^{-\alpha-1}\right];$$

(ii) *for each p in $(1,\infty)$ and each η in $(0,1)$, there exists a constant C such that*

$$\sup_{0<t\le 1-\eta} t^{-p(\alpha+1)} \exp\left(-\frac{\delta}{t} Q_{\pm t}(x,y,s)\right) J_\pm^{p/p'}(x,t) \le C\left[(1+x)^{2\alpha+2} \wedge (x^2(1-s^2))^{-\alpha-1}\right].$$

PROOF. We claim that for each η in $(0,1)$ there exists a positive constant C such that for all x, y, s and $t \ge -1+\eta$

$$Q_t(x,y,s) \ge Cx^2(1-s^2). \tag{16}$$

Moreover for all $(x,y,s) \in G$ and $t < (1+x)^{-2}/8$

$$Q_t(x,y,s) \ge C\frac{1}{(1+x)^2}. \tag{17}$$

Assuming the claim for the moment, we prove the lemma. To obtain (i) first, we observe that (16) implies

$$t^{-\alpha-1}\exp\left(-\frac{\delta}{t}Q_t(x,y,s)\right) \le C(Q_t(x,y,s))^{-\alpha-1} \le C(x^2(1-s^2))^{-\alpha-1}.$$

Next, we observe that on the one hand, if $t \ge (1+x)^{-2}/8$, it is enough to majorise the exponential by 1, to get

$$t^{-\alpha-1}\exp\left(-\frac{\delta}{t}Q_t(x,y,s)\right) \le C(1+x)^{2\alpha+2}.$$

On the other hand, if $t < (1+x)^{-2}/8$, by (17)

$$t^{-\alpha-1}\exp\left(-\frac{\delta}{t}Q_t(x,y,s)\right) \leq C(1+x)^{2\alpha+2}$$

and this concludes the proof of (i). Next we prove (ii). By Remark 2.4, we have

$$J_\pm(x,y) \leq C\int_{\mathsf{R}_+} e^{-\frac{\delta}{t}(1-t)^2w^2}\,\mathrm{d}m_\alpha(w) \leq Ct^{\alpha+1}, \tag{18}$$

since $1 \mp t > \eta$. Hence by (16)

$$t^{-p(\alpha+1)}\exp\left(-\frac{\delta}{t}Q_{\pm t}(x,y,s)\right)J_\pm^{p/p'}(x,t) \leq C(x^2(1-s^2))^{-\alpha-1}.$$

To prove the other inequality, we majorise the exponential by 1 and we consider separately the cases $t \geq (1+x)^{-2}/8$ and $t < (1+x)^{-2}/8$. In the first case, by (18) we obtain that

$$t^{-p(\alpha+1)}J_\pm^{p/p'}(x,t) \leq C(1+x)^{2\alpha+2}.$$

In the second case, by (17) and Remark 2.4 we get that

$$\begin{aligned} J_\pm^{p/p'}(x,t) &\leq C\left(\int_{w>C(1+x)^{-1}} e^{-w^2/t}\,\mathrm{d}m_\alpha(w)\right)^{p/p'} \\ &\leq Ct^{p/p'(\alpha+1)}\big(\sqrt{t}(1+x)\big)^{2\alpha+2}, \end{aligned}$$

which, again, implies (18).

We must finally prove the claims. To prove (16), consider two vectors $\bar{v}$ and $\bar{w}$ in R^2, such that $|\bar{v}| = (1+t)x$, $|\bar{w}| = (1-t)y$ and the angle between $\bar{v}$ and $\bar{w}$ is $\arccos s$. Then $|Q_t(x,y,s)|^{1/2} = |\bar{v}-\bar{w}|$ is minorised by the length of the projection of $\bar{v}$ on the direction orthogonal to $\bar{w}$. This gives the inequality $|Q_t(x,y,s)|^{1/2} \geq (1+t)x(1-s^2)^{1/2}$, which implies (16). For (17), it is quite straightforward to verify that, if $(x,y,s) \in G$, then

$$(x^2+y^2-2xys) > \frac{1}{4}\frac{1}{(1+x)^2}. \tag{19}$$

Namely, when $y \geq 1+x$, this follows from $x^2+y^2-2xys \geq 1$. If $y \leq 1+x$, we have that

$$(x^2+y^2+2xys)^{1/2} \leq x+y \leq 2(1+x),$$

and so $\min(1, (x^2+y^2+2xys)^{1/2}) > \frac{1}{2}\frac{1}{(1+x)}$. Since $(x, y, s) \in G$, this implies (19). Now to obtain (17), observe that, if $t < (1+x)^{-2}/8$, then $1-t > \frac{7}{8}$ and

$$\begin{aligned} Q_t(x, y, s)^{1/2} &\geq (1-t)(x^2+y^2-2xys)^{1/2} - 2tx \\ &\geq \frac{7}{16}\frac{1}{1+x} - \frac{1}{4}\frac{1}{1+x} \geq \frac{3}{16}\frac{1}{1+x}, \end{aligned}$$

where the first inequality follows from the geometric interpretation of $Q_t(x, y, s)$. This concludes the proof of the claims and of the lemma.

These estimates imply that the operator $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,1}$ is of weak type 1. This result is known (see [9]), but here we give a new proof, based on Lemma 4.2 below, which will also be useful to study $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$. Let $\mathscr{T}$ be the operator on $L^1(\tilde{\mu}_\alpha)$ defined by

$$\mathscr{T} f(x) = F_\alpha(x) \int_{\mathsf{R}_+} f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y),$$

where

$$F_\alpha(x) = e^{x^2} \int_{[-1,1]} (1+x)^{2\alpha+2} \wedge (x^2(1-s^2))^{-\alpha-1} \Pi_\alpha(s)\, \mathrm{d}s.$$

LEMMA 4.2. *The operator $\mathscr{T}$ is of weak type* 1.

PROOF. It is enough to prove that the function F_α

$$x \mapsto F_\alpha(x) = e^{x^2} \int_{[-1,1]} (1+x)^{2\alpha+2} \wedge (x^2(1-s^2))^{-\alpha-1} \Pi_\alpha(s)\, \mathrm{d}s$$

is in $L^{1,\infty}(\tilde{\mu}_\alpha)$. We can choose a constant C_0, such that $\lambda > C_0$ implies that the positive zero r_0 of the function

$$\xi \mapsto e^{\xi^2}(1+\xi)^{2\alpha+2} - \lambda$$

is greater than 1. Fix $\lambda > 0$ and let $E_\lambda = \{x : F_\alpha(x) \geq \lambda\}$. We must prove that $\tilde{\mu}_\alpha(E_\lambda) \leq \frac{C}{\lambda}$. Since $\tilde{\mu}_\alpha$ is a finite measure it is enough to assume that $\lambda > C_0$. Moreover, since $F_\alpha(x) \leq e^{x^2}(1+x)^{2\alpha+2}$, E_λ does not intersect the ball $B = \{x < r_0\}$. Finally, we need to consider the intersection of E_λ with the ring $R = \{r_0 < x < 2r_0\}$ only. In fact, the elementary relation $\int_M^\infty e^{-\rho^2}\rho^{2\alpha+1}d\rho \sim e^{-M^2}M^{2\alpha}$ for $M > 1$ implies

$$\begin{aligned} \tilde{\mu}_\alpha\{x > 2r_0\} &= \int_{2r_0}^\infty \frac{2x^{2\alpha+1}}{\Gamma(\alpha+1)} e^{-x^2}\, dx \\ &\leq Ce^{-4r_0^2} r_0^{2\alpha} \leq Ce^{-r_0^2}(1+r_0)^{2\alpha} \leq C\lambda^{-1}. \end{aligned}$$

Thus we need only to estimate $\tilde{\mu}_\alpha(E_\lambda \cap R)$. Let r' be the smallest point in $(r_0, 2r_0)$, such that $r' \in E_\lambda$. Then $F_\alpha(r') = \lambda$, by continuity. Hence

$$e^{r'^2} r_0^{-2\alpha-2} \int_{[-1,1]} \big[r_0^{2(2\alpha+2)} \wedge (1-s^2)^{-\alpha-1}\big] \Pi_\alpha(s)\, \mathrm{d}s \sim \lambda$$

and

$$\begin{aligned}\tilde{\mu}_\alpha(E_\lambda \cap R) &\le C \int_{r'}^{2r_0} e^{-\rho^2} \rho^{2\alpha+1} d\rho \le C e^{-r'^2} r_0^{2\alpha} \\ &\le C\lambda^{-1} r_0^{-2} \int_{[-1,1]} \big[r_0^{2(2\alpha+2)} \wedge (1-s^2)^{-\alpha-1}\big] \Pi_\alpha(s)\, \mathrm{d}s.\end{aligned}$$

Now it suffices to observe that

$$\int_{[-1,1]} \big[r_0^{2(2\alpha+2)} \wedge (1-s^2)^{-\alpha-1}\big] \Pi_\alpha(s)\, \mathrm{d}s \le C r_0^2$$

to conclude that $\tilde{\mu}_\alpha(E_\lambda \cap R) \le C\lambda^{-1}$, as desired.

THEOREM 4.3. *The operator $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,1}$ is of weak type* 1 *and of strong type q for every q in* $(1, \infty]$.

PROOF. Since the operator is clearly bounded on $L^\infty(\tilde{\mu}_\alpha)$, it is sufficient to prove that it is of weak type 1. By (15) and Lemma 4.1(i), $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,1}$ is controlled by the operator $\mathscr{T}$. So the conclusion follows by Lemma 4.2.

Now in order to study the L^q-boundedness of $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$, for each η in $(0, 1)$, we introduce the new maximal operator

(20)
$$\mathscr{A}_{p,\eta} f(x) = \sup_{0<t\le 1-\eta} t^{-\alpha-1} \int_{G(x)} e^{\frac{x^2+y^2}{2}} e^{-\frac{\cos\phi_p}{4}\left(t(x^2+y^2+2xys)+\frac{1}{t}(x^2+y^2-2xys)\right)} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y).$$

LEMMA 4.4. *For each p in $(1, 2)$, the following hold*

(i) *if $\eta > 0$ the operator $\mathscr{A}_{p,\eta}$ is of weak type p and p';*

(ii) *the operator $\mathscr{A}_{p,0}$ is of strong type q, whenever $p < q < p'$.*

PROOF. We claim that there exists a constant C such that for every $f \ge 0$

$$\mathscr{A}_{p,\eta} f(x) \le C \min(|\mathscr{T} f^p(x)|^{1/p}, |\mathscr{T} f^{p'}(x)|^{1/p'}). \tag{21}$$

Assuming the claim for the moment, we complete the proof of the lemma. First of all, statement (i) follows easily by the proof of Theorem 4.3. Next, to prove (ii), fix r in (p, p') and let $\lambda = \cos\phi_p/\cos\phi_r > 1$, so

$$\begin{aligned} t(x^2+y^2+2xys) &+ \frac{1}{t}(x^2+y^2-2xys) \\ &= \lambda\frac{\cos\phi_r}{\cos\phi_p}\Big(t(x^2+y^2+2xys)+\frac{1}{t}(x^2+y^2-2xys)\Big) \\ &\geq \frac{\cos\phi_r}{\cos\phi_p}\Big(\frac{t}{\lambda}(x^2+y^2+2xys)+\frac{\lambda}{t}(x^2+y^2-2xys)\Big). \end{aligned}$$

Thus

$$\mathscr{A}_{p,0}f(x) \leq \lambda^{\alpha+1}\mathscr{A}_{r,1-\frac{1}{\lambda}}f(x).$$

Then (i) implies that $\mathscr{A}_{p,0}$ is of weak type r and r' for each $r \in (p, p')$. By interpolation, it is of strong type q, with $p < q < p'$.

Now it remains to verify (21). By (15)

$$\begin{aligned} &\mathscr{A}_{p,\eta}f(x) \\ &\quad\leq C \sup_{0<t\leq 1-\eta} t^{-\alpha-1}e^{x^2/p}\int_{G(x)} e^{-\frac{\delta}{t}Q_t(x,y,s)}\Pi_\alpha(s)\,\mathrm{d}s f(y)y^{2\alpha+1}e^{-y^2/p}\,\mathrm{d}y, \end{aligned}$$

where δ is a positive constant, which depends on p. Applying Hölder's inequality, we see that the right hand side is controlled by

$$C\sup_{0<t\leq 1-\eta} t^{-\alpha-1}e^{x^2/p}\left(\int_{G(x)} e^{-\frac{\delta}{t}Q_t(x,y,s)}f^p(y)\Pi_\alpha(s)\,\mathrm{d}s\,\mathrm{d}\tilde{\mu}_\alpha(y)\right)^{1/p} J_+^{1/p'}(x,t).$$

So by Lemma 4.1(ii), $\mathscr{A}_{p,\eta}f(x) \leq C(\mathscr{T}f^p(x))^{1/p}$. The second inequality follows by the same arguments, observing that

$$\begin{aligned} &\mathscr{A}_{p,\eta}f(x) \\ &\quad\leq C \sup_{0<t\leq 1-\eta} t^{-\alpha-1}e^{x^2/p'}\int_{G(x)} e^{-\frac{\delta}{t}Q_{-t}(x,y,s)}\Pi_\alpha(s)\,\mathrm{d}s f(y)y^{2\alpha+1}e^{-y^2/p'}\mathrm{d}y. \end{aligned}$$

The rest of the proof is similar.

THEOREM 4.5. *For each p in $(1, 2)$ and σ in $(0, |z_p|)$, the following hold*

1. *the operator $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p}$ is of strong type q, whenever $p < q < p'$;*
2. *the operator $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$ is of weak type p and p'.*

PROOF. Assume that $f \geq 0$. The first statement is a straightforward consequence of the fact that $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p}$ is controlled by $\mathscr{A}_{p,0}$. Namely, since τ maps the sector $\mathbf{S}_{\phi_p}$ onto the set $\mathbf{F}_p$,

$$\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p}f(x) \leq \sup_{0<t\leq 1}\int_{G(x)}\sup_{|\phi|\leq\phi_p}|\widetilde{m}_{\alpha,\tau(te^{i\phi})}(x,y,s)|f(y)\,\mathrm{d}s\,\mathrm{d}\tilde{\mu}_\alpha(y),$$

for each $f \geq 0$. Combining (15) and (20),

$$\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p} f(x) \leq \mathscr{A}_{p,0} f(x).$$

The conclusion follows by Lemma 4.4(ii).

Next we prove that $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma}$ is of weak type p and p'. For each η in $(0, 1 \wedge \phi_p)$ let $\mathbf{S}_{\phi_p-\eta}$ and $\mathbf{T}_{p,\eta}$ be the sets

$$\mathbf{S}_{\phi_p-\eta} = \{\zeta \in \mathsf{C} : |\arg\zeta| < \phi_p - \eta\},$$
$$\mathbf{T}_{p,\eta} = \{\zeta \in \mathbf{S}_{\phi_p} : |\zeta| \leq 1 - \eta\}.$$

The transformation τ maps the point $e^{i\phi_p}$ to the point z_p, such that $\tau'(e^{i\phi_p}) \neq 0$. Thus for each $\sigma > 0$ there exists a small $\eta > 0$, such that $\mathbf{F}_{p,\sigma} \subset \tau(\mathbf{S}_{\phi_p-\eta} \cup \mathbf{T}_{p,\eta})$. Let $\mathscr{B}_{p,\eta}$ be the maximal operator

$$\mathscr{B}_{p,\eta} f(x) = \sup_{0<t\leq 1} \int_{G(x)} \sup_{|\phi|\leq\phi_p-\eta} |\widetilde{m}_{\alpha,\tau(te^{i\phi})}(x, y, s)|\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y).$$

Thus,

$$\begin{aligned}\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p,\sigma} f(x) &\leq \sup_{\zeta\in\mathbf{T}_{p,\eta}} \int_{G(x)} |\widetilde{m}_{\alpha,\tau(\zeta)}(x, y, s)|\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \\ &\quad + \sup_{\zeta\in\mathbf{S}_{\phi_p-\eta}} \int_{G(x)} |\widetilde{m}_{\alpha,\tau(\zeta)}(x, y, s)|\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \\ &\leq C\mathscr{A}_{p,\eta} f(x) + \mathscr{B}_{p,\eta} f(x).\end{aligned}$$

By Lemma 4.4, we know that $\mathscr{A}_{p,\eta}$ is of weak type p and p'. Moreover $\mathscr{B}_{p,\eta} f(x) \leq C\mathscr{A}_{r,0} f(x)$, with $\phi_r = \phi_p - \eta$ and r in $(1, p)$. Thus $\mathscr{B}_{p,\eta}$ is of strong type q, whenever $r < q < r'$. Hence $\mathscr{B}_{p,\eta}$ is of strong type p and p' and the theorem is proved.

REMARK 4.6. Theorems 3.4 and 4.3 can be extended to higher dimensions. As a consequence, (1), (2) and (4) of Theorem 2.1 hold in the multidimensional case. Now we briefly describe how to prove these results. Most of the arguments require only obvious changes. In particular, we point out that the expression of the kernel $m_{\alpha,z}$ is replaced by its higher-dimensional analogues (see [10]). The local region L becomes the subset of $\mathsf{R}^d_+ \times \mathsf{R}^d_+ \times [-1, 1]^d$, defined by

$$\left\{(x,y,s) : \left(|x|^2+|y|^2-2\sum_{i=1}^d x_i y_i s_i\right) \leq \min\left(1, \frac{1}{|x|^2+|y|^2+2\sum_{i=1}^d x_i y_i s_i}\right)\right\}.$$

The strategies in the proof of these theorems are the same. So the operator $\widetilde{\mathscr{M}}^{\mathrm{loc}}_{\alpha,p}$ is controlled by

$$T^* f(x) = \sup_{t>0} |T_t^\alpha f(x)|,$$

with $T_t^\alpha f(x) = \prod_{i=1}^d T_t^{\alpha_i} f(x)$. Moreover, it is quite straightforward to adapt the proof of Lemma 3.1 to verify the weak-type (1, 1) boundedness of T^* in higher dimensions. For the global part, the operator $\mathscr{T}$ which controls $\widetilde{\mathscr{M}}^*_{\alpha,1}$ is not so simple. Its expression is

$$\begin{aligned}\mathscr{T} f(x) = e^{|x|^2} \int_{\mathsf{R}^d_+} \int_{[-1,1]^d} & (1+|x|)^{2|\alpha|+2d} \\ & \wedge \left(|x|^2\left(1-\left(\frac{\sum_{i=1}^d x_i y_i s_i}{|x||y|}\right)^2\right)\right)^{-|\alpha|-d} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y),\end{aligned}$$

where $\Pi_\alpha(s) = \prod_{i=1}^d \Pi_{\alpha_i}(s_i)$ and $\mathrm{d}\tilde{\mu}_\alpha(y)$ is the Laguerre measure on R^d_+ obtained as the tensor product of d one-dimensional Laguerre measures. So its kernel depends on x and y in a more complicated way with respect to the one-dimensional case. This choice of $\mathscr{T}$ is suggested by the comparison with the Ornstein-Uhlenbeck case. Indeed, the maximal operator $\mathscr{H}_1^*$ is controlled by the operator

$$f \mapsto e^{|x|^2} \int_{\mathsf{R}^{|n|}} (1+|x|)^{|n|} \wedge (|x| \sin\theta)^{-|n|} f(y)\, \mathrm{d}\gamma(y), \tag{22}$$

where $\theta = \theta(x, y)$ is the angle between the vectors x and y [6].

Observe that when $\alpha = (\frac{n_1}{2} - 1 \ldots, \frac{n_d}{2} - 1)$, with $n_i \in \mathsf{N}$ and $n_i > 1$ for each $i = 1, \ldots, d$ and $|n| = n_1 + \cdots + n_d$, the operator $\mathscr{T}$ coincides with the operator defined in (22), acting on a polyradial function.

We set $\cos\theta = \frac{\sum_{i=1}^d x_i y_i s_i}{|x||y|}$ and $\sin\theta = \left(1 - \left(\frac{\sum_{i=1}^d x_i y_i s_i}{|x||y|}\right)^2\right)^{1/2}$, which can be interpreted as the expression of the cosine and the sine of the angle between two vectors by means of polyradial coordinates.

Now to prove that $\mathscr{T}$ is of weak type (1, 1), we use an adaptation of proof of Lemma 4.2. Assume that $\|f\|_1 = 1$. Choose a constant C_0, such that $\lambda > C_0$ implies that the positive zero r_0 of the function

$$\xi \mapsto e^{\xi^2}(1+\xi)^{2|\alpha|+2d} - \lambda$$

is greater than 1. Fix $\lambda > 0$ and let $E_\lambda = \{x \in \mathsf{R}^d_+ : \mathscr{T} f(x) \geq \lambda\}$. As in one dimensional case, to prove that $\tilde{\mu}_\alpha(E_\lambda) \leq \frac{C}{\lambda}$, it is sufficient to estimate the measure of the intersection of E_λ with the ring $R = \{r_0 \leq |x| \leq 2r_0\}$ and

$\lambda > C_0$. Write $x = \rho x'$, with $\rho > 0$ and $|x'| \in S^{d-1}$. We let E' denote the set of $x' \in S^{d-1}$ for which there exists a $\rho \in [r_0, 2r_0]$ with $\rho x' \in E$. For each $x' \in E'$ we let $r(x')$ be the smallest such ρ. Then $\mathscr{T} f(r(x')x')(r') = \lambda$, by continuity. Hence

$$e^{r(x')^2} r_0^{-2|\alpha|-2d} \int_{\mathsf{R}_+^d} \int_{[-1,1]^d} r_0^{2(2|\alpha|+2d)} \wedge (\sin\theta)^{-2|\alpha|-2d} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \sim \lambda.$$

Clearly

$$\begin{aligned}\tilde{\mu}_\alpha(E_\lambda \cap R) \\ &\le C\lambda^{-1} r_0^{-2} \int_{\mathsf{R}_+^d} \int_{[-1,1]^d} r_0^{2(2|\alpha|+2d)} \wedge (\sin\theta)^{-2|\alpha|-2d} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y).\end{aligned}$$

Now, it suffices to prove that

$$\int_{\mathsf{R}_+^d} \int_{A(x,y)} r_0^{2(2|\alpha|+2d)} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \le C r_0^2,$$

$$\int_{\mathsf{R}_+^d} \int_{A(x,y)^c} (\sin\theta)^{-2|\alpha|-2d} \Pi_\alpha(s)\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y) \le C r_0^2,$$

with $A(x, y) = \{s \in -1, 1^d : r_0^2 < (\sin\theta)^{-1}\}$. First one considers the case where $\alpha = (\frac{n_1}{2} - 1, \dots, \frac{n_d}{2} - 1)$, with $n_i \in \mathsf{N}$ and $n_i > 1$ for each $i = 1, \dots, d$. In this case the integrals can be interpreted as integrals over

$$\mathsf{R}^{|n|} = (\mathsf{R}_+ \times S^{n_1-1}) \times \cdots \times (\mathsf{R}_+ \times S^{n_d-1})$$

with respect to the Gaussian measure, in polyradial coordinates. In these cases, the desired estimates can be found in [6, Lemma 4.3]. The same estimates are obtained also $\alpha \in \frac{\mathsf{N}^d}{2} - 1 + i\mathsf{R}^d$. Finally the result for the other values of α are obtained via the multidimensional extension of Stein's complex interpolation theorem [1, Appendix A] (for more details, see [10, Theorem 3]). Now Theorem 4.5 follows by a simple adaptation of the one dimensional case.

Finally, we give our last positive result.

THEOREM 4.7. *For each $\alpha \ge 0$ and $p < \frac{2\alpha+2}{\alpha+3/2}$, the operator $\widetilde{\mathscr{M}}^{\mathrm{gl}}_{\alpha,p}$ is of weak type p.*

PROOF. By Theorem 4.5, we only need to prove that for any fixed σ, with $0 < \sigma < |z_p|$, the operator

$$\mathscr{N}_{p,\sigma} f(x) = \sup_{\zeta \in \mathbf{F}_p \setminus \mathbf{F}_{p,\sigma}} \int_{G(x)} |\widetilde{m}_{\alpha,\tau(\zeta)}(x, y, s)|\, \mathrm{d}s f(y)\, \mathrm{d}\tilde{\mu}_\alpha(y)$$

is of weak type p. This is equivalent to study the boundedness from $L^p(m_\alpha)$ to $L^{p,\infty}(\tilde{\mu}_\alpha)$ of the operator $\mathcal{N}^*_{p,\sigma} = \mathcal{N}_{p,\sigma}\Psi_p^{-1}$, where Ψ_p is the isometry from $L^p(\tilde{\mu}_\alpha)$ to $L^p(m_\alpha)$, defined by

$$\Psi_p f(x) = f(x)e^{-x^2/p} \tag{23}$$

and m_α is the measure, given in (7). Now, assume that $f \geq 0$ and $\|f\|_{L^p(m_\alpha)} = 1$. There exists a positive angle $\bar{\phi}$, such that

$$\mathcal{N}^*_{p,\sigma} f(x) \leq C \sup_{1-\sigma\leq t\leq 1} \sup_{\bar{\phi}<\phi\leq\phi_p} \bigg| \int_{G(x)} \exp\bigg[-\frac{y^2}{p'} + \frac{x^2+y^2}{2} - \frac{1}{4}\Big(te^{i\phi} + \frac{1}{t}e^{-i\phi}\Big)$$
$$(x^2+y^2)\bigg] J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)(ixy)^{-\alpha} f(y)\,\mathrm{d}m_\alpha(y)\bigg|.$$

Using the identities $1/p - 1/2 = 1/2 - 1/p' = \frac{\cos\phi_p}{2}$, we have

$$\mathcal{N}^*_{p,\sigma} f(x) \leq C \sup_{1-\sigma\leq t\leq 1} \int_{\mathsf{R}_+} \sup_{\bar{\phi}<|\phi|\leq\phi_p} \exp\left[-\frac{\cos\phi}{4t}((t-1)y + (t+1)x)^2\right]$$
$$\exp\left(-\frac{y^2}{2}(\cos\phi - \cos\phi_p)\right)\exp\left(\frac{x^2}{2}(1+\cos\phi)\right)\exp\left(-\frac{\cos\phi}{2t}(1-t^2)xy\right)$$
$$\left|J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)\right|(xy)^{-\alpha} f(y)\,\mathrm{d}m_\alpha(y).$$

The function E defined by

$$E(\phi) = e^{-\frac{\cos\phi}{4t}((t-1)y+(t+1)x)^2} e^{-\frac{y^2}{2}(\cos\phi-\cos\phi_p)} e^{\frac{x^2}{2}(1+\cos\phi)},$$

is increasing on $(0,\pi)$. Indeed

$$E'(\phi) = \sin\phi E(\phi)\left(\frac{1}{4t}(x-y)^2 + \frac{t}{4}(x+y)^2\right) > 0.$$

Thus $E(\phi) \leq E(\phi_p)$ for $|\phi| \leq \phi_p$, thus

$$\mathcal{N}^*_{p,\sigma} f(x) \leq C \sup_{1-\sigma\leq t\leq 1} e^{\frac{x^2}{p}} \int_{\mathsf{R}_+} \exp\left[-\frac{\cos\phi_p}{4t}((t-1)y + (t+1)x)^2\right] \tag{24}$$
$$\left\{\sup_{\bar{\phi}<\phi\leq\phi_p} e^{-\frac{\cos\phi}{2t}(1-t^2)xy}\left|J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)\right|\right\}(xy)^{-\alpha} f(y)\,\mathrm{d}m_\alpha(y).$$

Let Z be the region defined by

$$Z = \left\{ (x, y) \in \mathsf{R}_+ \times \mathsf{R}_+ : y \le \frac{1-\sigma}{\sigma} x \right\}$$

and let Z^c be its complement. We may control $\mathcal{N}^*_{p,\sigma}$ by the maximal operators $\mathcal{N}_Z$ and $\mathcal{N}_{Z^c}$ whose kernels are the product of the kernel of $\mathcal{N}^*_{p,\sigma}$ and the characteristic function of Z and Z^c respectively. Thus

$$\mathcal{N}^*_{p,\sigma} \le \mathcal{N}_Z + \mathcal{N}_{Z^c}.$$

Next we study these operators separately. First we analyse $\mathcal{N}_Z$. Since $(t-1)y + (t+1)x \ge x$ for all (x, y) in Z and t in $[1-\sigma, 1]$, $\mathcal{N}_Z f(x)$ is bounded by

$$C e^{x^2\left(\frac{1}{p} - \frac{\cos\phi_p}{4}\right)} \sup_{1-\sigma \le t \le 1} \int \left\{ \sup_{\bar\phi < |\phi| \le \phi_p} e^{\left(-\frac{\cos\phi}{2t}(1-t^2)xy\right)} \left| J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)\right| \right\} (xy)^{-\alpha} y^{2\alpha+1} f(y)\, \mathrm{d}y.$$

Applying Hölder's inequality

$$\mathcal{N}_Z f(x) \le C e^{x^2\left(\frac{1}{p} - \frac{\cos\phi_p}{4}\right)} \sup_{1-\sigma \le t \le 1} \left(\int \left\{ \sup_{\bar\phi < |\phi| \le \phi_p} e^{\left(-\frac{\cos\phi}{2t}(1-t^2)xy\right)} \left| J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)\right| \right\}^{p'} (xy)^{-\alpha p'}\, \mathrm{d}m_\alpha(y) \right)^{1/p'}.$$

We claim that

$$\sup_{1-\sigma \le t \le 1} \left(\int \left\{ \sup_{\bar\phi < |\phi| \le \phi_p} e^{\left(-\frac{\cos\phi}{2t}(1-t^2)xy\right)} \left| J_\alpha\left(\frac{ixy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right)\right| \right\}^{p'} (xy)^{-\alpha p'}\, \mathrm{d}m_\alpha(y) \right)^{1/p'} \le C x^{-\frac{2\alpha+2}{p'}}. \tag{25}$$

Assuming the claim, it follows that $\mathcal{N}_Z$ is of strong type p, because $x^{-\frac{2\alpha+2}{p'}} e^{x^2\left(\frac{1}{p} - \frac{\cos\phi_p}{4}\right)}$ is in $L^p(\tilde\mu_\alpha)$. It only remains to prove the claim. By the asymptotic behaviour of Bessel functions, there exists a $R > 0$, such that for each complex number w, with $|w| > R$

$$e^{-|\Im w|}\, |J_\alpha(w)| \le C|w|^{-1/2}. \tag{26}$$

Moreover $e^{-|\Im w|}\,|J_\alpha(w)| \le C|w|^\alpha$, for each $w \in \mathsf{C}$ (see [4]). The change of variable, $xy = z$, and the assumption $p < \frac{2\alpha+2}{\alpha+3/2}$ now yields (25).

Next we prove the weak boundedness of $\mathcal{N}_{Z^c}$. Observe that in Z^c, $y > \frac{1-\sigma}{\sigma}x$. Let $E_\lambda = \{x \in \mathsf{R}_+ : \mathcal{N}_{Z^c} f(x) \ge \lambda\}$. We want to show that $\sup_{\lambda>0} \lambda^p \tilde{\mu}_\alpha(E_\lambda) < \infty$. We claim that there exists a constant $C(\bar{\phi})$ such that

$$(27)\qquad \mathcal{N}_{Z^c} f(x) \le \begin{cases} Ce^{x^2/p}x^{-\frac{2\alpha+2}{p'}} & \text{for all } x > 0 \\ Ce^{x^2/p}x^{-\frac{2\alpha+2}{p}+1} & \text{for all } x \in [C(\bar{\phi}), \infty). \end{cases}$$

Indeed the first estimate may be proved by an argument similar to the one used in the estimate of $\mathcal{N}_Z f(x)$. To prove the second estimate, we observe that $y > \frac{1-\sigma}{\sigma}x$ in Z^c. Hence, for $1-\sigma < t \le 1$ and $\bar{\phi} \le \phi \le \phi_p$,

$$\left|\frac{xy}{2}\left(\frac{1}{t}e^{-i\phi} - te^{i\phi}\right)\right| = \frac{1-\sigma}{2\sigma}\frac{x^2}{t}(t^4 - 2t^2\cos 2\bar{\phi} + 1)^{1/2} > R,$$

if x is sufficiently large, say $x \ge C(\bar{\phi})$. thus by Hölder's inequality and (26)

$$\mathcal{N}_{Z^c} f(x) \le Ce^{x^2/p}\left(\int_{\frac{1-\sigma}{\sigma}x}^\infty (xy)^{(-\alpha-1/2)p'} y^{2\alpha+1}\,\mathrm{d}y\right)^{1/p'} \le Ce^{x^2/p}x^{-\frac{2\alpha+2}{p}+1},$$

for all $x \ge C(\bar{\phi})$. This proves (27). Now, let x_{min} be the minimum of the function $x \mapsto e^{x^2/p}x^{-\frac{2\alpha+2}{p}+1}$ and choose a constant $C > \max(C(\bar{\phi}), 1, x_{min})$. Since the function $x \mapsto e^{x^2/p}x^{-\frac{2\alpha+2}{p'}}\chi_{[0,C]}(x)$ is in $L^p(\tilde{\mu}_\alpha)$, we only need to check that the function

$$H(x) = e^{x^2/p}x^{-\frac{2\alpha+2}{p}+1}\chi_{[C,\infty)}(x)$$

is in $L^{p,\infty}(\tilde{\mu}_\alpha)$. For each $\lambda > H(C)$, let $x_\lambda > C$ be such that $H(x_\lambda) = \lambda$. Then

$$\tilde{\mu}_\alpha(x : H(x) > \lambda) = \int_{x_\lambda}^\infty \mathrm{d}\tilde{\mu}_\alpha(x) \le C(\alpha)x_\lambda^{2\alpha}e^{-x_\lambda^2} \le C\lambda^{-p},$$

because $p < 2$ and $x_\lambda \ge C > 1$. This concludes the proof.

5. Negative results

In this section we shall prove that $\widetilde{\mathcal{M}}^*_{\alpha,p,\sigma}$, with σ in $(0, |z_p|)$, is unbounded on $L^p(\tilde{\mu}_\alpha)$, whenever $1 \le p < 2$ and that $\widetilde{\mathcal{M}}^*_{\alpha,2,\sigma}$ is not of weak type $(2, 2)$, whenever $\alpha \notin \frac{2\mathsf{N}-1}{2}$. Finally we shall also prove that $\widetilde{\mathcal{M}}^*_{\alpha,p}$ is not of weak type p, for $p > \frac{2\alpha+2}{\alpha+3/2}$.

We start noticing that for each σ, with $0 < \sigma < |z_p|$, the set $\mathbf{F}_{p,\sigma}$ contains the arc $\{\tau(te^{i\phi_p}) : a \leq t \leq b\}$, with $0 < a < b < 1$. Since $\widetilde{\mathscr{M}}^*_{\alpha,p,\sigma}$ is bounded from below by the operator

$$f \mapsto \widetilde{\mathscr{M}}^*_{p,a,b} f = \sup_{a \leq t \leq b} |\widetilde{M}^\alpha_{\tau(te^{i\phi_p})} f|,$$

it is enough to prove that the latter operator is not of strong type p.

In the following we use the measure m_α on R_+ defined in (7).

THEOREM 5.1. *The operator $\widetilde{\mathscr{M}}^*_{p,a,b}$ is unbounded on $L^p(\tilde{\mu}_\alpha)$, $1 \leq p < 2$, for each $0 < a < b < 1$.*

PROOF. Let Ψ_p be the isometry defined in (23). We shall prove that the operator $\mathscr{W}_{p,a,b} = \Psi_p \widetilde{\mathscr{M}}^*_{p,a,b} \Psi_p^{-1}$ is unbounded on $L^p(m_\alpha)$. Now

$$\mathscr{W}_{p,a,b} g(x) \geq C \sup_{a \leq t \leq b} \left| \int k_{te^{i\phi_p}}(x, y) g(y) \, \mathrm{d}m_\alpha(y) \right|,$$

where

$$k_\zeta(x, y) = \exp\left(-(x^2 + y^2) \frac{(1-\zeta)^2}{4\zeta} - \frac{y^2}{p'} - \frac{x^2}{p} \right)$$
$$\left(\frac{ixy}{2} \left(\frac{1}{\zeta} - \zeta \right) \right)^{-\alpha} J_\alpha \left(\frac{ixy}{2} \left(\frac{1}{\zeta} - \zeta \right) \right).$$

By the asymptotic behaviour of Bessel functions

$$J_\alpha(iz) = (2\pi z)^{-1/2} \big[e^{-z - i(\frac{\alpha}{2}\pi + \frac{\pi}{4})}(1 + R(z)) + e^{z + i(\frac{\alpha}{2}\pi + \frac{\pi}{4})}(1 + R(-z)) \big], \tag{28}$$

with $R(z) = O(|z|^{-1})$ as $z \to \infty$, (see, for instance, [4]). Thus, since $\frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p'} = \frac{\cos\phi_p}{2}$, the kernel $k_{te^{i\phi_p}}$ can be written as

$$k_{te^{i\phi_p}}(x, y) = \exp(q_{te^{i\phi_p}}(x, y)) G_\alpha \left(\frac{ixy}{2} \left(\frac{e^{-i\phi}}{t} - te^{i\phi} \right) \right),$$

with $q_\zeta(x, y) = \frac{1}{2} \cos\phi_p (y^2 - x^2) - \frac{1}{4}(\zeta(x+y)^2 + \zeta^{-1}(x-y)^2)$ and

$$\begin{aligned} G_\alpha(iz) &= (iz)^{-\alpha} J_\alpha(iz) e^{-z} \\ &= (2\pi)^{-1/2} (z)^{-\alpha - 1/2} [e^{-2z - i(\frac{\alpha}{2}\pi + \frac{\pi}{4})}(1 + R(z)) + e^{i(\frac{\alpha}{2}\pi + \frac{\pi}{4})}(1 + R(-z))]. \end{aligned} \tag{29}$$

Fix a smooth function ϕ such that $\phi(0) = 1$ and with support contained in the interval $[-1, 1]$. For $y_0 \geq 2$ and $1/p' < \delta < 1/p$, let $g(y) = |y - y_0|^{-\delta} \phi(y - y_0)$.

Consider first the case $1 < p < 2$. We shall prove that there exist positive constants c, C_1, C_2, with $C_1 < C_2$, such that if y_0 is large and $x \in [C_1 y_0, C_2 y_0]$ then

$$|\mathscr{W}_{p,a,b} g(x)| \geq c y_0^{\delta-1}(1 + O(y_0^{-1})). \tag{30}$$

Assuming for the moment that (30) holds, for y_0 large, we have that $\|\mathscr{W}_{p,a,b} g(x)\|_{L^p(m_\alpha)} \geq C y_0^{\delta-\frac{1}{p'}} y_0^{\frac{2\alpha+1}{p}}$. Since $\|g\|_{L^p(m_\alpha)} \leq C y_0^{\frac{2\alpha+1}{p}}$, the quotient $\|\mathscr{W}_{p,a,b} g\|_{L^p(m_\alpha)}/\|g\|_{L^p(m_\alpha)}$ diverges if we let y_0 tend to infinity. It remains to prove (30). Define $t_x = \frac{y_0-x}{y_0+x}$ and choose two constants C_1 and C_2 such that $(1 - b)(1+b)^{-1} \leq C_1 < C_2 \leq (1-a)(1+a)^{-1}$. Then $a \leq t_x \leq b$, for all x in $[C_1 y_0, C_2 y_0]$. Let $\mathscr{Q}(x, y) = q_{t_x e^{i\phi_p}}(x, y)$. Then

$$\begin{aligned} |\mathscr{W}_{p,a,b} g(x)| & \\ \geq C & \left| \int \exp \mathscr{Q}(x,y) G_\alpha\left(\frac{ixy}{2}\left(\frac{e^{-i\phi_p}}{t_x} - t_x e^{i\phi_p}\right)\right) g(y)\,\mathrm{d}m_\alpha(y)\right|. \end{aligned} \tag{31}$$

Write $\mathscr{Q}(x, y) = \mathscr{R}(x, y) + i\mathscr{I}(x, y)$, with $\mathscr{R}$ and $\mathscr{I}$ real. The functions $\mathscr{R}$ and $\mathscr{I}$ are quadratic polynomials in y. Let

$$\mathscr{R}(x, y) = a_0(x) + a_1(x)(y - y_0) + a_2(x)(y - y_0)^2, \tag{32}$$

$$\mathscr{I}(x, y) = b_0(x) + b_1(x)(y - y_0) + b_2(x)(y - y_0)^2 \tag{33}$$

be their expansions in powers of $y - y_0$. Observe that

$$\begin{aligned} a_0(x) = a_1(x) = b_0(x) = 0,& \\ b_1(x) = x \sin \phi_p,& \end{aligned}$$

and $|a_2(x)| + |b_2(x)| \leq C$ for all x in $[C_1 y_0, C_2 y_0]$. Now by (32), (33) and by the change of variable $y = u + y_0$, the right hand side of (31) is equal to

$$C \left| \int e^{ib_1(x)u} |u|^{-\delta} \varphi(x, u)\,\mathrm{d}u \right|,$$

where

$$\begin{aligned} &\varphi(x, u) \\ &\quad = e^{(a_2(x)+ib_2(x))u^2} \phi(u) G_\alpha\left(\frac{ix(u+y_0)}{2}\left(\frac{e^{-i\phi_p}}{t_x} - t_x e^{i\phi_p}\right)\right)(u+y_0)^{2\alpha+1}, \end{aligned}$$

and $|\varphi(x, 0)| = 1 + O(y_0^{-2})$ for y_0 large. By [6, Lemma 5.2]

$$\int e^{ib_1(x)u} |u|^{-\delta} \varphi(x, u)\,\mathrm{d}u = C\varphi(x, 0)|b_1(x)|^{\delta-1} + E_\delta(b_1(x), \varphi),$$

with $E_\delta(b_1(x),\varphi)\le C|b(x)|^{\delta-2}(\|\varphi''\|_{L^1(\mathrm{d}x)}+\|\varphi\|_{H_2^2})$. Now it is quite easy to prove that

$$|\mathscr{W}_{p,a,b}g(x)|\ge C|x|^{\delta-1}(1+O(y_0^{-1})),$$

as y_0 tends to ∞ uniformly for x in $[C_1y_0, C_2, y_0]$. Thus (30) is satisfied.

It remains to consider the case $p=1$. Remark that in this case $\mathscr{Q}$ is real. By arguing as in the previous case, we are led to

$$\left|\int \exp\mathscr{Q}(x,y)G_\alpha\left(\frac{ixy}{2}(t_x^{-1}-t_x)\right)g(y)\,\mathrm{d}m_\alpha(y)\right| \\ = C\left|\int \exp(a_2(x)u^2)|u|^{-\delta}\varphi(x,u)\,\mathrm{d}u\right|,$$

where

$$\varphi(x,u)=\phi(u)G_\alpha\left(\frac{ix(u+y_0)}{2}(t_x^{-1}-t_x)\right),$$

because b_1 and b_2 vanish identically. Moreover, a_2 is uniformly bounded. Since ϕ has compact support and $\varphi(x,u)$ is bounded from below, there exists a positive constant C, such that

$$\left|\int \exp(a_2(x)u^2)|u|^{-\delta}\varphi(u)\,\mathrm{d}u\right|\ge C,\qquad x\in[C_1y_0, C_2y_0],$$

for large y_0. Thus the unboundedness of $\mathscr{W}_{1,a,b}$ on $L^1(m_\alpha)$ follows from

$$\frac{\|\mathscr{W}_{1,a,b}g\|_{L^1(m_\alpha)}}{\|g\|_{L^1(m_\alpha)}}\ge Cy_0,$$

when y_0 tends to infinity. This concludes the proof of the theorem.

To prove the unboundedness of $\widetilde{\mathscr{M}}^*_{\alpha,2,\sigma}$, we introduce a technical lemma.

LEMMA 5.2. *Assume that* $a\in\mathbb{C}\setminus\{\pm1,\pm i\}$ *and* $|a|=1$. *There exists a function h in $C_c(\mathbb{R}_+)$ such that*

$$\limsup_{t\to0^+} t^{-1/2}\left|\int_{\mathbb{R}_+}\left[ae^{i\frac{(x-y)^2}{t}}+\bar a e^{i\frac{(x+y)^2}{t}}\right]h(y)\,\mathrm{d}y\right|=\infty,$$

for all x in a measurable subset A of $\mathbb{R}$ of positive measure.

PROOF. By a result of Carleson [2], there exists a continuous function g, with compact support contained in $\mathbb{R}_+$ such that

$$\limsup_{t\to0^+} t^{-1/2}\left|\int_{\mathbb{R}} e^{i\frac{(x-y)^2}{t}}g(y)\,\mathrm{d}y\right|=\infty, \tag{34}$$

for each x in R. Let $p(y) = 1/2(g(y) + g(-y))$ and $d(y) = 1/2(g(y) - g(-y))$. Let B the measurable set of all x in R such that

$$\limsup_{t\to 0^+} t^{-1/2}\left|\int_{\mathsf{R}} e^{i\frac{(x-y)^2}{t}} p(y)\,\mathrm{d}y\right| = \infty,$$

Since p is even, the set B is symmetric. Moreover, since $g = p + d$, by (34)

$$\limsup_{t\to 0^+} t^{-1/2}\left|\int_{\mathsf{R}} e^{i\frac{(x-y)^2}{t}} d(y)\,\mathrm{d}y\right| = \infty, \qquad x \in \mathsf{R}\setminus B.$$

Assume first that $\tilde{\mu}_\alpha(B) \neq 0$. Since $a \neq \pm i$, then $a + \bar{a} \neq 0$ and

$$\limsup_{t\to 0^+} t^{-1/2}\left|\int_{\mathsf{R}} e^{i\frac{(x-y)^2}{t}} (a+\bar{a}) p(y)\,\mathrm{d}y\right| = \infty, \qquad x \in B.$$

Let C the measurable set of all x in B such that

$$\limsup_{t\to 0^+} t^{-1/2}\left|\int_{\mathsf{R}_+} \left[a e^{i\frac{(x-y)^2}{t}} + \bar{a} e^{i\frac{(x+y)^2}{t}}\right] p(y)\,\mathrm{d}y\right| = \infty.$$

Since

$$\begin{aligned}(a+\bar{a})\int_{\mathsf{R}} e^{i\frac{(x-y)^2}{t}} p(y)\,\mathrm{d}y &= \int_{\mathsf{R}_+} \left[a e^{i\frac{(x-y)^2}{t}} + \bar{a} e^{i\frac{(x+y)^2}{t}}\right] p(y)\,\mathrm{d}y \\ &\quad + \int_{\mathsf{R}_+} \left[\bar{a} e^{i\frac{(x-y)^2}{t}} + a e^{i\frac{(x+y)^2}{t}}\right] p(y)\,\mathrm{d}y,\end{aligned}$$

we have that

$$\limsup_{t\to 0^+} t^{-1/2}\left|\int_{\mathsf{R}_+} \left[\bar{a} e^{i\frac{(x-y)^2}{t}} + a e^{i\frac{(x+y)^2}{t}}\right] p(y)\,\mathrm{d}y\right| = \infty, \qquad x \in B\setminus C. \tag{35}$$

Now if $\tilde{\mu}_\alpha(C) \neq 0$, the conclusion follows by choosing $h = p$ and $A = C$. On the other hand, if $\tilde{\mu}_\alpha(C) = 0$, by (35) and the symmetry of B, the lemma is proved setting again $h = p$ and $A = B$.

Next, if $\tilde{\mu}_\alpha(B) = 0$, then we may conclude with similar arguments by choosing $h = d$ and using that for $a \neq \pm 1$ then $a - \bar{a} \neq 0$.

THEOREM 5.3. *Suppose that $\alpha \notin \frac{2\mathsf{N}-1}{2}$. The operator $\widetilde{\mathscr{M}}^*_{\alpha,2,\sigma}$, with $0 \leq \sigma < \pi$, is not of weak type 2.*

PROOF. Observe that the region $\mathbf{E}_2$ contains the imaginary axis. Thus it suffices to find a continuous function f with compact support such that

$$\limsup_{t\to 0^+}\left|\int \widetilde{m}_{\alpha,2it}(x,y) f(y)\,\mathrm{d}\tilde{\mu}_\alpha(y)\right| = \infty, \tag{36}$$

for all x ia a subset of R_+ of positive measure. Moreover, since $m_{\alpha,t}(x, y) = m_{\alpha,t}(-x, y)$ it is enough to show that (36) holds for all x in a measurable subset A of R, of positive measure. Since $\tau^{-1}(2it) = i\tan(t/2)$, we have

$$\begin{aligned}\widetilde{m}_{\alpha,2it}(x, y) = {} & C_\alpha(i\tan t/2)^{-1-\alpha}(1 + i\tan t/2)^{2\alpha+2} e^{\frac{x^2+y^2}{2}} e^{-i\frac{x^2+y^2}{4}\left(\tan t/2 - \frac{1}{\tan t/2}\right)} \\ & \left(\frac{xy}{2}\left(\tan t/2 + \frac{1}{\tan t/2}\right)\right)^{-\alpha} J_\alpha\left(\frac{xy}{2}\left(\tan t/2 + \frac{1}{\tan t/2}\right)\right) \\ = {} & C_\alpha(i\tan t/2)^{-1}(1 + O(t)) e^{\frac{x^2+y^2}{2}} e^{-i\frac{x^2+y^2}{4}\left(\tan t/2 - \frac{1}{\tan t/2}\right)} \\ & \left(\frac{xy}{2}\right)^{-\alpha} J_\alpha\left(\frac{xy}{2}\left(\tan t/2 + \frac{1}{\tan t/2}\right)\right),\end{aligned}$$

as $t \to 0^+$. Observe that if t tends to 0^+, then $\left(\frac{xy}{2}\left(\tan t/2 + \frac{1}{\tan t/2}\right)\right) \to +\infty$, for a.e. $x, y \in \mathsf{R}_+$. By (28), we may write

$$\begin{aligned}\widetilde{m}_{\alpha,2it}(x, y) = {} & C_\alpha(i\tan t/2)^{-1/2}(1 + O(t)) e^{\frac{x^2+y^2}{2}} e^{-i\frac{x^2+y^2}{4}\left(\tan t/2 - \frac{1}{\tan t/2}\right)} \\ \left(\frac{xy}{2}\right)^{-\alpha-1/2} & \left[a e^{i\frac{xy}{2}(\tan^{-1} t/2 + \tan t/2)}(1+O(t)) + \bar{a} e^{-i\frac{xy}{2}(\tan^{-1} t/2 + \tan t/2)}(1+O(t))\right],\end{aligned}$$

as $t \to 0^+$, with $a = e^{-\pi i(\alpha/2+1/4)}$. Thus

$$\begin{aligned}|\widetilde{\mathcal{M}}^*_{\alpha,2,\sigma} f(x)| \geq {} & \limsup_{t\to 0} C x^{-\alpha-1/2} e^{x^2/2} t^{-1/2} \\ & \left|\int_{\mathsf{R}}\left(a e^{i\frac{(x-y)^2}{t}} + \bar{a} e^{i\frac{(x+y)^2}{t}}\right) f(y) y^{\alpha+1/2} e^{-y^2/2}\, dy\right|.\end{aligned}$$

By Lemma 5.2, the conclusion follows by choosing $f(y) = h(y)e^{y^2} y^{-\alpha-1/2}$, since $a \neq \pm 1, \pm i$ whenever $\alpha \notin \frac{2\mathsf{N}-1}{2}$.

REMARK 5.4. By the proof of Lemma 5.2, Theorem 5.3 is still true either for $\alpha \in 2\mathsf{N} - 1/2$ or for $\alpha \in 2\mathsf{N} + 1/2$, which correspond to the values $a = \pm 1$ or $a = \pm i$ respectively.

At last we study the weak type p unboundedness of $\widetilde{\mathcal{M}}^*_{\alpha,p}$. Observe that for $\alpha \notin \frac{2\mathsf{N}-1}{2}$, we can extend the following result to the case $p = 2$, because $\widetilde{\mathcal{M}}^*_{\alpha,2,\sigma}$ is not of weak type 2 and $\widetilde{\mathcal{M}}^*_{\alpha,p} f > \widetilde{\mathcal{M}}^*_{\alpha,2,\sigma} f$ for each nonnegative function f.

THEOREM 5.5. *The operator $\widetilde{\mathcal{M}}^*_{\alpha,p}$ is not of weak type p for any p in* $(1, 2)$ *and* $p > \frac{2\alpha+2}{\alpha+3/2}$.

PROOF. Since the maximal operator $\widetilde{\mathscr{M}}^*_{\alpha,p}$ is bounded from below by the operator

$$f \mapsto \mathscr{N}_{p,\varepsilon} f = \sup_{1-\varepsilon \le t < 1} |\widetilde{M}^\alpha_{\tau(t^{e^{i\phi_p}})} f|,$$

for any $\varepsilon \in [0, 1]$, we only need to study $\mathscr{N}_{p,\varepsilon}$. We composed $\mathscr{N}_{p,\varepsilon}$ with the isometry Ψ_p, defined in (23). Therefore it suffices to prove that the operator $\mathscr{N}^*_{p,\varepsilon} = \mathscr{N}_{p,\varepsilon}\Psi_p^{-1}$ is unbounded from $L^p(m_\alpha)$ to $L^{p,\infty}(\tilde{\mu}_\alpha)$. Assume that for any x_0, y_0, with x_0 sufficiently large and $y_0 \ge x_0^\delta$, for $\delta > 0$ fixed, there exists a function g such that

$$\|g\|_{L^p(m_\alpha)} \le C \left(\frac{y_0}{x_0}\right)^{1/p} y_0^{\frac{2\alpha+1}{p}}, \tag{37}$$

$$|\mathscr{N}^*_{p,\varepsilon} g(x)| \ge C e^{x_0^2/p} \left(\frac{y_0}{x_0}\right)^{\alpha+3/2}, \qquad \forall x \in \left[x_0, x_0 + \frac{1}{x_0}\right]. \tag{38}$$

Assuming the claims for the moment, we conclude the proof. If we suppose that $\mathscr{N}^*_{p,\varepsilon}$ is bounded from $L^p(m_\alpha)$ to $L^{p,\infty}(\tilde{\mu}_\alpha)$, for some $p > \frac{2\alpha+2}{\alpha+3/2}$, then

$$\begin{aligned}
x_0^{2\alpha} e^{-x_0^2} &\le C\tilde{\mu}_\alpha \left\{\left[x_0, x_0 + \frac{1}{x_0}\right]\right\} \\
&\le C\tilde{\mu}_\alpha \left\{x : |\mathscr{N}^*_{p,\varepsilon} g(x)| \ge C e^{x_0^2/p} \left(\frac{y_0}{x_0}\right)^{\alpha+3/2}\right\} \\
&\le C e^{-x_0^2} \left(\frac{y_0}{x_0}\right)^{-(\alpha+3/2)p} \|g\|_{L^p(m_\alpha)}.
\end{aligned}$$

Choosing $y_0 = x_0^\delta$, with

$$\delta > \frac{(\alpha + 3/2)p - 2\alpha - 1}{(\alpha + 3/2)p - 2\alpha - 2}$$

and letting y_0 tend to infinity, we find a contradiction. We now get back to the claims. Arguing as in the proof of Theorem 5.1, for each function g and for every x in $\left[x_0, x_0 + \frac{1}{x_0}\right]$,

$$|\mathscr{N}^*_{p,\varepsilon} g(x)| \ge C e^{x_0^2/p} \left| \int \exp q_{t_x e^{i\phi_p}}(x, y)(xy)^{-\alpha-1/2} G'_\alpha\left(\frac{ixy}{2}(t_x^{-1} e^{-i\phi_p} - t_x e^{i\phi_p})\right) g(y)\, \mathrm{d}m_\alpha(y) \right|. \tag{39}$$

Here $G'_\alpha(z) = z^{\alpha+1/2}G_\alpha(z)$ and G_α is defined in (29). In particular, we may choose

$$t_x = \frac{\sqrt{y_0^2 + x_0^2 - x^2} - x_0}{y_0 + x},$$

such that $1 - \varepsilon \leq t_x < 1$, for each x in $\left[x_0, x_0 + \frac{1}{x_0}\right]$ and $x_0 \to +\infty$. Next we choose g as follows

$$g(y+y_0) = \chi_{[-1,1]}\left(\left(\frac{x_0}{y_0}\right)y\right)\exp\left(-i\sin\phi_p\left(yx_0 + \left(\frac{x_0}{y_0} + \frac{1}{2x_0y_0}\right)y^2\right)\right).$$

An easy calculation implies the estimate (37) of the $L^p(m_\alpha)$–norm of g. Moreover (38) follows from the asymptotic estimates of the kernel of the right hand side of (39). Namely, setting $u = \frac{x_0}{y_0}(y - y_0)$, we may write

$$\begin{aligned}|\mathcal{N}^*_{p,\varepsilon}g(x)| \geq Ce^{x_0^2/p}\frac{y_0}{x_0}e^{a_0(x)}\bigg|\int_{-1}^{1} & e^{a_1(x)\left(\frac{y_0}{x_0}\right)u + a_2(x)\left(\frac{y_0}{x_0}\right)^2u^2}e^{ib(x)u^2\left(\frac{y_0}{x_0}\right)^2}\\ &\left(\frac{u\frac{y_0}{x_0} + y_0}{x}\right)^{\alpha+1/2}G'_\alpha\left(\frac{ix(u\frac{y_0}{x_0} + y_0)}{2}(t_x^{-1}e^{-i\phi_p} - t_xe^{i\phi_p})\right)\mathrm{d}u\bigg|.\end{aligned}$$

A rather easy computation gives

$$\begin{aligned}&a_0(x) = O(1), && a_1(x) = O(1/y_0),\\ &a_2(x) = -\cos\phi_p(x_0^2/y_0^2) + O(x^4/y_0^4), && b(x) = O((x_0/y_0)^3) + O(x_0/y_0^3),\end{aligned}$$

as x in $\left[x_0, \frac{1}{x_0}\right]$, $y_0 \geq x_0^\delta$, $\delta > 1$. As z tends to infinity, $G'_\alpha(z) = 1 + O(|z|^{-1})$. By these asymptotic estimates, if x_0 is sufficiently large and $y_0 \geq x_0^\delta$, with $\delta > 1$, there exists a positive constant C such that the right hand side is bounded from below, i.e.

$$\begin{aligned}|\mathcal{N}^*_{p,\varepsilon}g(x)| &\geq Ce^{x_0^2/p}\left(\frac{y_0}{x_0}\right)^{\alpha+3/2}\left|\int_{-1}^{1}\exp(-\cos\phi_pu^2)\,\mathrm{d}u\right|\\ &\geq Ce^{x_0^2/p}\left(\frac{y_0}{x_0}\right)^{\alpha+3/2}.\end{aligned}$$

Thus (38) holds and the theorem is proved.

REMARK 5.6. Several cases remain to study, as the $L^{p'}(\tilde{\mu}_\alpha)$-boundedness of $\widetilde{\mathcal{M}}^{\mathrm{gl}}_{\alpha,p}$, with $1 < p < 2$. Recently P. Sjögren proved that the maximal operator $\mathcal{H}^*_p$ associated to the Ornstein-Uhlenbeck semigroup is of weak-type p', but not of strong type p' with respect to the Gaussian measure (see [11]). Nevertheless, these results cannot be easily extended to the corresponding maximal operators associated to the Laguerre semigroup, by means of the same techniques used in this work.

REFERENCES

1. Anker, J. P., *L_p Fourier multipliers on Riemannian symmetric spaces of the noncompact type*, Ann. of Math. 132 (1990), 597–628.
2. Carleson, L., *Some analytic problems related to statistical mechanics, in Euclidean harmonic Analysis*, Lecture Notes in Math. 779 (1979), 5–45.
3. Dinger, U., *Weak Type* (1, 1) *Estimates of the maximal function for the Laguerre semigroup in finite dimensions*, Revista Matemática Iberoamericana, vol. 8, no. 1 (1992), 93–118.
4. Erdélyi, M., Magnus, Oberhettinger, Tricomi, *Higher Trascendental functions*, Vol II, New York, 1953.
5. Garcìa-Cuerva, J., Mauceri, G., Sjögren, P., and Torrea, J., *Spectral multipliers for the Ornstein-Uhlenbeck semigroup*, J. D'Analyse Mat. 78 (1999), 281–305.
6. Garcìa-Cuerva, J., Mauceri, G., Meda, S., Sjögren, P., and Torrea, J., *Maximal operators for the holomorphic Ornstein-Uhlenbeck semigroup*, J. London Math. Soc. (2) 67 (2003), no. 1, 219–234.
7. Gutiérrez, C., Incognito, A., and Torrea, J., *Riesz Transforms, g-functions and multipliers for the Laguerre semigroup*, Houston J. Math. 27 (2001), no. 3, 579–592.
8. Guy, D. L., *Hankel Multplier Transformations and weighted p-norms*, Trans. Amer. Math. Soc. 95 (1960), 137–189.
9. Muckenhoupt, B., *Poisson integral for Hermite and Laguerre expansions*, Trans. Amer. Math. Soc. 139 (1969), 231–242.
10. Sasso, E., *Functional calculus for the Laguerre operator*, Math. Z. 249 (2005), no. 3, 683–711.
11. Sjögren, P., *Maximal operators related to the Ornstein-Uhlenbeck semigroup with complex time parameter*, preprint.
12. Stein, E. M., *Topics in Harmonic Analysis related to the Littlewood-Paley Theory*, Princeton Univ. Press, 1970.

DIPARTIMENTO DI MATEMATICA
VIA DODECANESO 35
16146 GENOVA
ITALY
E-mail: sasso@dima.unige.it